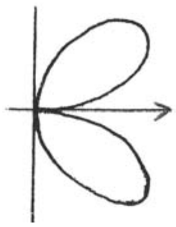
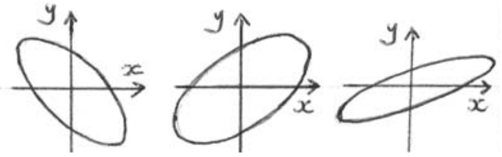
4756 (FP2) Further Methods for Advanced Mathematics

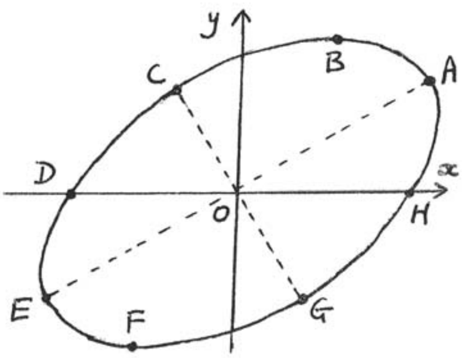
1(a)(i)	$x = r\cos\theta,\ y = r\sin\theta$ $(r^2\cos^2\theta + r^2\sin^2\theta)^2 = 3(r\cos\theta)(r\sin\theta)^2$ $r^4 = 3r^3\cos\theta\sin^2\theta$ $r = 3\cos\theta\sin^2\theta$	M1 A1 A1 ag **3**	(M0 for $x = \cos\theta,\ y = \sin\theta$)
(ii)		B1 B1 B1 **3**	Loop in 1st quadrant Loop in 4th quadrant Fully correct curve *Curve may be drawn using continuous or broken lines in any combination*
(b)	$\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,dx = \left[\frac{1}{\sqrt{3}}\arcsin\frac{\sqrt{3}\,x}{2}\right]_0^1$ $= \frac{1}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2}$ $= \frac{\pi}{3\sqrt{3}}$	M1 A1A1 M1 A1 **5**	For arcsin For $\frac{1}{\sqrt{3}}$ and $\frac{\sqrt{3}\,x}{2}$ Exact numerical value *Dependent on first M1* (M1A0 for $60/\sqrt{3}$)
	OR Put $\sqrt{3}\,x = 2\sin\theta$ — M1, A1 $\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,dx = \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{3}}\,d\theta$ — A1 $= \frac{\pi}{3\sqrt{3}}$ — M1A1		Any sine substitution For $\int \frac{1}{\sqrt{3}}\,d\theta$ *M1 dependent on first M1*
(c)(i)	$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \ldots$ $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \ldots$	B1 B1 **2**	*Accept unsimplified forms*
(ii)	$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$ $= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \ldots$	M1 A1 **2**	Obtained from two correct series *Terms need not be added* If M0, then B1 for $2x + \frac{2}{3}x^3 + \frac{2}{5}x^5$

(iii)	$\sum_{r=0}^{\infty}\frac{1}{(2r+1)4^r}=1+\frac{1}{3\times4}+\frac{1}{5\times4^2}+\dots$ $=2\times\frac{1}{2}+\frac{2}{3}\times(\frac{1}{2})^3+\frac{2}{5}\times(\frac{1}{2})^5+\dots$ $=\ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)=\ln 3$	B1 B1 B1 ag **3**	*Terms need not be added* For $x=\frac{1}{2}$ seen or implied Satisfactory completion
2 (i)	$\lvert z\rvert=8,\quad \arg z=\frac{1}{4}\pi$ $\lvert z^*\rvert=8,\quad \arg z^*=-\frac{1}{4}\pi$ $\lvert zw\rvert=8\times8=64$ $\arg(zw)=\frac{1}{4}\pi+\frac{7}{12}\pi=\frac{5}{6}\pi$ $\left\lvert\frac{z}{w}\right\rvert=\frac{8}{8}=1$ $\arg(\frac{z}{w})=\frac{1}{4}\pi-\frac{7}{12}\pi=-\frac{1}{3}\pi$	B1B1 B1 ft B1 ft B1 ft B1 ft B1 ft **7**	*Must be given separately* *Remainder may be given in exponential or* $r\,\mathrm{cjs}\,\theta$ *form* (B0 for $\frac{7}{4}\pi$) (B0 if left as $8/8$)
(ii)	$\frac{z}{w}=\cos(-\frac{1}{3}\pi)+\mathrm{j}\sin(-\frac{1}{3}\pi)$ $=\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{j}$ $a=\frac{1}{2},\quad b=-\frac{1}{2}\sqrt{3}$	M1 A1 **2**	If M0, then B1B1 for $\frac{1}{2}$ and $-\frac{\sqrt{3}}{2}$
(iii)	$r=\sqrt[3]{8}=2$ $\theta=\frac{1}{12}\pi$ $\theta=\frac{\pi}{12}+\frac{2k\pi}{3}$ $\theta=-\frac{7}{12}\pi,\quad \frac{3}{4}\pi$	B1 ft B1 M1 A1 **4**	Accept $\sqrt[3]{8}$ Implied by one further correct (ft) value *Ignore values outside the required range*
(iv)	$w^*=8\mathrm{e}^{-\frac{7}{12}\pi\mathrm{j}},\quad$ so $\;2\mathrm{e}^{-\frac{7}{12}\pi\mathrm{j}}=\frac{1}{4}w^*$ $k_1=\frac{1}{4}$ $z^*=8\mathrm{e}^{-\frac{1}{4}\pi\mathrm{j}}=-8\mathrm{e}^{\frac{3}{4}\pi\mathrm{j}}$ So $\;2\mathrm{e}^{\frac{3}{4}\pi\mathrm{j}}=-\frac{1}{4}z^*$ $k_2=-\frac{1}{4}$ $\mathrm{j}w=8\mathrm{e}^{(\frac{1}{2}\pi+\frac{7}{12}\pi)\mathrm{j}}=8\mathrm{e}^{\frac{13}{12}\pi\mathrm{j}}$ $=-8\mathrm{e}^{\frac{1}{12}\pi\mathrm{j}},\quad$ so $\;2\mathrm{e}^{\frac{1}{12}\pi\mathrm{j}}=-\frac{1}{4}\mathrm{j}w$ $k_3=-\frac{1}{4}$	B1 ft M1 A1 ft M1 A1 ft **5**	Matching *w** to a cube root with argument $-\frac{7}{12}\pi$ and $k_1=\frac{1}{4}$ or ft ft is $\frac{r}{8}$ Matching *z** to a cube root with argument $\frac{3}{4}\pi$ *May be implied* ft is $-\frac{r}{\lvert z^*\rvert}$ Matching j*w* to a cube root with argument $\frac{1}{12}\pi$ *May be implied* OR M1 for $\arg(\mathrm{j}w)=\frac{1}{2}\pi+\arg w$ *(implied by* $\frac{13}{12}\pi$ *or* $-\frac{11}{12}\pi$*)* ft is $-\frac{r}{8}$

3 (i)	$\mathbf{Q}^{-1}=\frac{1}{k-3}\begin{pmatrix}-1 & k+2 & -1\\ 1 & 4-3k & k-2\\ 1 & -5 & 1\end{pmatrix}$ When $k=4$, $\mathbf{Q}^{-1}=\begin{pmatrix}-1 & 6 & -1\\ 1 & -8 & 2\\ 1 & -5 & 1\end{pmatrix}$	M1 A1 M1 A1 M1 A1 **6**	Evaluation of determinant *(must involve k)* For $(k-3)$ Finding at least four cofactors *(including one involving k)* Six signed cofactors correct *(including one involving k)* Transposing and dividing by det *Dependent on previous M1M1* $\mathbf{Q}^{-1}$ correct (in terms of *k*) and result for $k=4$ stated After 0, SC1 for $\mathbf{Q}^{-1}$ when $k=4$ obtained correctly with some working
(ii)	$\mathbf{P}=\begin{pmatrix}2 & -1 & 4\\ 1 & 0 & 1\\ 3 & 1 & 2\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 3\end{pmatrix}$ $\mathbf{M}=\mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ $=\begin{pmatrix}2 & -1 & 4\\ 1 & 0 & 1\\ 3 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 3\end{pmatrix}\begin{pmatrix}-1 & 6 & -1\\ 1 & -8 & 2\\ 1 & -5 & 1\end{pmatrix}$ $=\begin{pmatrix}2 & 1 & 12\\ 1 & 0 & 3\\ 3 & -1 & 6\end{pmatrix}\begin{pmatrix}-1 & 6 & -1\\ 1 & -8 & 2\\ 1 & -5 & 1\end{pmatrix}$ $=\begin{pmatrix}11 & -56 & 12\\ 2 & -9 & 2\\ 2 & -4 & 1\end{pmatrix}$	B1B1 B2 M1 A2 **7**	For B2, order must be consistent Give B1 for $\mathbf{M}=\mathbf{P}^{-1}\mathbf{D}\mathbf{P}$ or $\begin{pmatrix}2 & -1 & 4\\ 1 & 0 & 1\\ 3 & 1 & 2\end{pmatrix}\begin{pmatrix}-1 & 6 & -1\\ -1 & 8 & -2\\ 3 & -15 & 3\end{pmatrix}$ Good attempt at multiplying two matrices (no more than 3 errors), leaving third matrix in correct position Give A1 for five elements correct Correct **M** implies B2M1A2 5-8 elements correct implies B2M1A1
(iii)	Characteristic equation is $(\lambda-1)(\lambda+1)(\lambda-3)=0$ $\lambda^3-3\lambda^2-\lambda+3=0$ $\mathbf{M}^3=3\mathbf{M}^2+\mathbf{M}-3\mathbf{I}$ $\mathbf{M}^4=3\mathbf{M}^3+\mathbf{M}^2-3\mathbf{M}$ $=3(3\mathbf{M}^2+\mathbf{M}-3\mathbf{I})+\mathbf{M}^2-3\mathbf{M}$ $=10\mathbf{M}^2-9\mathbf{I}$ $a=10,\ b=0,\ c=-9$	B1 M1 A1 M1 A1 **5**	In any correct form *(Condone omission of =0)* **M** satisfies the characteristic equation Correct expanded form *(Condone omission of* **I** *)*

4 (i)	$\cosh^2 x = [\frac{1}{2}(e^x + e^{-x})]^2 = \frac{1}{4}(e^{2x} + 2 + e^{-2x})$ $\sinh^2 x = [\frac{1}{2}(e^x - e^{-x})]^2 = \frac{1}{4}(e^{2x} - 2 + e^{-2x})$ $\cosh^2 x - \sinh^2 x = \frac{1}{4}(2+2) = 1$	B1 B1 B1 ag **3**	For completion
	OR $\cosh x + \sinh x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = e^x$ B1 $\cosh x - \sinh x = \frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x}) = e^{-x}$ B1 $\cosh^2 x - \sinh^2 x = e^x \times e^{-x} = 1$ B1		Completion
(ii)	$4(1+\sinh^2 x) + 9\sinh x = 13$ $4\sinh^2 x + 9\sinh x - 9 = 0$ $\sinh x = \frac{3}{4},\ -3$ $x = \ln 2,\ \ln(-3+\sqrt{10})$	M1 M1 A1A1 A1A1 ft **6**	(M0 for $1-\sinh^2 x$) Obtaining a value for $\sinh x$ Exact logarithmic form *Dep on M1M1* Max A1 if any extra values given
	OR $2e^{4x} + 9e^{3x} - 22e^{2x} - 9e^x + 2 = 0$ $(2e^{2x} - 3e^x - 2)(e^{2x} + 6e^x - 1) = 0$ M1 M1 $e^x = 2,\ -3+\sqrt{10}$ A1A1 $x = \ln 2,\ \ln(-3+\sqrt{10})$ A1A1 ft		Quadratic and / or linear factors Obtaining a value for e^x Ignore extra values *Dependent on M1M1* Max A1 if any extra values given *Just* $x = \ln 2$ *earns M0M1A1A0A0A0*
(iii)	$\frac{dy}{dx} = 8\cosh x \sinh x + 9\cosh x$ $= \cosh x(8\sinh x + 9)$ $= 0$ only when $\sinh x = -\frac{9}{8}$ $\cosh^2 x = 1 + (-\frac{9}{8})^2 = \frac{145}{64}$ $y = 4\times\frac{145}{64} + 9\times(-\frac{9}{8}) = -\frac{17}{16}$	B1 B1 M1 A1 **4**	Any correct form *or* $y = (2\sinh x + \frac{9}{4})^2 + \ldots\ (-\frac{17}{16})$ Correctly showing there is only one solution Exact evaluation of *y* or $\cosh^2 x$ or $\cosh 2x$ Give B2 (replacing M1A1) for -1.06 or better
(iv)	$\int_0^{\ln 2} (2 + 2\cosh 2x + 9\sinh x)\,dx$ $= [\ 2x + \sinh 2x + 9\cosh x\ ]_0^{\ln 2}$ $= \left\{ 2\ln 2 + \frac{1}{2}\left(4 - \frac{1}{4}\right) + \frac{9}{2}\left(2 + \frac{1}{2}\right)\right\} - 9$ $= 2\ln 2 + \frac{33}{8}$	M1 A2 M1 A1 ag **5**	Expressing in integrable form Give A1 for two terms correct $\sinh(2\ln 2) = \frac{1}{2}(4 - \frac{1}{4})$ *Must see both terms for M1* *Must also see* $\cosh(\ln 2) = \frac{1}{2}(2 + \frac{1}{2})$ *for A1*

	OR $\int_0^{\ln 2}(e^{2x}+2+e^{-2x}+\frac{9}{2}(e^x-e^{-x}))\,dx$ M1 $=\left[\frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x}+\frac{9}{2}e^x+\frac{9}{2}e^{-x}\right]_0^{\ln 2}$ A2 $=\left(2+2\ln 2-\frac{1}{8}+9+\frac{9}{4}\right)-\left(\frac{1}{2}-\frac{1}{2}+\frac{9}{2}+\frac{9}{2}\right)$ M1 $=2\ln 2+\frac{33}{8}$ A1 ag		Expanded exponential form (M0 if the 2 is omitted) Give A1 for three terms correct $e^{2\ln 2}=4$ and $e^{-2\ln 2}=\frac{1}{4}$ *both seen* *Must also see* $e^{\ln 2}=2$ and $e^{-\ln 2}=\frac{1}{2}$ *for A1*
5 (i)	$\lambda=0.5$ $\lambda=3$ $\lambda=5$ 	B1B1B1 **3**	
(ii)	Ellipse	B1 **1**	
(iii)	$y=\sqrt{2}\cos(\theta-\frac{1}{4}\pi)$ Maximum $y=\sqrt{2}$ when $\theta=\frac{1}{4}\pi$	M1 A1 ag **2**	or $\sqrt{2}\sin(\theta+\frac{1}{4}\pi)$
	OR $\frac{dy}{d\theta}=-\sin\theta+\cos\theta=0$ when $\theta=\frac{1}{4}\pi$ M1 $y=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}$ A1		
(iv)	$x^2+y^2=\lambda^2\cos^2\theta-2\cos\theta\sin\theta+\frac{1}{\lambda^2}\sin^2\theta$ $+\cos^2\theta+2\cos\theta\sin\theta+\sin^2\theta$ $=(\lambda^2+1)(1-\sin^2\theta)+(\frac{1}{\lambda^2}+1)\sin^2\theta$ $=1+\lambda^2+(\frac{1}{\lambda^2}-\lambda^2)\sin^2\theta$ When $\sin^2\theta=0$, $x^2+y^2=1+\lambda^2$ When $\sin^2\theta=1$, $x^2+y^2=1+\frac{1}{\lambda^2}$ Since $0\le\sin^2\theta\le 1$, distance from O, $\sqrt{x^2+y^2}$, is between $\sqrt{1+\frac{1}{\lambda^2}}$ and $\sqrt{1+\lambda^2}$	M1 M1 A1 ag M1 M1 A1 ag **6**	Using $\cos^2\theta=1-\sin^2\theta$
(v)	When $\lambda=1$, $x^2+y^2=2$ Curve is a circle (centre O) with radius $\sqrt{2}$	M1 A1 **2**	

(vi)		B4 **4**	A, E at maximum distance from O C, G at minimum distance from O B, F are stationary points D, H are on the *x*-axis Give ½ mark for each point, then round down Special properties must be clear from diagram, or stated *Max 3 if curve is not the correct shape*