1(a)(i)		B1 B1 **2**	Correct shape for $0 \le \theta \le \frac{1}{2}\pi$ Correct shape for $\frac{1}{2}\pi \le \theta \le \pi$ *Requires decreasing r on at least one axis* *Ignore other values of* θ
(ii)	Area is $\int \frac{1}{2} r^2 \,d\theta = \int_0^{\pi} \frac{1}{2} a^2 (e^{-k\theta})^2 \,d\theta$ $= \left[-\frac{a^2}{4k} e^{-2k\theta} \right]_0^{\pi}$ $= \frac{a^2}{4k}(1 - e^{-2k\pi})$	M1 A1 M1 A1 **4**	For $\int (e^{-k\theta})^2 \,d\theta$ For a correct integral expression including limits *(may be implied by later work) (Condone reversed limits)* Obtaining a multiple of $e^{-2k\theta}$ as the integral
(b)	$\int_0^{\frac{1}{2}} \frac{1}{3+4x^2} dx = \left[\frac{1}{2\sqrt{3}} \arctan\left(\frac{2x}{\sqrt{3}}\right) \right]_0^{\frac{1}{2}}$ $= \frac{1}{2\sqrt{3}} \arctan\left(\frac{1}{\sqrt{3}}\right)$ $= \frac{\pi}{12\sqrt{3}}$	M1 A1A1 M1 A1 **5**	For arctan For $\frac{1}{2\sqrt{3}}$ and $\frac{2x}{\sqrt{3}}$ *Dependent on first M1*
	OR Putting $2x = \sqrt{3}\tan\theta$ M1 A1 Integral is $\int_0^{\frac{1}{6}\pi} \frac{1}{2\sqrt{3}} d\theta$ A1 M1 $= \frac{\pi}{12\sqrt{3}}$ A1		For any tan substitution For $\int \frac{1}{2\sqrt{3}} d\theta$ For changing to limits of θ *Dependent on first M1*
(c)(i)	$f(x) = \tan x$, $f(0) = 0$ $f'(x) = \sec^2 x$, $f'(0) = 1$ $f''(x) = 2\sec^2 x \tan x$, $f''(0) = 0$ $f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$, $f'''(0) = 2$ $\tan x = x + \frac{x^3}{3!}(2) + \dots \quad (= x + \frac{1}{3}x^3 + \dots)$	B1 M1 A1 B1 ft **4**	Obtaining $f'''(x)$ For $f''(0)$ and $f'''(0)$ correct ft requires x^3 term and at least one other to be non-zero
(ii)	$\int_h^{4h} \frac{\tan x}{x} dx \approx \int_h^{4h} (1 + \frac{1}{3}x^2) \,dx$ $= \left[x + \frac{1}{9}x^3 \right]_h^{4h}$ $= (4h + \frac{64}{9}h^3) - (h + \frac{1}{9}h^3)$ $= 3h + 7h^3$	M1 A1 ft A1 ag **3**	Obtaining a polynomial to integrate For $x + \frac{1}{9}x^3$ ft requires at least two non-zero terms

2(a)(i)	$\lvert w \rvert = 3$, $\arg w = -\frac{1}{12}\pi$ $\lvert z \rvert = 2$, $\arg z = -\frac{1}{3}\pi$ $\left\lvert \frac{w}{z} \right\rvert = \frac{3}{2}$, $\arg \frac{w}{z} = (-\frac{1}{12}\pi) - (-\frac{1}{3}\pi) = \frac{1}{4}\pi$	B1 B1B1 B1B1 ft **5**	*Deduct 1 mark if answers given in form $r(\cos\theta + \mathrm{j}\sin\theta)$ but modulus and argument not stated.* Accept degrees and decimal approxs
(ii)	$\frac{w}{z} = \frac{3}{2}(\cos\frac{1}{4}\pi + \mathrm{j}\sin\frac{1}{4}\pi)$ $= \frac{3}{2\sqrt{2}} + \frac{3}{2\sqrt{2}}\mathrm{j}$	M1 A1 **2**	Accept $\sqrt{1.125} + \sqrt{1.125}\,\mathrm{j}$
(b)(i)	$\mathrm{e}^{-\frac{1}{2}\mathrm{j}\theta} + \mathrm{e}^{\frac{1}{2}\mathrm{j}\theta}$ $= (\cos\frac{1}{2}\theta - \mathrm{j}\sin\frac{1}{2}\theta) + (\cos\frac{1}{2}\theta + \mathrm{j}\sin\frac{1}{2}\theta)$ $= 2\cos\frac{1}{2}\theta$	M1 A1	For either bracketed expression
	$1 + \mathrm{e}^{\mathrm{j}\theta} = \mathrm{e}^{\frac{1}{2}\mathrm{j}\theta}(\mathrm{e}^{-\frac{1}{2}\mathrm{j}\theta} + \mathrm{e}^{\frac{1}{2}\mathrm{j}\theta})$ $= \mathrm{e}^{\frac{1}{2}\mathrm{j}\theta}(2\cos\frac{1}{2}\theta)$	M1 A1 ag **4**	
	OR $1 + \mathrm{e}^{\mathrm{j}\theta} = 1 + \cos\theta + \mathrm{j}\sin\theta$ $= 2\cos^2\frac{1}{2}\theta + 2\mathrm{j}\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$ M1 $= 2\cos\frac{1}{2}\theta(\cos\frac{1}{2}\theta + \mathrm{j}\sin\frac{1}{2}\theta)$ $= 2\mathrm{e}^{\frac{1}{2}\mathrm{j}\theta}\cos\frac{1}{2}\theta$ A1		
(ii)	$C + \mathrm{j}S = 1 + \binom{n}{1}\mathrm{e}^{\mathrm{j}\theta} + \binom{n}{2}\mathrm{e}^{2\mathrm{j}\theta} + \ldots + \binom{n}{n}\mathrm{e}^{n\mathrm{j}\theta}$ $= (1 + \mathrm{e}^{\mathrm{j}\theta})^n$ $= 2^n \mathrm{e}^{\frac{1}{2}n\theta\mathrm{j}} \cos^n \frac{1}{2}\theta$ $C = 2^n \cos(\frac{1}{2}n\theta)\cos^n\frac{1}{2}\theta$ $S = 2^n \sin(\frac{1}{2}n\theta)\cos^n\frac{1}{2}\theta$ $\frac{S}{C} = \frac{2^n \sin(\frac{1}{2}n\theta)\cos^n\frac{1}{2}\theta}{2^n \cos(\frac{1}{2}n\theta)\cos^n\frac{1}{2}\theta} = \frac{\sin(\frac{1}{2}n\theta)}{\cos(\frac{1}{2}n\theta)} = \tan(\frac{1}{2}n\theta)$	M1 M1A1 M1 A1 A1 B1 ag **7**	Using (i) to obtain a form from which the real and imaginary parts can be written down Allow ft from $C + \mathrm{j}S = \mathrm{e}^{\frac{1}{2}n\theta\mathrm{j}} \times$ any real function of *n* and θ

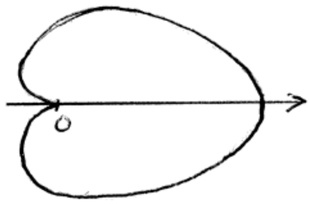
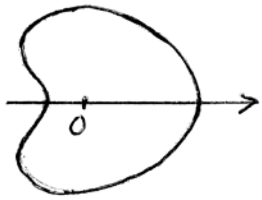
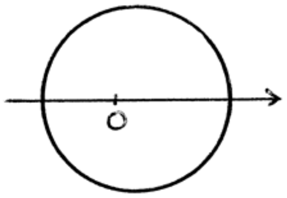
3 (i)	$\det \mathbf{P} = 1(6-k) - 1(4-2)$	M1	
	$= 4 - k$	A1	
	$\mathbf{P}^{-1} = \frac{1}{4-k}\begin{pmatrix} -1 & 2 & 6-k \\ 4 & -4-k & k-12 \\ -1 & 2 & 2 \end{pmatrix}$	M1 M1 A1 ft	Evaluating at least three cofactors Fully correct method for inverse Ft from wrong determinant
	When $k = 2$, $\mathbf{P}^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 2 & 4 \\ 4 & -6 & -10 \\ -1 & 2 & 2 \end{pmatrix}$	B1 ag **6**	Correctly obtained
(ii)	$\mathbf{M}\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$ $\quad \mathbf{M}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$	M1	For one evaluation
	$\mathbf{M}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$		
	Eigenvalues are 0, 1, 2	A1A1A1 **4**	
	OR M1		Obtaining an eigenvalue (e.g. by solving $-\lambda^3 + 3\lambda^2 - 2\lambda = 0$)
	Eigenvalues are 0, 1, 2 A2 A1		Give A1 for one correct Verifying given eigenvectors, linking with eigenvalues correctly
(iii)		B1B1	For $\begin{pmatrix} 4 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix}$ seen (for B2, these must be consistent)
	$\mathbf{M}^n = \begin{pmatrix} 4 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix}\frac{1}{2}\begin{pmatrix} -1 & 2 & 4 \\ 4 & -6 & -10 \\ -1 & 2 & 2 \end{pmatrix}$	M1A1	For $\mathbf{SD}^n\mathbf{S}^{-1}$ (M1A0 if order wrong)
	$= \frac{1}{2}\begin{pmatrix} 0 & 2 & 2^{n+1} \\ 0 & 1 & 3\times 2^n \\ 0 & 0 & -2^n \end{pmatrix}\begin{pmatrix} -1 & 2 & 4 \\ 4 & -6 & -10 \\ -1 & 2 & 2 \end{pmatrix}$	B1 ft	*or* $\frac{1}{2}\begin{pmatrix} 4 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 4 & -6 & -10 \\ -2^n & 2^{n+1} & 2^{n+1} \end{pmatrix}$
	$= \begin{pmatrix} 4-2^n & -6+2^{n+1} & -10+2^{n+1} \\ 2-3\times 2^{n-1} & -3+3\times 2^n & -5+3\times 2^n \\ 2^{n-1} & -2^n & -2^n \end{pmatrix}$	M1 A1	Evaluating product of 3 matrices Any correct form
	$= \begin{pmatrix} 4 & -6 & -10 \\ 2 & -3 & -5 \\ 0 & 0 & 0 \end{pmatrix} + 2^{n-1}\begin{pmatrix} -2 & 4 & 4 \\ -3 & 6 & 6 \\ 1 & -2 & -2 \end{pmatrix}$	A1 ag **8**	

	OR Prove $\mathbf{M}^n = \mathbf{A} + 2^{n-1}\mathbf{B}$ by induction When $n = 1$, $\mathbf{A} + \mathbf{B} = \mathbf{M}$	B1		
	Assuming $\mathbf{M}^k = \mathbf{A} + 2^{k-1}\mathbf{B}$, $\mathbf{M}^{k+1} = \mathbf{A}\mathbf{M} + 2^{k-1}\mathbf{B}\mathbf{M}$	M1A2		or $\mathbf{M}^{k+1} = \mathbf{M}\mathbf{A} + 2^{k-1}\mathbf{M}\mathbf{B}$
	$= \mathbf{A} + 2^{k-1}(2\mathbf{B})$	A1A1		
	$= \mathbf{A} + 2^k\mathbf{B}$	A1		
	True for $n = k \Rightarrow$ True for $n = k+1$; hence true for all positive integers *n*	A1		*Dependent on previous 7 marks*

4 (i)	If $y = \text{arcosh}\,x$, $x = \cosh y = \frac{1}{2}(e^y + e^{-y})$	M1	$\frac{1}{2}$ and + must be correct
	$e^{2y} - 2x e^y + 1 = 0$	M1	
	$e^y = \dfrac{2x \pm \sqrt{4x^2 - 4}}{2}$	M1	
	$= x \pm \sqrt{x^2 - 1}$	A1	
	Since $y \geq 0$, $e^y \geq 1$, so $e^y = x + \sqrt{x^2 - 1}$ $\text{arcosh}\,x = y = \ln(x + \sqrt{x^2 - 1})$	A1 ag **5**	
(ii)	$\displaystyle\int_{2.5}^{3.9} \frac{1}{\sqrt{4x^2 - 9}}\,dx = \left[\frac{1}{2}\text{arcosh}\left(\frac{2x}{3}\right)\right]_{2.5}^{3.9}$	M1 A1A1	For arcosh (or any cosh substitution) For $\dfrac{1}{2}$ and $\dfrac{2x}{3}$ (or $2x = 3\cosh u$ and $\int \frac{1}{2}\,du$)
	$= \frac{1}{2}(\text{arcosh}\,2.6 - \text{arcosh}\,\frac{5}{3})$ $= \frac{1}{2}\left(\ln(2.6 + \sqrt{2.6^2 - 1}) - \ln(\frac{5}{3} + \sqrt{\frac{25}{9} - 1})\right)$	M1	(or limits of u in logarithmic form)
	$= \frac{1}{2}(\ln 5 - \ln 3)$ $= \frac{1}{2}\ln\frac{5}{3}$	A1 **5**	
	OR M2		For $\ln(kx + \sqrt{k^2x^2 - ...})$ Give M1 for $\ln(k_1x + \sqrt{k_2{}^2x^2 - ...})$
	$\left[\frac{1}{2}\ln(2x + \sqrt{4x^2 - 9})\right]_{2.5}^{3.9}$ A1A1		For $\frac{1}{2}$ and $\ln(2x + \sqrt{4x^2 - 9})$ (or $\ln(x + \sqrt{x^2 - \frac{9}{4}})$)
	$= \frac{1}{2}\ln 15 - \frac{1}{2}\ln 9$ $= \frac{1}{2}\ln\frac{5}{3}$ A1		
(iii)	$\dfrac{dy}{dx} = \dfrac{(2 + \sinh x)\sinh x - (\cosh x)(\cosh x)}{(2 + \sinh x)^2}$ $= \dfrac{2\sinh x - 1}{(2 + \sinh x)^2}$	M1 A1	Using quotient rule Any correct form
	$\dfrac{dy}{dx} = \dfrac{1}{9}$ when $18\sinh x - 9 = (2 + \sinh x)^2$ $\sinh^2 x - 14\sinh x + 13 = 0$	M1	Quadratic in $\sinh x$ (or product of two quadratics in e^x)
	$\sinh x = 1,\ 13$	M1	Solving quadratic to obtain at least one value of $\sinh x$ (or e^x)
	When $\sinh x = 1$, $\cosh x = \sqrt{2}$, $x = \ln(1 + \sqrt{2})$	M1	Obtaining x in logarithmic form (must use a correct formula for arsinh)
	Point is $\left(\ln(1 + \sqrt{2}),\ \dfrac{\sqrt{2}}{3}\right)$	A1 ag	
	When $\sinh x = 13$, $\cosh x = \sqrt{170}$, $x = \ln(13 + \sqrt{170})$ Point is $\left(\ln(13 + \sqrt{170}),\ \dfrac{\sqrt{170}}{15}\right)$	A1A1 **8**	*SR* B1B1 for verifying $y = \frac{1}{3}\sqrt{2}$ and $\dfrac{dy}{dx} = \dfrac{1}{9}$ when $x = \ln(1 + \sqrt{2})$

Alternatives for Q4 (i)

	$\cosh\ln(x+\sqrt{x^2-1}) = \frac{1}{2}(e^{\ln(x+\sqrt{x^2-1})} + e^{-\ln(x+\sqrt{x^2-1})})$ $= \frac{1}{2}(x+\sqrt{x^2-1}+\frac{1}{x+\sqrt{x^2-1}})$ $= \frac{1}{2}(x+\sqrt{x^2-1}+x-\sqrt{x^2-1})$ $= x$ Since $\ln(x+\sqrt{x^2-1})>0$, $\text{arcosh}\, x = \ln(x+\sqrt{x^2-1})$	M1 M1 M1 A1 A1 **5**	
	If $y = \text{arcosh}\, x$ then $\ln(x+\sqrt{x^2-1}) = \ln(\cosh y + \sqrt{\cosh^2 y - 1})$ $= \ln(\cosh y + \sinh y)$ since $\sinh y > 0$ $= \ln(e^y)$ $= y$	M1 M1 A1 M1 A1 **5**	

5 (i)	$k=1$ 	B1 B1	General shape correct Cusp at O clearly shown
	$k=1.5$ 	B1 B1	General shape correct 'Dimple' correctly shown
	$k=4$ 	B1 **5**	
(ii)	Cusp	B1 **1**	
(iii)	When $k=1$, there are 3 points When $k=1.5$, there are 4 points When $k=4$, there are 2 points	B2 **2**	Give B1 for two cases correct
(iv)	$x=k\cos\theta+\cos^2\theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta}=-k\sin\theta-2\cos\theta\sin\theta$ $=-\sin\theta(k+2\cos\theta)$ $=0$ when $\theta=0,\ \pi,$ or $\cos\theta=-\tfrac{1}{2}k$ For just two points, $k\ge 2$	B1 B1 M1 A1 **4**	Allow $k>2$
(v)	$d^2=r^2+1^2-2r\cos\theta$ $=(k+\cos\theta)^2+1-2(k+\cos\theta)\cos\theta$ $=k^2+1-\cos^2\theta \quad (=k^2+\sin^2\theta)$ Since $0\le\cos^2\theta\le 1$, $k^2\le d^2\le k^2+1$	M1 A1 M1 A1 ag **4**	or $0\le\sin^2\theta\le 1$
(vi)	When *k* is large, $\sqrt{k^2+1}\approx k$, so $d\approx k$ Curve is very nearly a circle, with centre $(1,\ 0)$ and radius *k*	M1 A1 **2**	