GCE

Mathematics (MEI)

Advanced GCE

Unit **4756:** Further Methods for Advanced Mathematics

Mark Scheme for January 2012

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This mark scheme is published as an aid to teachers and students, to indicate the requirements of the examination. It shows the basis on which marks were awarded by examiners. It does not indicate the details of the discussions which took place at an examiners' meeting before marking commenced.

All examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the report on the examination.

OCR will not enter into any discussion or correspondence in connection with this mark scheme.

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Annotations

Annotation	Meaning
✓and ✗	
BOD	Benefit of doubt
FT	Follow through
ISW	Ignore subsequent working
M0, M1	Method mark awarded 0, 1
A0, A1	Accuracy mark awarded 0, 1
B0, B1	Independent mark awarded 0, 1
SC	Special case
^	Omission sign
MR	Misread
Highlighting	
Other abbreviations in mark scheme	**Meaning**
E1	Mark for explaining
U1	Mark for correct units
G1	Mark for a correct feature on a graph
M1 dep*	Method mark dependent on a previous mark, indicated by *
cao	Correct answer only
oe	Or equivalent
rot	Rounded or truncated
soi	Seen or implied
www	Without wrong working

Subject-specific Marking Instructions

a Annotations should be used whenever appropriate during your marking.

The A, M and B annotations must be used on your standardisation scripts for responses that are not awarded either 0 or full marks. It is vital that you annotate standardisation scripts fully to show how the marks have been awarded.

For subsequent marking you must make it clear how you have arrived at the mark you have awarded.

b An element of professional judgement is required in the marking of any written paper. Remember that the mark scheme is designed to assist in marking incorrect solutions. Correct *solutions* leading to correct answers are awarded full marks but work must not be judged on the answer alone, and answers that are given in the question, especially, must be validly obtained; key steps in the working must always be looked at and anything unfamiliar must be investigated thoroughly.

Correct but unfamiliar or unexpected methods are often signalled by a correct result following an *apparently* incorrect method. Such work must be carefully assessed. When a candidate adopts a method which does not correspond to the mark scheme, award marks according to the spirit of the basic scheme; if you are in any doubt whatsoever (especially if several marks or candidates are involved) you should contact your Team Leader.

c The following types of marks are available.

M
A suitable method has been selected and *applied* in a manner which shows that the method is essentially understood. Method marks are not usually lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, eg by substituting the relevant quantities into the formula. In some cases the nature of the errors allowed for the award of an M mark may be specified.

A
Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated Method mark is earned (or implied). Therefore M0 A1 cannot ever be awarded.

B
Mark for a correct result or statement independent of Method marks.

E
A given result is to be established or a result has to be explained. This usually requires more working or explanation than the establishment of an unknown result.

Unless otherwise indicated, marks once gained cannot subsequently be lost, eg wrong working following a correct form of answer is ignored. Sometimes this is reinforced in the mark scheme by the abbreviation isw. However, this would not apply to a case where a candidate passes through the correct answer as part of a wrong argument.

d When a part of a question has two or more 'method' steps, the M marks are in principle independent unless the scheme specifically says otherwise; and similarly where there are several B marks allocated. (The notation 'dep *' is used to indicate that a particular mark is dependent on an earlier, asterisked, mark in the scheme.) Of course, in practice it may happen that when a candidate has once gone wrong in a part of a question, the work from there on is worthless so that no more marks can sensibly be given. On the other hand, when two or more steps are successfully run together by the candidate, the earlier marks are implied and full credit must be given.

e The abbreviation ft implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A and B marks are given for correct work only — differences in notation are of course permitted. A (accuracy) marks are not given for answers obtained from incorrect working. When A or B marks are awarded for work at an intermediate stage of a solution, there may be various alternatives that are equally acceptable. In such cases, exactly what is acceptable will be detailed in the mark scheme rationale. If this is not the case please consult your Team Leader.

Sometimes the answer to one part of a question is used in a later part of the same question. In this case, A marks will often be 'follow through'. In such cases you must ensure that you refer back to the answer of the previous part question even if this is not shown within the image zone. You may find it easier to mark follow through questions candidate-by-candidate rather than question-by-question.

f Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise. Candidates are expected to give numerical answers to an appropriate degree of accuracy, with 3 significant figures often being the norm. Small variations in the degree of accuracy to which an answer is given (e.g. 2 or 4 significant figures where 3 is expected) should not normally be penalised, while answers which are grossly over- or under-specified should normally result in the loss of a mark. The situation regarding any particular cases where the accuracy of the answer may be a marking issue should be detailed in the mark scheme rationale. If in doubt, contact your Team Leader.

g Rules for replaced work

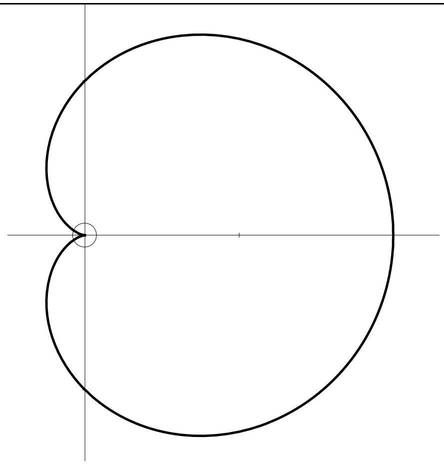
If a candidate attempts a question more than once, and indicates which attempt he/she wishes to be marked, then examiners should do as the candidate requests.

If there are two or more attempts at a question which have not been crossed out, examiners should mark what appears to be the last (complete) attempt and ignore the others.

NB Follow these maths-specific instructions rather than those in the assessor handbook.

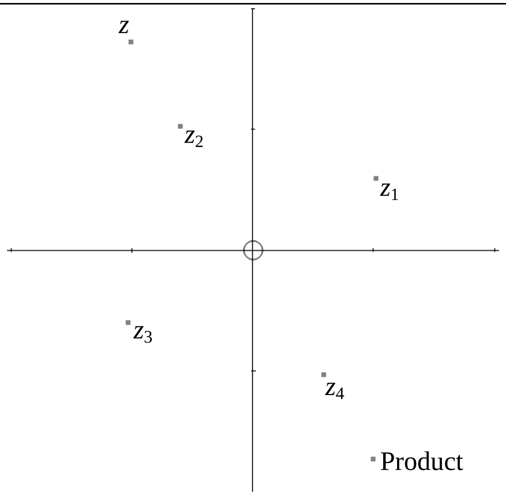
h For a *genuine* misreading (of numbers or symbols) which is such that the object and the difficulty of the question remain unaltered, mark according to the scheme but following through from the candidate's data. A penalty is then applied; 1 mark is generally appropriate, though this may differ for some units. This is achieved by withholding one A mark in the question.

Note that a miscopy of the candidate's own working is not a misread but an accuracy error.

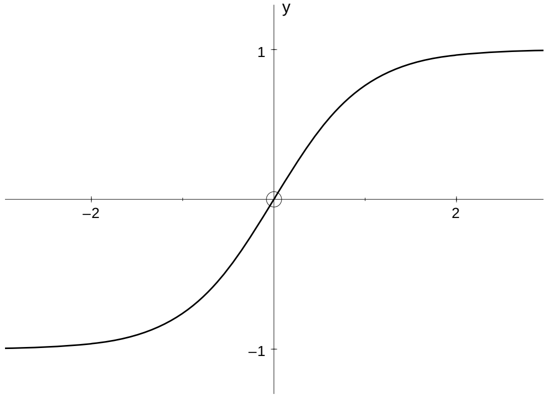
Question			Answer	Marks	Guidance	
1	(a)	(i)		G2 [2]	A fully correct curve Give G1 for one error, e.g. incorrect form at O, lack of clear symmetry, sharp point at RH extremity	
1	(a)	(ii)	Area $=\frac{1}{2}\int_0^{2\pi}(1+\cos\theta)^2\,d\theta$	M1	Integral expression involving $(1 + \cos\theta)^2$	
			$=\frac{1}{2}\int_0^{2\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$	A1	Correct expanded integral expression, incl. limits	Limits may be implied by later work. Penalise missing ½ here (max. 4/6)
			$=\frac{1}{2}\int_0^{2\pi}\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$	M1	Using $\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos 2\theta$	Allow sign or factor errors
			$=\frac{1}{2}\left[\frac{3}{2}\theta+2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{2\pi}$	A2	Correct result of integration	Give A1 for one error in this expression
			$=\frac{3}{2}\pi$	A1 [6]	Dependent on previous A2	
1	(b)		$\sin x = x-\frac{1}{6}x^3\ldots$ $\cos x = 1-\frac{1}{2}x^2\ldots$	B1	Both series correct as far as second term	Ignore higher-order terms. Allow denominators left as 2!, 3!
			$\tan x \approx \left(x-\frac{1}{6}x^3\right)\left(1-\frac{1}{2}x^2\right)^{-1}$	M1	Using $\tan x = \frac{\sin x}{\cos x}$	Allow even if no further progress but must be used, not just stated

Question			Answer	Marks	Guidance	
			$=\left(x-\frac{1}{6}x^3\right)\left(1+\frac{1}{2}x^2+\ldots\right)$	M1	Using binomial expansion. Dependent on first M1	If methods mixed, mark to benefit of candidate
			$= x+\frac{1}{2}x^3-\frac{1}{6}x^3+\ldots$	M1	Expanding brackets. Dependent on previous M1	
			$= x+\frac{1}{3}x^3\ldots$	A1A1	$a = 1$, $b = \frac{1}{3}$ correctly obtained	Dependent on both M1s. Deduct 1 for each additional term (*)
			OR $\dfrac{x-\frac{1}{6}x^3}{1-\frac{1}{2}x^2}=ax+bx^3$ M1		Using $\tan x=\dfrac{\sin x}{\cos x}$	
			$\Rightarrow x-\frac{1}{6}x^3=\left(1-\frac{1}{2}x^2\right)\left(ax+bx^3\right)=ax+\left(b-\frac{1}{2}a\right)x^3+\ldots$ M1		Attempting to compare coeffs.	
			$\Rightarrow a=1$ A1		Correctly obtained	As (*)
			$b-\frac{1}{2}a=-\frac{1}{6}$ M1		Obtaining b	
			$\Rightarrow b=\frac{1}{3}$ A1		Correctly obtained	As (*)
			OR $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$			
			$f''(x) = 2\sec^2 x \tan x$ M1		Attempting first two derivatives	
			$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$ M1		Attempting third derivative	Using the product rule
			$f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = 2$			
			$f(x)=f(0)+xf'(0)+\dfrac{x^2f''(0)}{2!}+\ldots$ M1		Applying Maclaurin series. Dependent on first M1	
			$=x+\frac{1}{3}x^3\ldots$ A1A1		Correctly obtained	As (*)
				[6]		
1	(c)		$\displaystyle\int_0^1 \frac{1}{\sqrt{1-\frac{1}{4}x^2}}\,dx=\int_0^1\frac{2}{\sqrt{4-x^2}}\,dx=\left[2\arcsin\frac{x}{2}\right]_0^1$	M1	arcsin alone, or any sine substitution	
				A1	2 and $\dfrac{x}{2}$	
			$=2\left(\dfrac{\pi}{6}-0\right)$	M1	Using limits. Dependent on first M1	No need to see explicit use of $x = 0$ Limits wrong way round M0
			$=\dfrac{\pi}{3}$	A1	Evaluated in terms of π	
				[4]		

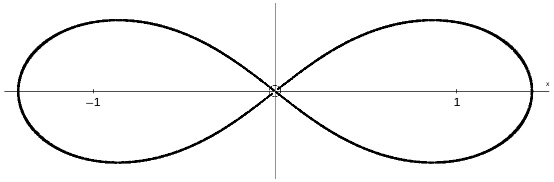
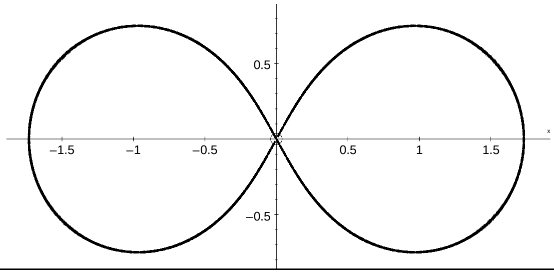
Question			Answer	Marks	Guidance	
2	(a)		$C + jS = 1 + ae^{j\theta} + a^2e^{2j\theta} + \ldots$	M1	Forming $C + jS$ as a series of powers	$\ldots a^2(\cos 2\theta + j\sin 2\theta)$ insufficient. Powers must be correct
			This is a geometric series with $r = ae^{j\theta}$	M1	Identifying G.P. and attempting sum. Dependent on first M1	
			Sum to infinity $= \dfrac{1}{1-ae^{j\theta}}$	A1		
			$= \dfrac{1}{1-ae^{j\theta}} \times \dfrac{1-ae^{-j\theta}}{1-ae^{-j\theta}}$	M1*	Multiplying numerator and denominator by $1-ae^{-j\theta}$ o.e.	
			$= \dfrac{1-ae^{-j\theta}}{1-ae^{j\theta}-ae^{-j\theta}+a^2}$	M1	Multiplying out denominator. Dependent on M1*	Use of FOIL with powers combined correctly (allow one slip)
			$= \dfrac{1-a(\cos\theta - j\sin\theta)}{1-2a\cos\theta+a^2}$	M1	Introducing trig functions. Dependent on M1*	Condone e.g. $e^{-j\theta} = \cos\theta + j\sin\theta$
			$= \dfrac{1-a\cos\theta}{1-2a\cos\theta+a^2} + \dfrac{aj\sin\theta}{1-2a\cos\theta+a^2}$			
			$\Rightarrow C = \dfrac{1-a\cos\theta}{1-2a\cos\theta+a^2}$	E1		Answer given. www which leads to C
			and $S = \dfrac{a\sin\theta}{1-2a\cos\theta+a^2}$	A1		
				[8]		
2	(b)		Modulus = 2	B1		
			Argument $= \dfrac{2\pi}{3}$	B1		
			$\Rightarrow -1 + j\sqrt{3} = 2e^{j\frac{2\pi}{3}}$			
			$\Rightarrow$ fourth roots have $r = \sqrt[4]{2}$	B1		Allow 1.19 or better
			and θ $= \dfrac{\pi}{6}$			
			$\Rightarrow$ roots are $\sqrt[4]{2}e^{j\frac{\pi}{6}}, \sqrt[4]{2}e^{j\frac{2\pi}{3}}, \sqrt[4]{2}e^{j\frac{7\pi}{6}}, \sqrt[4]{2}e^{j\frac{5\pi}{3}}$	M1 A1	÷ arg z by 4 and adding $\dfrac{\pi}{2}$ All arguments correct	$\theta = \dfrac{\pi}{6} + \dfrac{2k\pi}{4}$ scores M1; k = 0, 1, 2, 3 (or −2, −1, 0, 1) A1

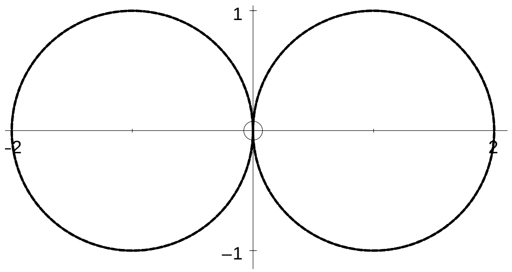
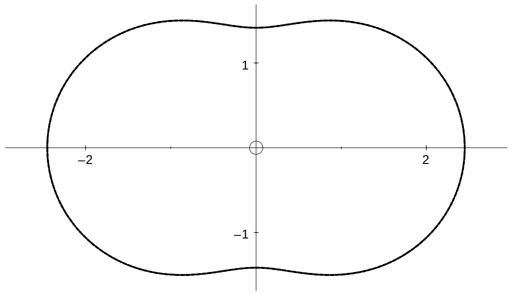
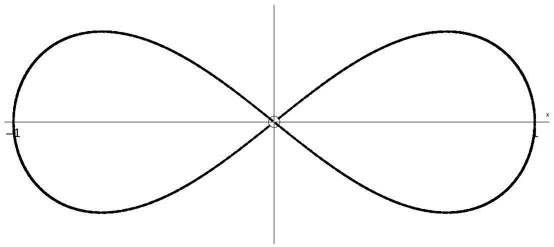
Question			Answer	Marks	Guidance	
				G1 G1ft G1ft	Position of z Roots forming square Position of product	In 2nd quadrant Ignore marked angles Correct or their $-z$
			Product of 4th roots $= 2e^{j(1+4+7+10)\frac{\pi}{6}}$	M1	Attempting to find product	Must consider both modulus and argument
			$= 2e^{j\frac{5\pi}{3}}$	A1 [10]	Or $-\dfrac{\pi}{3}$ o.e.	
3	(i)		$\mathbf{M} - \lambda\mathbf{I} = \begin{pmatrix} 3-\lambda & -1 & 2 \\ -4 & 3-\lambda & 2 \\ 2 & 1 & -1-\lambda \end{pmatrix}$			
			$\det(\mathbf{M} - \lambda\mathbf{I}) = (3-\lambda)[(3-\lambda)(-1-\lambda) - 2]$ $+ 1[-4(-1-\lambda) - 4] + 2[-4 - 2(3-\lambda)]$ $= (3-\lambda)(\lambda^2 - 2\lambda - 5) + 4\lambda + 2(2\lambda - 10)$ $= -\lambda^3 + 5\lambda^2 - \lambda - 15 + 4\lambda + 4\lambda - 20$	M1 A1 M1	Obtaining $\det(\mathbf{M} - \lambda\mathbf{I})$ Any correct form Multiplying out. Dep. on first M1	
			$\Rightarrow \lambda^3 - 5\lambda^2 - 7\lambda + 35 = 0$	E1 [4]		Answer given
3	(ii)		$\lambda^3 - 5\lambda^2 - 7\lambda + 35 = 0$ $\Rightarrow (\lambda - 5)(\lambda^2 - 7) = 0$	M1 A1 M1	Factorising, obtaining a quadratic Correct quadratic Solving quadratic	If M0, give B1 for substituting $\lambda = 5$
			$\lambda = \pm\sqrt{7}$	A1 [4]		Allow 2.65 or better

Question			Answer	Marks	Guidance	
3	(iii)		$\lambda = 5 \Rightarrow \begin{pmatrix} -2 & -1 & 2 \\ -4 & -2 & 2 \\ 2 & 1 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$			
			$\Rightarrow -2x - y + 2z = 0$ $-4x - 2y + 2z = 0$ $2x + y - 6z = 0$	M1	Two independent equations	Need to multiply out, unless implied by later work
			$\Rightarrow z = 0,\ y = -2x$	M1	Obtaining a non-zero eigenvector	
			$\Rightarrow$ eigenvector is $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$	A1		
			$\Rightarrow$ eigenvector of unit length is $\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$	A1ft	$\frac{1}{\sqrt{5}}$ f.t. their eigenvector	
			$\mathbf{M}^2\mathbf{v}$ has magnitude 25 in direction of $\mathbf{v}$ $\mathbf{M}^{-1}\mathbf{v}$ has magnitude $\frac{1}{5}$ in direction of $\mathbf{v}$	B1 B1 [6]	Both magnitudes c.a.o. Directions c.a.o.	May be given as column vectors
3	(iv)		$\lambda^3 - 5\lambda^2 - 7\lambda + 35 = 0$			
			$\Rightarrow \mathbf{M}^3 - 5\mathbf{M}^2 - 7\mathbf{M} + 35\mathbf{I} = \mathbf{0}$	M1	Using Cayley-Hamilton Theorem	Condone omitted $\mathbf{I}$
			$\Rightarrow \mathbf{M}^4 = 5\mathbf{M}^3 + 7\mathbf{M}^2 - 35\mathbf{M}$	A1	Correct expression involving $\mathbf{M}^4$ and non-negative powers of $\mathbf{M}$	
			$= 5(5\mathbf{M}^2 + 7\mathbf{M} - 35\mathbf{I}) + 7\mathbf{M}^2 - 35\mathbf{M}$	M1	Substituting for $\mathbf{M}^3$ and obtaining expression in required form	
			$= 32\mathbf{M}^2 - 175\mathbf{I}$	A1 [4]	$a = 32,\ b = 0,\ c = -175$	

Question			Answer	Marks	Guidance	
4	(i)		$\tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}}$	B1	Or $\frac{e^{2t} - 1}{e^{2t} + 1}$	Condone other variables used
				G1 G1 [3]	Correct shape Asymptotes at $y = \pm 1$. Dependent on first G1	 If text and graph conflict, mark what is shown on the graph
4	(ii)		$y = \text{artanh } x \Rightarrow x = \tanh y$ $\Rightarrow x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$ $\Rightarrow x(e^y + e^{-y}) = e^y - e^{-y}$	M1	First step in rearrangement	Or $x = \frac{e^{2y} - 1}{e^{2y} + 1}$ Variables the right way round at some stage, and clearing fractions
			$\Rightarrow xe^y + xe^{-y} = e^y - e^{-y}$ $\Rightarrow xe^{-y} + e^{-y} = e^y - xe^y$ $\Rightarrow e^{-y}(1+x) = e^y(1-x)$ $\Rightarrow e^{2y} = \frac{1+x}{1-x}$	M1 A1	Obtaining e^{2y} in terms of x	Dependent on first M1
			$\Rightarrow 2y = \ln\left(\frac{1+x}{1-x}\right)$ $\Rightarrow \text{artanh } x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	E1		Answer given
			Valid for $-1 < x < 1$	B1 [5]	Independent	

Question			Answer	Marks	Guidance	
4	(iii)		$\tanh y = x$ $\Rightarrow \operatorname{sech}^2 y \dfrac{dy}{dx} = 1$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\operatorname{sech}^2 y} = \dfrac{1}{1-\tanh^2 y}$ $= \dfrac{1}{1-x^2}$ $y = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) = \dfrac{1}{2}\ln(1+x) - \dfrac{1}{2}\ln(1-x)$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2} \times \dfrac{1}{1+x} - \dfrac{1}{2} \times \dfrac{-1}{1-x}$ $= \dfrac{1}{2} \times \dfrac{1-x+1+x}{(1+x)(1-x)}$ $= \dfrac{1}{1-x^2}$	M1 E1 M1 A1 E1 [5]	Differentiating and explicitly attempting to express in terms of tanh y Correctly obtained Attempting logarithmic diff. Any correct form Convincing manipulation	
4	(iv)		$\int_0^{\frac{1}{2}} \operatorname{artanh} x \, dx = [x \operatorname{artanh} x]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \dfrac{x}{1-x^2} dx$ $= \dfrac{1}{2}\operatorname{artanh}\dfrac{1}{2} - \left[-\dfrac{1}{2}\ln(1-x^2)\right]_0^{\frac{1}{2}}$ $= \dfrac{1}{4}\ln\left(\dfrac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) + \dfrac{1}{2}\ln\dfrac{3}{4}$ $= \dfrac{1}{4}\ln 3 + \dfrac{1}{4}\ln\dfrac{9}{16}$ $= \dfrac{1}{4}\ln\dfrac{27}{16}$	M1 A1 A1 M1 E1 [5]	Using integration by parts This line correct $-\dfrac{1}{2}\ln(1-x^2)$ Applying limits and using (ii) in result of integration Convincing manipulation www	With u = artanh x, $v' = 1$ and $v = x$ Condone omitted limits Or $-\dfrac{1}{2}\ln(1-x) - \dfrac{1}{2}\ln(1+x)$ Must be exact Answer given

Question			Answer	Marks	Guidance	
5	(i)		$((x+1)^2+y^2)((x-1)^2+y^2)=1$ $\Rightarrow (x^2+2x+1+y^2)(x^2-2x+1+y^2)=1$ $\Rightarrow (r^2+2r\cos\theta+1)(r^2-2r\cos\theta+1)=1$ $\Rightarrow (r^2+1)^2-4r^2\cos^2\theta=1$ $\Rightarrow r^4+2r^2=4r^2\cos^2\theta$ $\Rightarrow r^4=2r^2(2\cos^2\theta-1)$ $\Rightarrow r^4=2r^2\cos 2\theta$ $\Rightarrow r^2=2\cos 2\theta$	M1 E1 M1 E1 [4]	Using $r^2=x^2+y^2$ and $x=r\cos\theta$ Correctly obtained Using $\cos 2\theta = 2\cos^2\theta - 1$ Correctly obtained	
5	(ii)		$r^2 \ge 0 \Rightarrow \cos 2\theta \ge 0$ $\Rightarrow 0 \le \theta \le \frac{\pi}{4}, \frac{3\pi}{4} \le \theta \le \frac{5\pi}{4}, \frac{7\pi}{4} \le \theta \le 2\pi$ 	M1 A1 G2 [4]	Considering $\cos 2\theta \ge 0$ Or B2. Inequalities must be non-strict Curve must be complete. Award G1 for a curve with one error	Or $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}, \frac{3\pi}{4} \le \theta \le \frac{5\pi}{4}$
5	(iii)	(*A*)	$k=1$: 	G1	Curve must be complete	

Question			Answer	Marks	Guidance	
			$k = 2$:  	G2	Curve must be complete. Award G1 for a curve with one error	For G2, curve must appear sufficiently different from case $k = 1$
			For $k = 1$, the gradients at the pole are finite For $k = 2$, they appear to be infinite	B1		
		(*B*)	$k = 4$:  	G2	Curve must be complete. Award G1 for a curve with one error	
			Tends to circle as k tends to infinity	B1 [7]		
5	(iv)		$k = -1$: 	G2	Curve must be complete. Award G1 for a curve with one error	
			As $k \to -2$, the curve retains its figure-of-eight shape, but contracts towards the origin	B1 [3]		