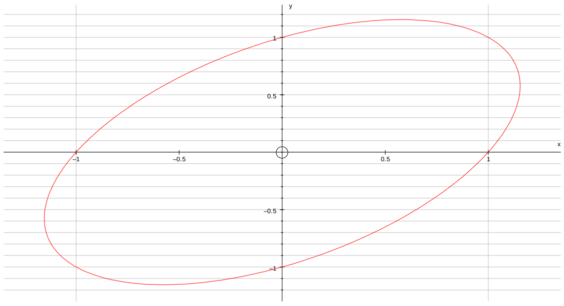
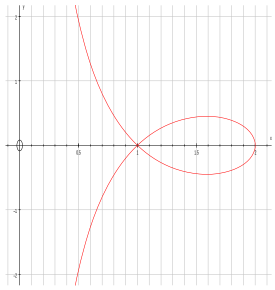
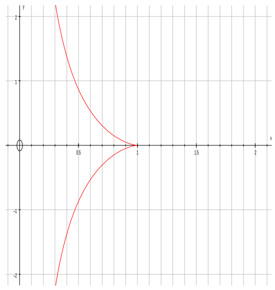
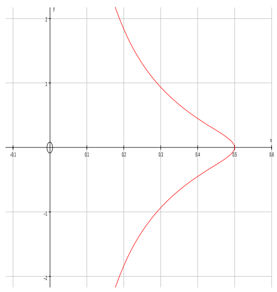
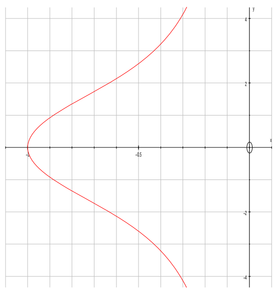
4756 (FP2) Further Methods for Advanced Mathematics

1 (a)	$y = \arctan\sqrt{x}$ $u = \sqrt{x}$, $y = \arctan u$ $\Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}}, \frac{dy}{du} = \frac{1}{1+u^2}$ $\Rightarrow \frac{dy}{dx} = \frac{1}{1+u^2} \times \frac{1}{2\sqrt{x}}$ $= \frac{1}{1+x} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}(x+1)}$	M1 A1 A1	Using Chain Rule Correct derivative in any form Correct derivative in terms of x
	OR $\tan y = \sqrt{x}$ $\Rightarrow \sec^2 y \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ M1A1 $\sec^2 y = 1 + \tan^2 y = 1 + x$ $\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}(x+1)}$ A1		Rearranging for $\sqrt{x}$ or x and differentiating implicitly
	$\Rightarrow \int_0^1 \frac{1}{\sqrt{x}(x+1)} dx = \left[2\arctan\sqrt{x}\right]_0^1$ $= 2 \arctan 1 - 2 \arctan 0$ $= 2 \times \frac{\pi}{4} = \frac{\pi}{2}$	M1 A1 A1 (ag)	Integral in form $k \arctan\sqrt{x}$ $k = 2$
(b)(i)	$x = r\cos\theta, y = r\sin\theta, x^2 + y^2 = r^2$ $x^2 + y^2 = xy + 1$ $\Rightarrow r^2 = r^2\cos\theta\sin\theta + 1$ $\Rightarrow r^2 = \frac{1}{2}r^2\sin 2\theta + 1$ $\Rightarrow 2r^2 = r^2\sin 2\theta + 2$ $\Rightarrow r^2(2 - \sin 2\theta) = 2$ $\Rightarrow r^2 = \frac{2}{2-\sin 2\theta}$	M1 A1 A1 A1 (ag) **4**	Using at least one of these LHS RHS Clearly obtained *SR*: $x = r\sin\theta$, $y = r\cos\theta$ used M1A1A0A0 max.
(ii)	Max r is $\sqrt{2}$ Occurs when $\sin 2\theta = 1$ $\Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$ Min $r = \sqrt{\frac{2}{3}}$ Occurs when $\sin 2\theta = -1$ $\Rightarrow \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$	B1 M1 A1 B1 M1 A1 **6**	 Attempting to solve Both. Accept degrees. A0 if extras in range $\frac{\sqrt{6}}{3}$ Attempting to solve (must be –1) Both. Accept degrees. A0 if extras in range

(iii)		G1 G1 **2**	Closed curve, roughly elliptical, with no points or dents Major axis along $y = x$ **18**
2 (a)	$\cos 5\theta + \mathrm{j}\sin 5\theta = (\cos\theta + \mathrm{j}\sin\theta)^5$ $= \cos^5\theta + 5\cos^4\theta\,\mathrm{j}\sin\theta + 10\cos^3\theta\,\mathrm{j}^2\sin^2\theta$ $+ 10\cos^2\theta\,\mathrm{j}^3\sin^3\theta + 5\cos\theta\,\mathrm{j}^4\sin^4\theta + \mathrm{j}^5\sin^5\theta$ $= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta + \mathrm{j}(\ldots)$ $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ $= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$ $= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$	M1 M1 A1 M1 M1 A1 **6**	Using de Moivre Using binomial theorem appropriately Correct real part. Must evaluate powers of j Equating real parts Replacing $\sin^2\theta$ by $1 - \cos^2\theta$ $a = 16$, $b = -20$, $c = 5$
(b)	$C + \mathrm{j}S$ $= e^{\mathrm{j}\theta} + e^{\mathrm{j}\left(\theta + \frac{2\pi}{n}\right)} + \ldots + e^{\mathrm{j}\left(\theta + \frac{(2n-2)\pi}{n}\right)}$ This is a G.P. $a = e^{\mathrm{j}\theta}$, $r = e^{\mathrm{j}\frac{2\pi}{n}}$ Sum $= \dfrac{e^{\mathrm{j}\theta}\left(1 - \left(e^{\mathrm{j}\frac{2\pi}{n}}\right)^n\right)}{1 - e^{\mathrm{j}\frac{2\pi}{n}}}$ Numerator $= e^{\mathrm{j}\theta}\left(1 - e^{2\pi\mathrm{j}}\right)$ and $e^{2\pi\mathrm{j}} = 1$ so sum = 0 $\Rightarrow$ $C = 0$ and $S = 0$	M1 A1 M1 A1 A1 E1 E1 **7**	Forming series $C + \mathrm{j}S$ as exponentials Need not see whole series Attempting to sum finite or infinite G.P. Correct a, r used or stated, and n terms Must see j Convincing explanation that sum = 0 $C = S = 0$. Dep. on previous E1 Both E marks dep. on 5 marks above
(c)	$e^t \approx 1 + t + \frac{1}{2}t^2$ $\dfrac{t}{e^t - 1} \approx \dfrac{t}{t + \frac{1}{2}t^2}$ $\dfrac{t}{t + \frac{1}{2}t^2} = \dfrac{1}{1 + \frac{1}{2}t} = \left(1 + \frac{1}{2}t\right)^{-1} = 1 - \frac{1}{2}t + \ldots$	B1 M1 A1 M1	Ignore terms in higher powers Substituting Maclaurin series Suitable manipulation and use of binomial theorem
	OR $\dfrac{1}{1 + \frac{1}{2}t} = \dfrac{1}{1 + \frac{1}{2}t} \times \dfrac{1 - \frac{1}{2}t}{1 - \frac{1}{2}t} = \dfrac{1 - \frac{1}{2}t}{1 - \frac{1}{4}t^2}$ M1		
	Hence $\dfrac{t}{e^t - 1} \approx 1 - \frac{1}{2}t$	A1 (ag)	
	OR $\left(e^t - 1\right)\left(1 - \frac{1}{2}t\right) = \left(t + \frac{1}{2}t^2 + \ldots\right)\left(1 - \frac{1}{2}t\right)$ M1 A1 $\approx t$ + terms in t^3 M1 $\Rightarrow$ $\dfrac{t}{e^t - 1} \approx 1 - \frac{1}{2}t$ A1		Substituting Maclaurin series Correct expression Multiplying out Convincing explanation
		5	**18**

3 (i)	$\mathbf{M}^{-1} = \frac{1}{4-a}\begin{pmatrix} 2 & -2-2a & 2+a \\ 2 & 2-3a & 2a-2 \\ -1 & 5 & -3 \end{pmatrix}$ When $a = -1$, $\mathbf{M}^{-1} = \frac{1}{5}\begin{pmatrix} 2 & 0 & 1 \\ 2 & 5 & -4 \\ -1 & 5 & -3 \end{pmatrix}$	M1 A1 M1 A1 M1 A1 **6**	Evaluating determinant $4 - a$ Finding at least four cofactors Six signed cofactors correct Transposing and dividing by det $\mathbf{M}^{-1}$ correct (in terms of a) and result for $a = -1$ stated *SR*: After 0 scored, SC1 for $\mathbf{M}^{-1}$ when $a = -1$, obtained correctly with some working
(ii)	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & 0 & 1 \\ 2 & 5 & -4 \\ -1 & 5 & -3 \end{pmatrix}\begin{pmatrix} -2 \\ b \\ 1 \end{pmatrix}$ $\Rightarrow \quad x = -\frac{3}{5},\ y = b - \frac{8}{5},\ z = b - \frac{1}{5}$	M2 M1 A2	Attempting to multiply $(-2 \ \ b \ \ 1)^T$ by given matrix (M0 if wrong order) Multiplying out A1 for one correct
	OR $4x + y = b - 4$ $x - y = 1 - b$ o.e. M1 M1 $\Rightarrow \quad x = -\frac{3}{5}$ A1 M1 $\Rightarrow \quad y = b - \frac{8}{5},\ z = b - \frac{1}{5}$ A1		Eliminating one unknown in 2 ways Or e.g. $3x + z = b - 2$, $5x = -3$ Or e.g. $3y - 4z = -b - 4$, $5y - 5z = -7$ Solve to obtain one value. Dep. on M1 above One unknown correct After M0, SC1 for value of x Finding the other two unknowns Both correct
		5	
(iii)	e.g. $3x - 3y = 2b + 2$ $5x - 5y = 4$ Consistent if $\frac{2b+2}{3} = \frac{4}{5}$ $\Rightarrow \quad b = \frac{1}{5}$ Solution is a line	M1 A1A1 M1 A1 B2 **7**	Eliminating one unknown in 2 ways Two correct equations Or e.g. $3x + 6z = b - 2$, $5x + 10z = -3$ Or e.g. $3y + 6z = -b - 4$, $5y + 10z = -7$ Attempting to find b **18**

4 (i)	$\sinh x = \frac{e^x - e^{-x}}{2} \Rightarrow \sinh^2 x = \frac{(e^x - e^{-x})^2}{4}$		
	$= \frac{e^{2x} - 2 + e^{-2x}}{4}$	B1	$e^{2x} - 2 + e^{-2x}$
	$\Rightarrow 2\sinh^2 x + 1 = \frac{e^{2x} - 2 + e^{-2x}}{2} + 1$		
	$= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x$	B1	Correct completion
	$\Rightarrow 2\sinh 2x = 4\sinh x\cosh x$	B1	Both correct derivatives
	$\Rightarrow \sinh 2x = 2\sinh x\cosh x$	B1	Correct completion
		4	
(ii)	$2\cosh 2x + 3\sinh x = 3$		
	$\Rightarrow 2(1 + 2\sinh^2 x) + 3\sinh x = 3$	M1	Using identity
	$\Rightarrow 4\sinh^2 x + 3\sinh x - 1 = 0$	A1	Correct quadratic
	$\Rightarrow (4\sinh x - 1)(\sinh x + 1) = 0$	M1	Solving quadratic
	$\Rightarrow \sinh x = \frac{1}{4}, -1$	A1	Both
		M1	Use of arsinh $x = \ln(x + \sqrt{x^2+1})$ o.e. Must obtain at least one value of x
	$\Rightarrow x = \text{arsinh}(\frac{1}{4}) = \ln(\frac{1+\sqrt{17}}{4})$	A1	Must evaluate $\sqrt{x^2+1}$
	$x = \text{arsinh}(-1) = \ln(-1+\sqrt{2})$	A1	
	OR $2e^{4x} + 3e^{3x} - 6e^{2x} - 3e^x + 2 = 0$		
	$\Rightarrow (2e^{2x} - e^x - 2)(e^{2x} + 2e^x - 1) = 0$ M1A1		Factorising quartic
	$\Rightarrow e^x = \frac{1 \pm \sqrt{17}}{4}$ or $-1 \pm \sqrt{2}$ M1A1		Solving either quadratic
	$\Rightarrow x = \ln(\frac{1+\sqrt{17}}{4})$ or $\ln(-1+\sqrt{2})$ M1A1A1		Using ln (dependent on first M1)
		7	
(iii)	$\cosh t = \frac{5}{4} \Rightarrow \frac{e^t + e^{-t}}{2} = \frac{5}{4}$		
	$\Rightarrow 2e^{2t} - 5e^t + 2 = 0$	M1	Forming quadratic in e^t
	$\Rightarrow (2e^t - 1)(e^t - 2) = 0$	M1	Solving quadratic
	$\Rightarrow e^t = \frac{1}{2}, 2$	A1	
	$\Rightarrow t = \pm\ln 2$	A1 (ag)	Convincing working
	$\int_4^5 \frac{1}{\sqrt{x^2-16}}\,dx = \left[\text{arcosh}\frac{x}{4}\right]_4^5$	B1	
	$= \text{arcosh}\frac{5}{4} - \text{arcosh } 1$	M1	Substituting limits
	$= \ln 2$	A1	A0 for $\pm\ln 2$
	OR $\int_4^5 \frac{1}{\sqrt{x^2-16}}\,dx = \left[\ln\left(x + \sqrt{x^2-16}\right)\right]_4^5$ B1		
	$= \ln 8 - \ln 4$ M1		Substituting limits
	$= \ln 2$ A1		
		7	18

5 (i)	Horz. projection of QP = $k \cos\theta$ Vert. projection of QP = $k \sin\theta$ Subtract OQ = $\tan\theta$	B1 B1 B1 **3**	 Clearly obtained
(ii)	$k = 2$  $k = 1$  $k = ½$  $k = −1$ 	G1 G1 G1 G1 **4**	Loop Cusp
(iii)(A) **(B)** **(C)**	for all k, y axis is an asymptote $k = 1$ $k > 1$	B1 B1 B1 **3**	Both
(iv)	Crosses itself at (1, 0) $k = 2 \Rightarrow \cos\theta = ½ \Rightarrow \theta = 60°$ $\Rightarrow$ curve crosses itself at 120°	 M1 A1 **2**	 Obtaining a value of θ Accept 240°
(v)	$y = 8\sin\theta - \tan\theta$ $\Rightarrow \dfrac{dy}{d\theta} = 8\cos\theta - \sec^2\theta$ $\Rightarrow 8\cos\theta - \dfrac{1}{\cos^2\theta} = 0$ at highest point $\Rightarrow \cos^3\theta = \dfrac{1}{8} \Rightarrow \cos\theta = \pm\dfrac{1}{2} \Rightarrow \theta = 60°$ at top $\Rightarrow x = 4$ $y = 3\sqrt{3}$	 M1 A1 A1 **3**	 Complete method giving θ Both
(vi)	RHS = $\dfrac{(k\cos\theta-1)^2}{k^2\cos^2\theta}\left(k^2-k^2\cos^2\theta\right)$ $= \dfrac{(k\cos\theta-1)^2}{k^2\cos^2\theta}\times k^2\sin^2\theta$ $= (k\cos\theta-1)^2\tan^2\theta$ $= \left((k\cos\theta-1)\tan\theta\right)^2$ $= (k\sin\theta-\tan\theta)^2$ = LHS	M1 M1 E1 **3**	Expressing one side in terms of θ Using trig identities **18**