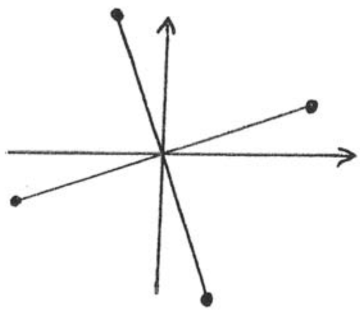
4756 (FP2) Further Methods for Advanced Mathematics

1(a)	Area is $\int_0^{\pi} \frac{1}{2}a^2(1-\cos 2\theta)^2\,d\theta$ $= \int_0^{\pi} \frac{1}{2}a^2\left(1-2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right)d\theta$ $= \frac{1}{2}a^2\left[\frac{3}{2}\theta-\sin 2\theta+\frac{1}{8}\sin 4\theta\right]_0^{\pi}$ $= \frac{3}{4}\pi a^2$	M1 A1 B1 B1B1B1 ft A1 **7**	For $\int(1-\cos 2\theta)^2\,d\theta$ Correct integral expression including limits (may be implied by later work) For $\cos^2 2\theta = \frac{1}{2}(1+\cos 4\theta)$ Integrating $a+b\cos 2\theta+c\cos 4\theta$ *[Max B2 if answer incorrect and no mark has previously been lost]*
(b)(i)	$f'(x) = \frac{1}{1+(\sqrt{3}+x)^2}$ $f''(x) = \frac{-2(\sqrt{3}+x)}{\left(1+(\sqrt{3}+x)^2\right)^2}$	M1 A1 M1 A1 **4**	Applying $\frac{d}{du}\arctan u = \frac{1}{1+u^2}$ or $\frac{dy}{dx}=\frac{1}{\sec^2 y}$ Applying chain (or quotient) rule
(ii)	$f(0)=\frac{1}{3}\pi$ $f'(0)=\frac{1}{4},\quad f''(0)=-\frac{1}{8}\sqrt{3}$ $\arctan(\sqrt{3}+x)=\frac{1}{3}\pi+\frac{1}{4}x-\frac{1}{16}\sqrt{3}\,x^2+\ldots$	B1 M1 A1A1 ft **4**	Stated; or appearing in series *Accept* 1.05 Evaluating $f'(0)$ or $f''(0)$ For $\frac{1}{4}x$ and $-\frac{1}{16}\sqrt{3}\,x^2$ *ft provided coefficients are non-zero*
(iii)	$\int_{-h}^{h}(\frac{1}{3}\pi x+\frac{1}{4}x^2-\frac{1}{16}\sqrt{3}\,x^3+\ldots)\,dx$ $=\left[\frac{1}{6}\pi x^2+\frac{1}{12}x^3-\frac{1}{64}\sqrt{3}\,x^4+\ldots\right]_{-h}^{h}$ $\approx(\frac{1}{6}\pi h^2+\frac{1}{12}h^3-\frac{1}{64}\sqrt{3}\,h^4)$ $-(\frac{1}{6}\pi h^2-\frac{1}{12}h^3-\frac{1}{64}\sqrt{3}\,h^4)$ $=\frac{1}{6}h^3$	M1 A1 ft A1 ag **3**	Integrating (award if *x* is missed) for $\frac{1}{12}x^3$ Allow ft from $a+\frac{1}{4}x+cx^2$ provided that $a \neq 0$ Condone a proof which neglects h^4

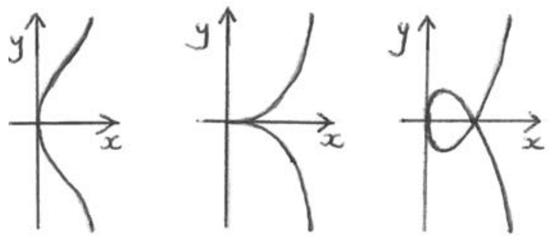
2(a)	4th roots of $16\mathrm{j} = 16\mathrm{e}^{\frac{1}{2}\pi\mathrm{j}}$ are $r\mathrm{e}^{\mathrm{j}\theta}$ where $r = 2$ $\theta = \frac{1}{8}\pi$ $\theta = \frac{\pi}{8} + \frac{2k\pi}{4}$ $\theta = -\frac{7}{8}\pi,\ -\frac{3}{8}\pi,\ \frac{5}{8}\pi$ 	B1 B1 M1 A1 M1 A1 **6**	*Accept* $16^{\frac{1}{4}}$ Implied by at least two correct (ft) further values or stating $k = -2,\ -1,\ (0),\ 1$ Points at vertices of a square centre O *or* 3 correct points (ft) *or* 1 point in each quadrant
(b)(i)	$(1-2\mathrm{e}^{\mathrm{j}\theta})(1-2\mathrm{e}^{-\mathrm{j}\theta}) = 1-2\mathrm{e}^{\mathrm{j}\theta}-2\mathrm{e}^{-\mathrm{j}\theta}+4$ $= 5-2(\mathrm{e}^{\mathrm{j}\theta}+\mathrm{e}^{-\mathrm{j}\theta})$ $= 5-4\cos\theta$	M1 A1 A1 ag **3**	For $\mathrm{e}^{\mathrm{j}\theta}\mathrm{e}^{-\mathrm{j}\theta} = 1$
	OR $(1-2\cos\theta-2\mathrm{j}\sin\theta)(1-2\cos\theta+2\mathrm{j}\sin\theta)$ M1 $= (1-2\cos\theta)^2+4\sin^2\theta$ A1 $= 1-4\cos\theta+4(\cos^2\theta+\sin^2\theta)$ $= 5-4\cos\theta$ A1		
(ii)	$C+\mathrm{j}S = 2\mathrm{e}^{\mathrm{j}\theta}+4\mathrm{e}^{2\mathrm{j}\theta}+8\mathrm{e}^{3\mathrm{j}\theta}+\ldots+2^n\mathrm{e}^{n\mathrm{j}\theta}$ $= \frac{2\mathrm{e}^{\mathrm{j}\theta}\left(1-(2\mathrm{e}^{\mathrm{j}\theta})^n\right)}{1-2\mathrm{e}^{\mathrm{j}\theta}}$ $= \frac{2\mathrm{e}^{\mathrm{j}\theta}(1-2^n\mathrm{e}^{n\mathrm{j}\theta})(1-2\mathrm{e}^{-\mathrm{j}\theta})}{(1-2\mathrm{e}^{\mathrm{j}\theta})(1-2\mathrm{e}^{-\mathrm{j}\theta})}$ $= \frac{2\mathrm{e}^{\mathrm{j}\theta}-4-2^{n+1}\mathrm{e}^{(n+1)\mathrm{j}\theta}+2^{n+2}\mathrm{e}^{n\mathrm{j}\theta}}{5-4\cos\theta}$ $C = \frac{2\cos\theta-4-2^{n+1}\cos(n+1)\theta+2^{n+2}\cos n\theta}{5-4\cos\theta}$ $S = \frac{2\sin\theta-2^{n+1}\sin(n+1)\theta+2^{n+2}\sin n\theta}{5-4\cos\theta}$	M1 M1 A1 M1 A2 M1 A1 ag A1 **9**	Obtaining a geometric series Summing (M0 for sum to infinity) Give A1 for two correct terms in numerator Equating real (or imaginary) parts

3 (i)	Characteristic equation is $(7-\lambda)(-1-\lambda)+12=0$ $\lambda^2-6\lambda+5=0$ $\lambda=1,\ 5$	M1 A1A1	
	When $\lambda=1$, $\begin{pmatrix}7 & 3\\ -4 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$	M1	or $\begin{pmatrix}6 & 3\\ -4 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$ *can be awarded for either eigenvalue*
	$7x+3y=x$ $-4x-y=y$	M1	Equation relating *x* and *y*
	$y=-2x$, eigenvector is $\begin{pmatrix}1\\ -2\end{pmatrix}$	A1	or any (non-zero) multiple
	When $\lambda=5$, $\begin{pmatrix}7 & 3\\ -4 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=5\begin{pmatrix}x\\ y\end{pmatrix}$ $7x+3y=5x$ $-4x-y=5y$	M1	
	$y=-\frac{2}{3}x$, eigenvector is $\begin{pmatrix}3\\ -2\end{pmatrix}$	A1 **8**	*SR* $(\mathbf{M}-\lambda\mathbf{I})\mathbf{x}=\lambda\mathbf{x}$ can earn M1A1A1M0M1A0M1A0
(ii)	$\mathbf{P}=\begin{pmatrix}1 & 3\\ -2 & -2\end{pmatrix}$	B1 ft	B0 if **P** is singular
	$\mathbf{D}=\begin{pmatrix}1 & 0\\ 0 & 5\end{pmatrix}$	B1 ft **2**	For B2, the order must be consistent

(iii)	$\mathbf{M}=\mathbf{P}\mathbf{D}\mathbf{P}^{-1}$	M1	*May be implied*
	$\mathbf{M}^n=\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$	M1	
	$=\mathbf{P}\begin{pmatrix}1 & 0\\ 0 & 5^n\end{pmatrix}\mathbf{P}^{-1}$	A1 ft	*Dependent on M1M1*
	$=\begin{pmatrix}1 & 3\\ -2 & -2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 5^n\end{pmatrix}\frac{1}{4}\begin{pmatrix}-2 & -3\\ 2 & 1\end{pmatrix}$	B1 ft	For $\mathbf{P}^{-1}$
	$=\begin{pmatrix}1 & 3\times5^n\\ -2 & -2\times5^n\end{pmatrix}\frac{1}{4}\begin{pmatrix}-2 & -3\\ 2 & 1\end{pmatrix}$		or $\begin{pmatrix}1 & 3\\ -2 & -2\end{pmatrix}\frac{1}{4}\begin{pmatrix}-2 & -3\\ 2\times5^n & 5^n\end{pmatrix}$
	$=\frac{1}{4}\begin{pmatrix}-2+6\times5^n & -3+3\times5^n\\ 4-4\times5^n & 6-2\times5^n\end{pmatrix}$	M1	Obtaining at least one element in a product of three matrices
	$a=-\frac{1}{2}+\frac{3}{2}\times5^n$	A1 ag	
	$b=-\frac{3}{4}+\frac{3}{4}\times5^n$		
	$c=1-5^n$ $\quad d=\frac{3}{2}-\frac{1}{2}\times5^n$	A2 **8**	Give A1 for one of *b, c, d* correct *SR* If $\mathbf{M}^n=\mathbf{P}^{-1}\mathbf{D}^n\mathbf{P}$ is used, max marks are M0M1A0B1M1A0A1 (*d* should be correct) *SR* If their **P** is singular, max marks are M1M1A1B0M0

4 (i)	$\frac{1}{2}(e^x + e^{-x}) = k$ $e^{2x} - 2k\,e^x + 1 = 0$ $e^x = \dfrac{2k \pm \sqrt{4k^2 - 4}}{2} = k \pm \sqrt{k^2 - 1}$ $x = \ln(k + \sqrt{k^2 - 1})$ or $\ln(k - \sqrt{k^2 - 1})$ $(k + \sqrt{k^2 - 1})(k - \sqrt{k^2 - 1}) = k^2 - (k^2 - 1) = 1$ $\ln(k - \sqrt{k^2 - 1}) = \ln(\dfrac{1}{k + \sqrt{k^2 - 1}}) = -\ln(k + \sqrt{k^2 - 1})$ $x = \pm\ln(k + \sqrt{k^2 - 1})$	M1 M1 A1 M1 A1 ag **5**	or $\cosh x + \sinh x = e^x$ or $k \pm \sqrt{k^2 - 1} = e^x$ One value sufficient or $\cosh x$ is an even function *(or equivalent)*
(ii)	$\displaystyle\int_1^2 \frac{1}{\sqrt{4x^2 - 1}}\,dx = \left[\frac{1}{2}\text{arcosh}\,2x\right]_1^2$ $= \frac{1}{2}(\,\text{arcosh}\,4 - \text{arcosh}\,2\,)$ $= \frac{1}{2}\left(\ln(4 + \sqrt{15}) - \ln(2 + \sqrt{3})\right)$	M1 A1 A1 M1 A1 **5**	For arcosh *or* $\ln(\lambda x + \sqrt{\lambda^2 x^2 - ...})$ or any cosh substitution For $\text{arcosh}\,2x$ or $2x = \cosh u$ or $\ln(2x + \sqrt{4x^2 - 1})$ or $\ln(x + \sqrt{x^2 - \frac{1}{4}})$ For $\frac{1}{2}$ or $\int \frac{1}{2}\,du$ Exact numerical logarithmic form
(iii)	$6\sinh x - 2\sinh x\cosh x = 0$ $\cosh x = 3$ (or $\sinh x = 0$) $x = 0$ $x = \pm\ln(3 + \sqrt{8})$	M1 M1 B1 A1 **4**	 Obtaining a value for $\cosh x$ or $x = \ln(3 \pm \sqrt{8})$
	OR $e^{4x} - 6e^{3x} + 6e^x - 1 = 0$ $(e^{2x} - 1)(e^{2x} - 6e^x + 1) = 0$ M2 $x = 0$ B1 $x = \ln(3 \pm \sqrt{8})$ A1		 or $(e^x - e^{-x})(e^x + e^{-x} - 6) = 0$
(iv)	$\dfrac{dy}{dx} = 6\cosh x - 2\cosh 2x$ If $\dfrac{dy}{dx} = 5$ then $6\cosh x - 2(2\cosh^2 x - 1) = 5$ $4\cosh^2 x - 6\cosh x + 3 = 0$ Discriminant $D = 6^2 - 4\times4\times3 = -12$ Since $D < 0$ there are no solutions	B1 M1 M1 A1 **4**	 Using $\cosh 2x = 2\cosh^2 x - 1$ Considering *D*, or completing square, or considering turning point

	OR Gradient $g = 6\cosh x - 2\cosh 2x$	B1		
	$g' = 6\sinh x - 4\sinh 2x = 2\sinh x(3 - 4\cosh x)$			
	$= 0$ when $x = 0$ (only)	M1		
	$g'' = 6\cosh x - 8\cosh 2x = -2$ when $x = 0$	M1		
	Max value $g = 4$ when $x = 0$			
	So *g* is never equal to 5	A1		Final A1 requires a complete proof showing this is the only turning point

5 (i)	$\lambda = -1$ $\lambda = 0$ $\lambda = 1$  cusp loop	B1B1B1 B1B1 **5**	 Two different features (cusp, loop, asymptote) correctly identified
(ii)	$x = 1$	B1 **1**	
(iii)	Intersects itself when $y = 0$ $t = (\pm)\sqrt{\lambda}$ $\left(\frac{\lambda}{1+\lambda},\ 0 \right)$	M1 A1 A1 **3**	
(iv)	$\frac{dy}{dt} = 3t^2 - \lambda = 0$ $t = \pm\sqrt{\frac{\lambda}{3}}$ $x = \frac{\lambda/3}{1+\lambda/3} = \frac{\lambda}{3+\lambda}$ $y = \pm\left((\frac{\lambda}{3})^{\frac{3}{2}} - \lambda (\frac{\lambda}{3})^{\frac{1}{2}} \right)$ $= \pm\lambda^{\frac{3}{2}}\left(\frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}} \right) = \pm\lambda^{\frac{3}{2}}\left(-\frac{2}{3\sqrt{3}} \right)$ $= \pm\sqrt{\frac{4\lambda^3}{27}}$	M1 A1 ag M1 A1 ag **4**	 One value sufficient
(v)	From asymptote, $a = 8$ From intersection point, $\frac{a\lambda}{1+\lambda} = 2$ $\lambda = \frac{1}{3}$ From maximum point, $b\sqrt{\frac{4\lambda^3}{27}} = 2$ $b = 27$	B1 M1 A1 M1 A1 **5**	