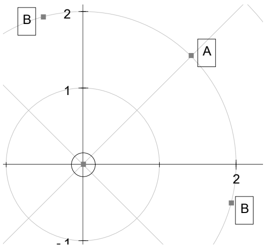
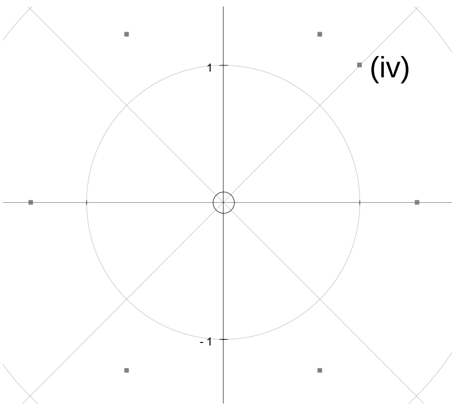
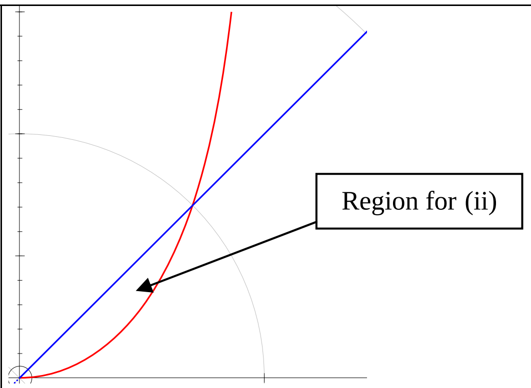
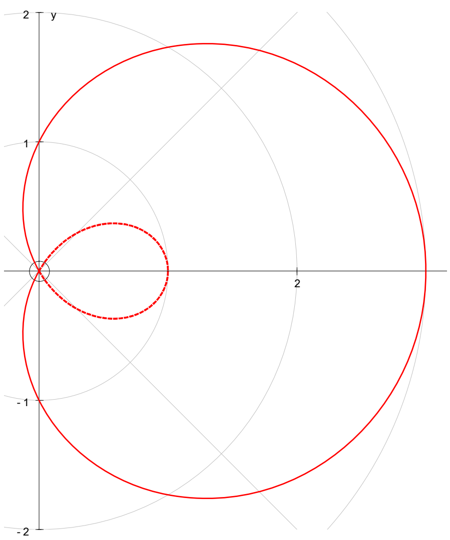
4756 (FP2) Further Methods for Advanced Mathematics

1 (a)(i)	$f(x) = \cos x$ $\quad f(0) = 1$ $f'(x) = -\sin x$ $\quad f'(0) = 0$ $f''(x) = -\cos x$ $\quad f''(0) = -1$ $f'''(x) = \sin x$ $\quad f'''(0) = 0$ $f''''(x) = \cos x$ $\quad f''''(0) = 1$ $\Rightarrow \cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \ldots$	M1 A1 A1 A1 (ag) **4**	Derivatives cos, sin, cos, sin, cos Correct signs Correct values. Dep on previous A1 www
(ii)	$\cos x \times \sec x = 1$ $\Rightarrow \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right)\left(1 + ax^2 + bx^4\right) = 1$ $\Rightarrow 1 + \left(a - \frac{1}{2}\right)x^2 + \left(b - \frac{1}{2}a + \frac{1}{24}\right)x^4 = 1$ $\Rightarrow a - \frac{1}{2} = 0,\ b - \frac{1}{2}a + \frac{1}{24} = 0$ $\Rightarrow a = \frac{1}{2}$ $b = \frac{5}{24}$	E1 M1 A1 B1 B1 **5**	o.e. Multiply to obtain terms in x^2 and x^4 Terms correct in any form (may not be collected) Correctly obtained by any method: must not just be stated Correctly obtained by any method
(b)(i)	$y = \arctan \frac{x}{a}$ $\Rightarrow x = a \tan y$ $\Rightarrow \frac{dx}{dy} = a \sec^2 y$ $\Rightarrow \frac{dx}{dy} = a(1 + \tan^2 y)$ $\Rightarrow \frac{dy}{dx} = \frac{a}{a^2 + x^2}$	M1 A1 A1 A1 (ag) **4**	(a) tan y = and attempt to differentiate both sides Or $\sec^2 y \frac{dy}{dx} = \frac{1}{a}$ Use $\sec^2 y = 1 + \tan^2 y$ o.e. www SC1: Use derivative of arctan x and Chain Rule (properly shown)
(ii)(A)	$\int_{-2}^{2} \frac{1}{4 + x^2} dx = \left[\frac{1}{2}\arctan \frac{x}{2}\right]_{-2}^{2}$ $= \frac{\pi}{4}$	M1 A1 A1 **3**	arctan alone, or any tan substitution $\frac{1}{2}$ and $\frac{x}{2}$, or $\int \frac{1}{2} d\theta$ without limits Evaluated in terms of π
(ii)(B)	$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{4}{1 + 4x^2} dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\frac{1}{4} + x^2} dx$ $= \left[2\arctan(2x)\right]_{-\frac{1}{2}}^{\frac{1}{2}}$ $= \pi$	M1 A1 A1 **3**	arctan alone, or any tan substitution 2 and $2x$, or $\int 2 d\theta$ without limits Evaluated in terms of π **19**

2 (i)	Modulus = 1 Argument = $\frac{\pi}{3}$	B1 B1 **2**	Must be separate Accept 60°, 1.05^c
(ii)	 $a = 2e^{\frac{j\pi}{4}}$ $\arg b = \frac{\pi}{4} \pm \frac{\pi}{3}$ $b = 2e^{-\frac{j\pi}{12}}, 2e^{\frac{7j\pi}{12}}$	G2,1,0 B1 M1 A1ft **5**	G2: A in first quadrant, argument $\approx \frac{\pi}{4}$ B in second quadrant, same mod B′ in fourth quadrant, same mod Symmetry G1: 3 points and at least 2 of above, or B, B′ on axes, or BOB′ straight line, or BOB′ reflex Must be in required form (accept $r = 2$, $\theta = \pi/4$) Rotate by adding (or subtracting) $\pi/3$ to (or from) argument. Must be $\pi/3$ Both. Ft value of r for a. Must be in required form, but don't penalise twice
(iii)	$z_1^6 = \left(\sqrt{2}e^{\frac{j\pi}{3}}\right)^6 = \left(\sqrt{2}\right)^6 e^{2j\pi}$ $= 8$ Others are $re^{j\theta}$ where $r = \sqrt{2}$ and $\theta = -\frac{2\pi}{3}, -\frac{\pi}{3}, 0, \frac{2\pi}{3}, \pi$  	M1 A1 (ag) M1 A1 G1 G1 **6**	$\left(\sqrt{2}\right)^6 = 8$ or $\frac{\pi}{3} \times 6 = 2\pi$ seen www "Add" $\frac{\pi}{3}$ to argument more than once Correct constant r and five values of θ. Accept θ in [0, 2π] or in degrees 6 points on vertices of regular hexagon Correctly positioned (2 roots on real axis). Ignore scales SC1 if G0 and 5 points correctly plotted
(iv)	$w = z_1 e^{-\frac{j\pi}{12}} = \sqrt{2}e^{\frac{j\pi}{3}}e^{-\frac{j\pi}{12}} = \sqrt{2}e^{\frac{j\pi}{4}}$ $= \sqrt{2}\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)$ $= 1 + j$	M1 A1 G1 **3**	$\arg w = \frac{\pi}{3} - \frac{\pi}{12}$ Or B2 Same modulus as z_1
(v)	$w^6 = \left(\sqrt{2}e^{\frac{j\pi}{4}}\right)^6 = 8e^{\frac{3j\pi}{2}}$ $= -8j$	M1 A1 **2**	Or $z_1^6 e^{-\frac{j\pi}{2}} = 8e^{-\frac{j\pi}{2}}$ cao. Evaluated **18**

3(a)(i)		G1 G1 G1 **3**	r increasing with θ Correct for $0 \le \theta \le \pi/3$ (ignore extra) Gradient less than 1 at O
(ii)	Area = $\int_0^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta = \frac{1}{2} a^2 \int_0^{\frac{\pi}{4}} \tan^2 \theta \, d\theta$ $= \frac{1}{2} a^2 \int_0^{\frac{\pi}{4}} \sec^2 \theta - 1 \, d\theta$ $= \frac{1}{2} a^2 [\tan \theta - \theta]_0^{\frac{\pi}{4}}$ $= \frac{1}{2} a^2 \left(1 - \frac{\pi}{4}\right)$	M1 M1 A1 A1 G1 **5**	Integral expression involving $\tan^2\theta$ Attempt to express $\tan^2\theta$ in terms of $\sec^2\theta$ $\tan\theta - \theta$ and limits 0, $\frac{\pi}{4}$ A0 if e.g. triangle − this answer Mark region on graph
(b)(i)	Characteristic equation is $(0.2 - \lambda)(0.7 - \lambda) - 0.24 = 0$ $\Rightarrow \lambda^2 - 0.9\lambda - 0.1 = 0$ $\Rightarrow \lambda = 1, -0.1$ When $\lambda = 1$, $\begin{pmatrix} -0.8 & 0.8 \\ 0.3 & -0.3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Rightarrow -0.8x + 0.8y = 0,\ 0.3x - 0.3y = 0$ $\Rightarrow x - y = 0$, eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ o.e. When $\lambda = -0.1$, $\begin{pmatrix} 0.3 & 0.8 \\ 0.3 & 0.8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Rightarrow 0.3x + 0.8y = 0$ $\Rightarrow$ eigenvector is $\begin{pmatrix} 8 \\ -3 \end{pmatrix}$ o.e.	M1 A1 M1 A1 M1 A1 **6**	$(\mathbf{M} - \lambda\mathbf{I})\mathbf{x} = \mathbf{x}$ M0 below At least one equation relating x and y At least one equation relating x and y
(ii)	$\mathbf{Q} = \begin{pmatrix} 1 & 8 \\ 1 & -3 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & -0.1 \end{pmatrix}$	B1ft B1ft B1 **3**	B0 if **Q** is singular. Must label correctly If order consistent. Dep on B1B1 earned **17**

4 (a)(i)	$\cosh^2 x = \left[\frac{1}{2}\left(e^x + e^{-x}\right)\right]^2 = \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)$ $\sinh^2 x = \left[\frac{1}{2}\left(e^x - e^{-x}\right)\right]^2 = \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)$ $\cosh^2 x - \sinh^2 x = \frac{1}{4}(2+2) = 1$	M1 A1 (ag) **2**	Both expressions (M0 if no "middle" term) and subtraction www
	OR $\cosh x + \sinh x = e^x$ $\cosh x - \sinh x = e^{-x}$ $\cosh^2 x - \sinh^2 x = e^x \times e^{-x} = 1$ A1		Both, and multiplication Completion
(ii)(A)	$\cosh x = \sqrt{1+\sinh^2 x} = \sqrt{1+\tan^2 y}$ $= \sec y$ $\Rightarrow \tanh x = \frac{\sinh x}{\cosh x} = \frac{\tan y}{\sec y} = \sin y$	M1 A1 A1 (ag) **3**	Use of $\cosh^2 x = 1 + \sinh^2 x$ and $\sinh x = \tan y$ www
(ii)(B)	$\text{arsinh } x = \ln(x + \sqrt{1+x^2})$ $\Rightarrow \text{arsinh}(\tan y) = \ln(\tan y + \sqrt{1+\tan^2 y})$ $\Rightarrow x = \ln(\tan y + \sec y)$	M1 A1 A1 (ag) **3**	Attempt to use ln form of arsinh www
	OR $\sinh x = \tan y \Rightarrow \frac{e^x - e^{-x}}{2} = \tan y$ $\Rightarrow e^{2x} - 2e^x \tan y - 1 = 0$ M1 $\Rightarrow e^x = \tan y \pm \sqrt{\tan^2 y + 1}$ A1 $\Rightarrow x = \ln(\tan y + \sec y)$ A1		 Arrange as quadratic and solve for e^x o.e. www
(b)(i)	$y = \text{artanh } x \Rightarrow x = \tanh y$ $\Rightarrow \frac{dx}{dy} = \text{sech}^2 y$ $\Rightarrow \frac{dy}{dx} = \frac{1}{\text{sech}^2 y} = \frac{1}{1-\tanh^2 y} = \frac{1}{1-x^2}$ Integral $= \left[\text{artanh } x\right]_{-\frac{1}{2}}^{\frac{1}{2}}$ $= 2 \text{ artanh} \frac{1}{2}$	M1 A1 M1 A1 (ag) **4**	tanh y = and attempt to differentiate Or $\text{sech}^2 y \frac{dy}{dx} = 1$ Or B2 for $\frac{1}{1-x^2}$ www artanh or any tanh substitution www
(ii)	$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$ $\Rightarrow 1 = A(1 + x) + B(1 - x)$ $\Rightarrow A = ½, B = ½$ $\Rightarrow \int \frac{1}{1-x^2} dx = \int \frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x} dx$ $= -\frac{1}{2}\ln\|1-x\| + \frac{1}{2}\ln\|1+x\| + c$ or $\frac{1}{2}\ln\left\|\frac{1+x}{1-x}\right\| + c$ o.e.	 M1 A1 M1 A1 **4**	 Correct form of partial fractions and attempt to evaluate constants Log integrals www. Condone omitted modulus signs and constant After 0 scored, SC1 for correct answer
(iii)	$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1-x^2} dx = \left[-\frac{1}{2}\ln\|1-x\| + \frac{1}{2}\ln\|1+x\|\right]_{-\frac{1}{2}}^{\frac{1}{2}} = \ln 3$ $\Rightarrow 2 \text{ artanh} \frac{1}{2} = \ln 3 \Rightarrow \text{artanh} \frac{1}{2} = \frac{1}{2} \ln 3$	M1 A1 (ag) **2**	Substitution of ½ **and** –½ seen anywhere (or correct use of 0, ½) www **18**

5 (i)		G1 G1 G1 **3**	Symmetry in horizontal axis (3, 0) to (0, 0) (0, 0) to (0, 1)
(ii)(A) **(ii)(B)** **(ii)(C)** **(ii)(D)**	$a > 0.5$ $a < -0.5$ Circle: r is constant The two loops get closer together The shape becomes more nearly circular Cusp $a = -0.5$	B1 B1 B1 B1 B1 B1 B1 **7**	 Shape and reason
(iii)	$1 + 2a\cos\theta = 0 \Rightarrow \cos\theta = -\frac{1}{2a}$ If $a > 0.5$, $-1 < -\frac{1}{2a} < 0$ and there are two values of θ in $[0, 2\pi]$, $\pi - \arccos\left(\frac{1}{2a}\right)$ and $\pi + \arccos\left(\frac{1}{2a}\right)$ These differ by $2\arccos\left(\frac{1}{2a}\right)$ $\arccos\left(\frac{1}{2a}\right) = \arctan\sqrt{4a^2-1}$ Tangents are $y = x\sqrt{4a^2-1}$ and $y = -x\sqrt{4a^2-1}$ $\sqrt{4a^2-1}$ is real for $a > 0.5$ if $a > 0$	B1 M1 A1 (ag) M1 A1 A1 A1ft E1 **8**	Equation Relating arccos to arctan by triangle or $\tan^2\theta = \sec^2\theta - 1$ Negative of above **18**