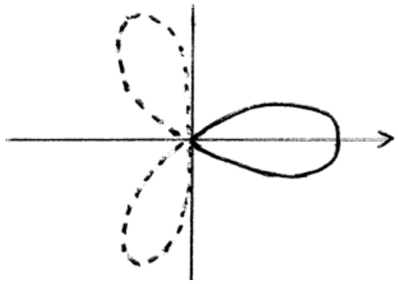
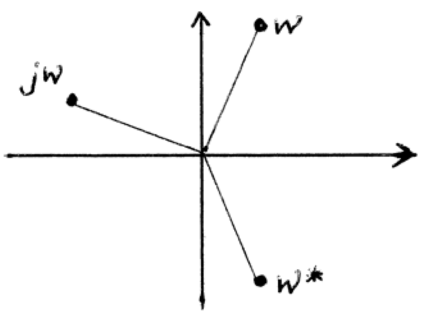
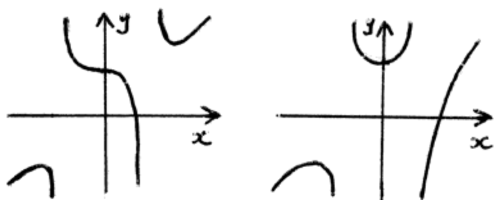
4756 **Mark Scheme** **January 2006**

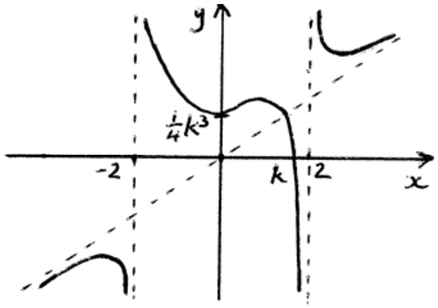
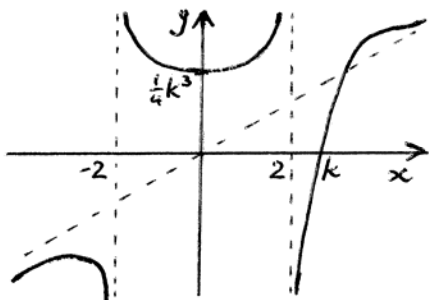
1(a)(i)		B1 B1 B1 **3**	For one loop in correct quadrant(s) For two more loops Continuous and broken lines *Dependent on previous B1B1*
(ii)	Area is $\int \frac{1}{2}r^2\,d\theta = \int_{-\frac{1}{6}\pi}^{\frac{1}{6}\pi} \frac{1}{2}a^2\cos^2 3\theta\,d\theta$ $= \int_{-\frac{1}{6}\pi}^{\frac{1}{6}\pi} \frac{1}{4}a^2(1+\cos 6\theta)\,d\theta$ $= \left[\ \frac{1}{4}a^2(\theta + \frac{1}{6}\sin 6\theta)\ \right]_{-\frac{1}{6}\pi}^{\frac{1}{6}\pi}$ $= \frac{1}{12}\pi a^2$	M1 A1 M1 A1 B1 **5**	For $\int \cos^2 3\theta\,d\theta$ For a correct integral expression including limits *(may be implied by later work)* For $\int \cos^2 3\theta\,d\theta = \frac{1}{2}\theta + \frac{1}{12}\sin 6\theta$ Accept $0.262a^2$
(b)	$\int_0^{\frac{3}{4}} \frac{1}{\sqrt{3-4x^2}}\,dx = \left[\ \frac{1}{2}\arcsin\left(\frac{2x}{\sqrt{3}}\right)\right]_0^{\frac{3}{4}}$ $= \frac{1}{2}\arcsin\left(\frac{3}{2\sqrt{3}}\right)$ $= \frac{1}{6}\pi$	M1 A1A1 M1 A1 **5**	For arcsin For $\frac{1}{2}$ and $\frac{2x}{\sqrt{3}}$ *Dependent on previous M1*
(c)	Putting $\sqrt{3}x = \tan\theta$ Integral is $\int_0^{\frac{1}{3}\pi} \frac{1}{\sec^3\theta}\left(\frac{\sec^2\theta}{\sqrt{3}}\right)d\theta$ $= \int_0^{\frac{1}{3}\pi} \frac{\cos\theta}{\sqrt{3}}\,d\theta = \left[\ \frac{\sin\theta}{\sqrt{3}}\ \right]_0^{\frac{1}{3}\pi}$ $= \frac{1}{2}$	M1 A1A1 M1 A1 **5**	For any tan substitution For $\frac{1}{(\sec^2\theta)^{\frac{3}{2}}}$ and $\frac{\sec^2\theta}{\sqrt{3}}$ Including limits of θ
	OR Putting $2x = \sqrt{3}\sin\theta$ M1 A1 Integral is $\int_0^{\frac{1}{3}\pi} \frac{1}{2}\,d\theta$ A1 M1 $= \frac{1}{6}\pi$ A1		For any sine substitution For $\int \frac{1}{2}\,d\theta$ For changing to limits of θ *Dependent on previous M1*

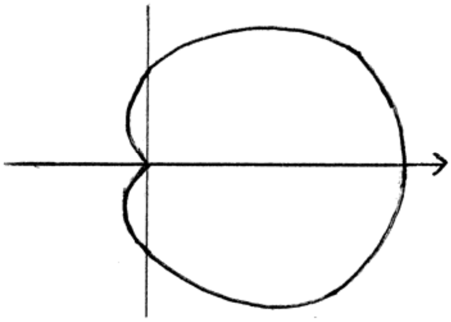
2 (i)	$\lvert w \rvert = \frac{1}{2}$, $\arg w = 3\theta$ $\lvert w^* \rvert = \frac{1}{2}$, $\arg w^* = -3\theta$ $\lvert \mathrm{j}w \rvert = \frac{1}{2}$, $\arg \mathrm{j}w = 3\theta + \frac{1}{2}\pi$  	B1 B1 ft B1B1 ft B2 **6**	w^* and $\mathrm{j}w$ in correct positions relative to their w in first quadrant Give B1 for at least two points in correct quadrants
(ii)	$(1+w)(1+w^*) = 1 + \frac{1}{2}\mathrm{e}^{3\mathrm{j}\theta} + \frac{1}{2}\mathrm{e}^{-3\mathrm{j}\theta} + (\frac{1}{2}\mathrm{e}^{3\mathrm{j}\theta})(\frac{1}{2}\mathrm{e}^{-3\mathrm{j}\theta})$ $= 1 + \frac{1}{2}(\cos 3\theta + \mathrm{j}\sin 3\theta) + \frac{1}{2}(\cos 3\theta - \mathrm{j}\sin 3\theta) + \frac{1}{4}$ $= \frac{5}{4} + \cos 3\theta$	M1 A1 M1 A1 (ag) **4**	for $w^* = \frac{1}{2}\mathrm{e}^{-3\mathrm{j}\theta}$ for $1 + \frac{1}{4}$ correctly obtained for $w = \frac{1}{2}(\cos 3\theta + \mathrm{j}\sin 3\theta)$ for $\cos 3\theta$ correctly obtained
(iii)	$C + \mathrm{j}S = \mathrm{e}^{2\mathrm{j}\theta} - \frac{1}{2}\mathrm{e}^{5\mathrm{j}\theta} + \frac{1}{4}\mathrm{e}^{8\mathrm{j}\theta} - \ldots$ $= \dfrac{\mathrm{e}^{2\mathrm{j}\theta}}{1 + \frac{1}{2}\mathrm{e}^{3\mathrm{j}\theta}}$ $= \dfrac{\mathrm{e}^{2\mathrm{j}\theta}(1 + \frac{1}{2}\mathrm{e}^{-3\mathrm{j}\theta})}{(1 + \frac{1}{2}\mathrm{e}^{3\mathrm{j}\theta})(1 + \frac{1}{2}\mathrm{e}^{-3\mathrm{j}\theta})}$ $= \dfrac{\mathrm{e}^{2\mathrm{j}\theta}(1 + \frac{1}{2}\mathrm{e}^{-3\mathrm{j}\theta})}{\frac{5}{4} + \cos 3\theta}$ $= \dfrac{\mathrm{e}^{2\mathrm{j}\theta} + \frac{1}{2}\mathrm{e}^{-\mathrm{j}\theta}}{\frac{5}{4} + \cos 3\theta} \quad \left(= \dfrac{4\mathrm{e}^{2\mathrm{j}\theta} + 2\mathrm{e}^{-\mathrm{j}\theta}}{5 + 4\cos 3\theta} \right)$ $C = \dfrac{4\cos 2\theta + 2\cos\theta}{5 + 4\cos 3\theta}$ $S = \dfrac{4\sin 2\theta - 2\sin\theta}{5 + 4\cos 3\theta}$	M1 M1 A1 M1 A1 M1 A1 (ag) A1 **8**	Obtaining a geometric series Summing an infinite geometric series Using complex conjugate of denom Equating real or imaginary parts Correctly obtained

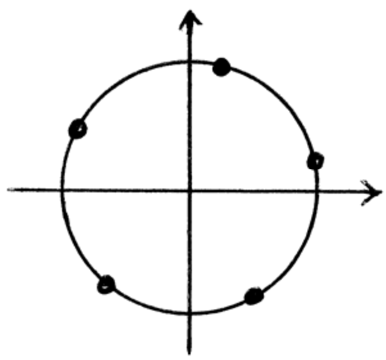
3 (i)	$(1-\lambda)[(-3-\lambda)(-4-\lambda)-12]$ $-2[-2(-4-\lambda)-12]+3[-4-2(-3-\lambda)]=0$ $(1-\lambda)(\lambda^2+7\lambda)-2(2\lambda-4)+3(2\lambda+2)=0$ $\lambda^3+6\lambda^2-9\lambda-14=0$	M1 A1 A1 (ag) **3**	Evaluating $\det(\mathbf{M}-\lambda\mathbf{I})$ *Allow one omission and two sign errors* $\det(\mathbf{M}-\lambda\mathbf{I})$ correct Correctly obtained (=0 is required)
(ii)	When $\lambda=-1$, $-1+6+9-14=0$ $(\lambda+1)(\lambda^2+5\lambda-14)=0$ $(\lambda+1)(\lambda-2)(\lambda+7)=0$ Other eigenvalues are $2, -7$	B1 M1 A1 **3**	or showing that $(\lambda+1)$ is a factor, and deducing that -1 is a root for $(\lambda+1)\times$quadratic factor
(iii)	$x+2y+3z=-x$ $-2x-3y+6z=-y$ $2x+2y-4z=-z$ $z=0,\ x+y=0$ An eigenvector is $\begin{pmatrix}1\\-1\\0\end{pmatrix}$	M1 M1 A1 **3**	At least two equations Solving to obtain an eigenvector
	OR $\begin{pmatrix}2\\2\\3\end{pmatrix}\times\begin{pmatrix}-2\\-2\\6\end{pmatrix}=\begin{pmatrix}18\\-18\\0\end{pmatrix}$ M1 M1 A1		Appropriate vector product Evaluation of vector product
(iv)	$\mathbf{M}\begin{pmatrix}3\\0\\1\end{pmatrix}=\begin{pmatrix}6\\0\\2\end{pmatrix}=2\begin{pmatrix}3\\0\\1\end{pmatrix}$ $\mathbf{M}\begin{pmatrix}0\\3\\-2\end{pmatrix}=\begin{pmatrix}0\\-21\\14\end{pmatrix}=-7\begin{pmatrix}0\\3\\-2\end{pmatrix}$	M1 A1A1 **3**	Any method for verifying or finding an eigenvector
(v)	$\mathbf{P}=\begin{pmatrix}1&3&0\\-1&0&3\\0&1&-2\end{pmatrix}$ $\mathbf{D}=\begin{pmatrix}-1&0&0\\0&2&0\\0&0&-7\end{pmatrix}^3$ $=\begin{pmatrix}-1&0&0\\0&8&0\\0&0&-343\end{pmatrix}$	B1 ft M1 A1 ft **3**	seen or implied (ft) *(condone eigenvalues in wrong order)* Order must be consistent with **P** (when B1 has been awarded)
(vi)	By CHT, $\mathbf{M}^3+6\mathbf{M}^2-9\mathbf{M}-14\mathbf{I}=\mathbf{0}$ $\mathbf{M}^2+6\mathbf{M}-9\mathbf{I}-14\mathbf{M}^{-1}=\mathbf{0}$ $\mathbf{M}^{-1}=\frac{1}{14}\mathbf{M}^2+\frac{3}{7}\mathbf{M}-\frac{9}{14}\mathbf{I}$	B1 M1 A1 **3**	Condone omission of **I** Condone dividing by **M**

4 (a)	$\frac{1}{2}(e^x - e^{-x}) + 2(e^x + e^{-x}) = 8$ $5e^{2x} - 16e^x + 3 = 0$ $(5e^x - 1)(e^x - 3) = 0$ $e^x = \frac{1}{5},\ 3$ $x = -\ln 5,\ \ \ln 3$	M1 M1 M1 A1A1 A1 ft **6**	Exponential form Quadratic in e^x Solving to obtain a value of e^x Exact logarithmic form from 2 positive values of e^x *Dependent on M3*
	OR $\sqrt{c^2-1} = 8 - 4c$ $15c^2 - 64c + 65 = 0$ M1 M1 $c = \frac{5}{3},\ \frac{13}{5}$ A1A1 $x = \pm\ln 3,\ \pm\ln 5$ M1 $x = \ln 3,\ -\ln 5$ A1		Obtaining quadratic in c (or s) $(15s^2 + 16s - 48 = 0)$ Solving to obtain a value of c (or s) or $s = \frac{4}{3},\ -\frac{12}{5}$ Logarithmic form (including $\pm$ if c) cao
(b)	$\int_0^2 \frac{1}{2}e^x(e^x - e^{-x})\,dx$ $= \left[\frac{1}{4}e^{2x} - \frac{1}{2}x\right]_0^2$ $= (\frac{1}{4}e^4 - 1) - (\frac{1}{4})$ $= \frac{1}{4}(e^4 - 5)$	M1 M1 A1 A1 **4**	Exponential form Integrating to obtain a multiple of e^{2x}
(c)(i)	$\dfrac{\frac{2}{3}}{\sqrt{1+(\frac{2}{3}x)^2}} \quad \left(= \dfrac{2}{\sqrt{9+4x^2}} \right)$	B2 **2**	Give B1 for any non-zero multiple of this
(ii)	$\left[x\,\text{arsinh}(\frac{2}{3}x)\right]_0^2 - \int_0^2 \dfrac{2x}{\sqrt{9+4x^2}}\,dx$ $= \left[x\,\text{arsinh}(\frac{2}{3}x) - \frac{1}{2}\sqrt{9+4x^2}\right]_0^2$ $= \left(2\,\text{arsinh}(\frac{4}{3}) - \frac{5}{2}\right) - (-\frac{3}{2})$ $= 2\ln\left(\frac{4}{3} + \sqrt{1+\frac{16}{9}}\right) - 1$ $= 2\ln 3 - 1$	M1 A1 ft B1 M1 M1 A1 (ag) **6**	Integration by parts applied to $\text{arsinh}(\frac{2}{3}x)\times 1$ for $\int \dfrac{x}{\sqrt{9+4x^2}}\,dx = \frac{1}{4}\sqrt{9+4x^2}$ Using both limits (provided both give non-zero values) Logarithmic form for arsinh (intermediate step required)

5 (i)	$x = 2, \quad x = -2$ $y = x + \dfrac{4x - k^3}{x^2 - 4}$ Asymptote is $y = x$	B1 M1 A1 **3**	 Dividing out or B2 for $y = x$ stated
(ii)	 $k < 2$ $\qquad$ $k > 2$	B1 B1 B1 B1 **4**	$k < 2$ for LH and RH sections for central section, with positive intercepts on both axes $k > 2$ for LH and central sections for RH section, crossing x-axis
(iii)	$\dfrac{dy}{dx} = \dfrac{(x^2 - 4)(3x^2) - (x^3 - k^3)(2x)}{(x^2 - 4)^2}$ $= \dfrac{x(2k^3 + x^3 - 12x)}{(x^2 - 4)^2}$ $\dfrac{dy}{dx} = 0$ when $x = 0$ When $x \approx 0$, $2k^3 + x^3 - 12x > 0$ $\dfrac{dy}{dx} < 0$ when $x < 0$, $\dfrac{dy}{dx} > 0$ when $x > 0$ Hence there is a minimum when $x = 0$	M1 A1 A1 (ag) M1 A1 (ag) **5**	Using quotient rule (or equivalent) Any correct form Correctly shown or evaluating $\dfrac{d^2y}{dx^2}$ when $x = 0$ or $\dfrac{d^2y}{dx^2} = \frac{1}{8}k^3 > 0$ when $x = 0$
(iv)	Curve crosses $y = x$ when $x^3 - k^3 = x(x^2 - 4)$ $x = \frac{1}{4}k^3$ So curve crosses this asymptote	M1 A1 (ag) **2**	

(v)	$k < 2$ 	B2		Asymptotes shown Intercepts $\frac{1}{4}k^3$ and k indicated Minimum on positive y-axis Maximum shown Give B1 for minimum and maximum on central section
	$k > 2$ 	B2	**4**	Asymptotes shown Intercepts $\frac{1}{4}k^3$ and k indicated Minimum on positive y-axis RH section crosses $y = x$ and approaches it from above Give B1 for RH section approaching both asymptotes correctly

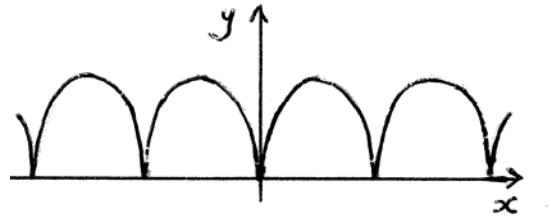
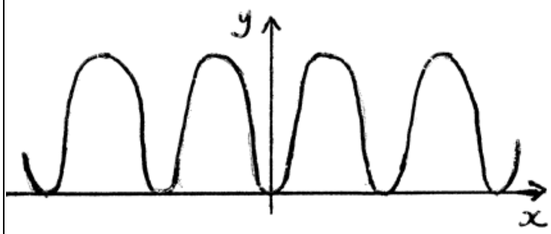
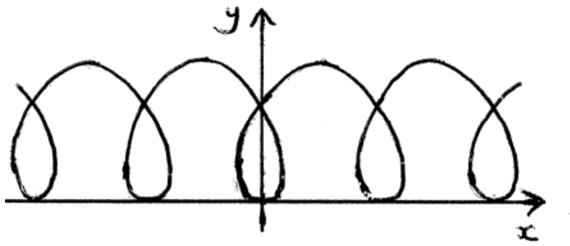
1(a)(i)		B1 B1 **2**	Correct shape for $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ including maximum in 1st quadrant Correct form at O and no extra sections
(ii)	Area is $\int \frac{1}{2} r^2 \, d\theta = \int_{-\frac{3}{4}\pi}^{\frac{3}{4}\pi} \frac{1}{2} a^2 (\sqrt{2} + 2\cos\theta)^2 \, d\theta$ $= \int_{-\frac{3}{4}\pi}^{\frac{3}{4}\pi} a^2 (1 + 2\sqrt{2}\cos\theta + 1 + \cos 2\theta) \, d\theta$ $= \Big[a^2 (2\theta + 2\sqrt{2}\sin\theta + \frac{1}{2}\sin 2\theta) \Big]_{-\frac{3}{4}\pi}^{\frac{3}{4}\pi}$ $= 3(\pi + 1)a^2$	M1 A1 B1 B1B1 ft M1 A1 **7**	For integral of $(\sqrt{2} + 2\cos\theta)^2$ For a correct integral expression including limits *(may be implied by later work)* Using $2\cos^2\theta = 1 + \cos 2\theta$ Integration of $\cos\theta$ and $\cos 2\theta$ Evaluation using $\sin\frac{3}{4}\pi = (\pm)\frac{1}{\sqrt{2}}$
(b)(i)	$f'(x) = \sec^2(\frac{1}{4}\pi + x)$ $f''(x) = 2\sec^2(\frac{1}{4}\pi + x)\tan(\frac{1}{4}\pi + x)$ $f(0) = 1, \quad f'(0) = 2, \quad f''(0) = 4$ $f(x) = 1 + 2x + 2x^2 + \ldots$	B1 B1 M1 B1A1A1 **6**	Any correct form Evaluating $f'(0)$ or $f''(0)$
	OR $g'(u) = \sec^2 u$ (where $g(u) = \tan u$) B1 $g''(u) = 2\sec^2 u \tan u$ B1 $g(\frac{1}{4}\pi) = 1, \; g'(\frac{1}{4}\pi) = 2, \; g''(\frac{1}{4}\pi) = 4$ M1 $f(x) = g(\frac{1}{4}\pi + x) = 1 + 2x + 2x^2 + \ldots$ B1A1A1		Condone $\sec^2 x$ etc Evaluating $g'(\frac{1}{4}\pi)$ or $g''(\frac{1}{4}\pi)$
(ii)	$\int_{-h}^{h} x^2 (1 + 2x + 2x^2 + \ \ldots) \, dx$ $= \Big[\frac{1}{3}x^3 + \frac{1}{2}x^4 + \frac{2}{5}x^5 + \ldots \Big]_{-h}^{h}$ $\approx (\frac{1}{3}h^3 + \frac{1}{2}h^4 + \frac{2}{5}h^5) - (-\frac{1}{3}h^3 + \frac{1}{2}h^4 - \frac{2}{5}h^5)$ $= \frac{2}{3}h^3 + \frac{4}{5}h^5$	M1 A1 ft A1 (ag) **3**	Using series and integrating (ft requires three non-zero terms) Correctly shown Allow ft from $1 + kx + 2x^2$ with $k \neq 0$

2 (a)(i)	$z^n + \frac{1}{z^n} = 2\cos n\theta$, $\quad z^n - \frac{1}{z^n} = 2\mathrm{j}\sin n\theta$	B1B1 **2**	
(ii)	$\left(z - \frac{1}{z}\right)^4 \left(z + \frac{1}{z}\right)^2 = 64\sin^4\theta\cos^2\theta$ $= z^6 - 2z^4 - z^2 + 4 - \frac{1}{z^2} - \frac{2}{z^4} + \frac{1}{z^6}$ $= 2\cos 6\theta - 4\cos 4\theta - 2\cos 2\theta + 4$ $\sin^4\theta\cos^2\theta = \frac{1}{32}\cos 6\theta - \frac{1}{16}\cos 4\theta - \frac{1}{32}\cos 2\theta + \frac{1}{16}$ $(A = \frac{1}{32},\ B = -\frac{1}{16},\ C = -\frac{1}{32},\ D = \frac{1}{16})$	B1 M1 A1 M1 A1 ft A1 **6**	Expansion $z^6 + \ldots + z^{-6}$ Using $z^n + \frac{1}{z^n} = 2\cos n\theta$ with $n = 2,\ 4$ or 6. Allow M1 if used in partial expansion, or if 2 omitted, etc
(b)(i)	$\lvert 4 + 4\mathrm{j} \rvert = \sqrt{32}$, $\quad \arg(4 + 4\mathrm{j}) = \frac{1}{4}\pi$	B1B1 **2**	Accept $5.7;\ \ 0.79,\ 45°$
(ii)	$r = \sqrt{2}$ $\theta = -\frac{3}{4}\pi,\ \ -\frac{7}{20}\pi,\ \ \frac{1}{20}\pi,\ \ \frac{9}{20}\pi,\ \ \frac{17}{20}\pi$ 	B1 B3 B2 **6**	Accept $32^{\frac{1}{10}},\ 1.4,\ \sqrt[5]{4\sqrt{2}}$ etc Accept $-2.4, -1.1, 0.16, 1.4, 2.7$ Give B2 for three correct Give B1 for one correct Deduct 1 mark (maximum) if degrees used $(-135°,\ -63°,\ 9°,\ 81°,\ 153°)$ $\frac{1}{20}\pi + \frac{2}{5}k\pi$ earns B2; with $k = -2,\ -1,\ 0,\ 1,\ 2$ earns B3 Give B1 for four points correct, or B1 ft for five points
(iii)	$\sqrt{2}\mathrm{e}^{-\frac{3}{4}\pi\mathrm{j}} = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\mathrm{j}\right)$ $= -1 - \mathrm{j}$ $p = -1,\ q = -1$	M1 A1 **2**	Exact evaluation of a fifth root Give B2 for correct answer stated or obtained by any other method

3 (i)	$\mathbf{M}^{-1} = \frac{1}{5-k}\begin{pmatrix} 1 & 5k-13 & 5-2k \\ 1 & 52-8k & 3k-20 \\ -1 & -12 & 5 \end{pmatrix}$	M1 A1 M1 A1 M1 A1 **6**	Evaluating determinant For $(5-k)$ *must be simplified* Finding at least four cofactors At least 6 signed cofactors correct Transposing matrix of cofactors and dividing by determinant Fully correct
	OR Elementary row operations applied to **M** (LHS) and **I** (RHS), and obtaining at least two zeros in LHS M1 Obtaining one row in LHS consisting of two zeros and a multiple of $(5-k)$ A1 Obtaining one row in RHS which is a multiple of a row of the inverse matrix A1 Obtaining two zeros in every row in LHS M1 Completing process to find inverse M1A1		or elementary column operations
(ii)	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 & 22 & -9 \\ 1 & -4 & 1 \\ -1 & -12 & 5 \end{pmatrix}\begin{pmatrix} 12 \\ m \\ 0 \end{pmatrix}$ $x = -11m-6,\quad y = 2m-6,\quad z = 6m+6$	M1 M1 M1 A2 ft **5**	Substituting $k=7$ into inverse Correct use of inverse Evaluating matrix product Give A1 ft for one correct *Accept unsimplified forms or solution left in matrix form*
	OR e.g. eliminating *x* , $3y - z = -24$ $5y - z = 4m - 36$ M2 $y = 2m - 6$ M1 $x = -11m-6,\quad y = 2m-6,\quad z = 6m+6$ A2		Eliminating one variable in two different ways Obtaining one of *x* , *y* , *z* Give M3 for any other valid method leading to one of *x* , *y* , *z* in terms of *m* Give A1 for one correct
(iii)	Eliminating *x* , $3y + 3z = -24$ $5y + 5z = 4p - 36$ For solutions, $4p - 36 = -24 \times \frac{5}{3}$	M2 A1 M1	Eliminating one variable in two different ways Two correct equations *Dependent on previous M2*
	OR Replacing one column of matrix with column from RHS, and evaluating determinant M2 determinant $12 + 12p$ or $-12 - 12p$ A1 For solutions, det $= 0$ M1		*Dependent on previous M2*

	OR Any other method leading to an equation from which *p* could be found M3 Correct equation A1		
	$p = -1$ Let $z = \lambda$, $x = 5 - \lambda,\quad y = -8 - \lambda,\; z = \lambda$	A1 M1 (or M3) A1 **7**	Obtaining a line of solutions Give M3 when M0 for finding *p* or $x = 13 + \lambda,\quad y = \lambda,\; z = -8 - \lambda$ or $x = \lambda,\quad y = -13 + \lambda,\; z = 5 - \lambda$ Accept $x = 5 - z,\; y = -8 - z$ or $x = y + 13 = 5 - z$ etc

4 (i)	$1+2\sinh^2 x = 1+2[\frac{1}{2}(e^x - e^{-x})]^2$ $= 1+\frac{1}{2}(e^{2x} - 2 + e^{-2x})$ $= \frac{1}{2}(e^{2x} + e^{-2x})$ $= \cosh 2x$	B1 B1 B1 (ag) **3**	For $(e^x - e^{-x})^2 = e^{2x} - 2 + e^{-2x}$ For $\cosh 2x = \frac{1}{2}(e^{2x} + e^{-2x})$ For completion
(ii)	$2(1+2\sinh^2 x) + \sinh x = 5$ $4\sinh^2 x + \sinh x - 3 = 0$ $(4\sinh x - 3)(\sinh x + 1) = 0$ $\sinh x = \frac{3}{4},\ -1$ $x = \text{arsinh}(\frac{3}{4}) = \ln(\frac{3}{4} + \sqrt{\frac{9}{16}+1}) = \ln 2$ $x = \text{arsinh}(-1) = \ln(-1+\sqrt{1+1}) = \ln(\sqrt{2}-1)$	M1 M1 A1A1 A1 ft A1 ft **6**	Using (i) Solving to obtain a value of $\sinh x$ or $-\ln(\sqrt{2}+1)$ *SR* Give A1 for $\pm\ln 2,\ \pm\ln(\sqrt{2}-1)$
	OR $2e^{4x} + e^{3x} - 10e^{2x} - e^x + 2 = 0$ M2 $(e^x - 2)(2e^x + 1)(e^{2x} + 2e^x - 1) = 0$ A1A1 $x = \ln 2,\ \ln(\sqrt{2}-1)$ A1A1 ft		Obtaining a linear or quadratic factor For $(e^x - 2)$ and $(e^{2x} + 2e^x - 1)$
(iii)	$\int_0^{\ln 3} \frac{1}{2}(\cosh 2x - 1)\,dx$ $= \left[\frac{1}{4}\sinh 2x - \frac{1}{2}x\right]_0^{\ln 3}$ $= \frac{1}{8}\left(9-\frac{1}{9}\right) - \frac{1}{2}\ln 3$ $= \frac{10}{9} - \frac{1}{2}\ln 3$	M1 A1A1 M1 A1 (ag) **5**	Expressing in integrable form or $\int \frac{1}{4}(e^{2x} - 2 + e^{-2x})\,dx$ or $(\frac{1}{8}e^{2x} - \frac{1}{8}e^{-2x}) - \frac{1}{2}x$ For $e^{2\ln 3} = 9$ and $e^{-2\ln 3} = \frac{1}{9}$ M0 for just stating $\sinh(2\ln 3) = \frac{40}{9}$ etc Correctly obtained
(iv)	Put $x = 3\cosh u$ when $x = 3,\ u = 0$ when $x = 5,\ u = \text{ar}\cosh\frac{5}{3} = \ln 3$ $\int_3^5 \sqrt{x^2 - 9}\,dx = \int_0^{\ln 3} (3\sinh u)(3\sinh u\,du)$ $= 9\int_0^{\ln 3} \sinh^2 u\,du$ $= 10 - \frac{9}{2}\ln 3$	M1 B1 A1 A1 **4**	Any cosh substitution For $\ln 3$ *Not awarded for* $\text{arcosh}\frac{5}{3}$ *Limits not required*

5 (i)	 Has cusps Periodic / Symmetrical in *y*-axis / Has maxima / Is never below the *x*-axis	B2 B1 B1 **4**	At least two cusps clearly shown Give B1 for at least two arches Any other feature
(ii)	 The curve has no cusps	B2 B1 **3**	At least two minima (zero gradient) clearly shown Give B1 for general shape correct (at least two cycles) For description of any *difference*
(iii)	(*A*) 	B2 **2**	At least two loops Give B1 for general shape correct (at least one cycle)
	(*B*) $\frac{dy}{dx} = \frac{\sin\theta}{1-2\cos\theta}$	M1 A1 **2**	Correct method of differentiation *Allow M1 if inverted* Allow $\frac{\sin\theta}{1-k\cos\theta}$
	(*C*) $\frac{dy}{dx}$ is infinite when $1-2\cos\theta=0$ $\theta=\frac{1}{3}\pi$ $x=\frac{1}{3}\pi-2\sin\frac{1}{3}\pi$ $=-(\sqrt{3}-\frac{1}{3}\pi)$ Hence width of loop is $2(\sqrt{3}-\frac{1}{3}\pi)$ $=2\sqrt{3}-\frac{2\pi}{3}$	M1 A1 M1 M1 A1 (ag) **5**	 Any correct value of θ Finding width of loop Correctly obtained *Condone negative answer*
(iv)	$k=4.6$	B2 **2**	Give B1 for a value between 4 and 5 (inclusive)

1(a)(i)		B1 B1 **2**	Correct shape for $0 \le \theta \le \frac{1}{2}\pi$ Correct shape for $\frac{1}{2}\pi \le \theta \le \pi$ *Requires decreasing r on at least one axis* *Ignore other values of* θ
(ii)	Area is $\int \frac{1}{2}r^2\,d\theta = \int_0^{\pi} \frac{1}{2}a^2(e^{-k\theta})^2\,d\theta$ $= \left[-\frac{a^2}{4k}e^{-2k\theta} \right]_0^{\pi}$ $= \frac{a^2}{4k}(1 - e^{-2k\pi})$	M1 A1 M1 A1 **4**	For $\int (e^{-k\theta})^2\,d\theta$ For a correct integral expression including limits *(may be implied by later work) (Condone reversed limits)* Obtaining a multiple of $e^{-2k\theta}$ as the integral
(b)	$\int_0^{\frac{1}{2}} \frac{1}{3+4x^2}\,dx = \left[\frac{1}{2\sqrt{3}}\arctan\left(\frac{2x}{\sqrt{3}}\right) \right]_0^{\frac{1}{2}}$ $= \frac{1}{2\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\right)$ $= \frac{\pi}{12\sqrt{3}}$	M1 A1A1 M1 A1 **5**	For arctan For $\frac{1}{2\sqrt{3}}$ and $\frac{2x}{\sqrt{3}}$ *Dependent on first M1*
	OR Putting $2x = \sqrt{3}\tan\theta$ — M1, A1 Integral is $\int_0^{\frac{1}{6}\pi} \frac{1}{2\sqrt{3}}\,d\theta$ — A1 M1 $= \frac{\pi}{12\sqrt{3}}$ — A1		For any tan substitution For $\int \frac{1}{2\sqrt{3}}\,d\theta$ For changing to limits of θ *Dependent on first M1*
(c)(i)	$f(x) = \tan x, \quad f(0) = 0$ $f'(x) = \sec^2 x, \quad f'(0) = 1$ $f''(x) = 2\sec^2 x \tan x, \quad f''(0) = 0$ $f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x, \quad f'''(0) = 2$ $\tan x = x + \frac{x^3}{3!}(2) + \ldots \quad (= x + \frac{1}{3}x^3 + \ldots)$	B1 M1 A1 B1 ft **4**	Obtaining $f'''(x)$ For $f''(0)$ and $f'''(0)$ correct ft requires x^3 term and at least one other to be non-zero
(ii)	$\int_h^{4h} \frac{\tan x}{x}\,dx \approx \int_h^{4h} (1 + \frac{1}{3}x^2)\,dx$ $= \left[x + \frac{1}{9}x^3 \right]_h^{4h}$ $= (4h + \frac{64}{9}h^3) - (h + \frac{1}{9}h^3)$ $= 3h + 7h^3$	M1 A1 ft A1 ag **3**	Obtaining a polynomial to integrate For $x + \frac{1}{9}x^3$ ft requires at least two non-zero terms

2(a)(i)	$\lvert w \rvert = 3$, $\arg w = -\frac{1}{12}\pi$ $\lvert z \rvert = 2$, $\arg z = -\frac{1}{3}\pi$ $\left\lvert \frac{w}{z} \right\rvert = \frac{3}{2}$, $\arg \frac{w}{z} = (-\frac{1}{12}\pi) - (-\frac{1}{3}\pi) = \frac{1}{4}\pi$	B1 B1B1 B1B1 ft **5**	*Deduct 1 mark if answers given in form* $r(\cos\theta + \mathrm{j}\sin\theta)$ *but modulus and argument not stated.* Accept degrees and decimal approxs
(ii)	$\frac{w}{z} = \frac{3}{2}(\cos\frac{1}{4}\pi + \mathrm{j}\sin\frac{1}{4}\pi)$ $= \frac{3}{2\sqrt{2}} + \frac{3}{2\sqrt{2}}\mathrm{j}$	M1 A1 **2**	Accept $\sqrt{1.125} + \sqrt{1.125}\,\mathrm{j}$
(b)(i)	$\mathrm{e}^{-\frac{1}{2}\mathrm{j}\theta} + \mathrm{e}^{\frac{1}{2}\mathrm{j}\theta}$ $= (\cos\frac{1}{2}\theta - \mathrm{j}\sin\frac{1}{2}\theta) + (\cos\frac{1}{2}\theta + \mathrm{j}\sin\frac{1}{2}\theta)$ $= 2\cos\frac{1}{2}\theta$	M1 A1	For either bracketed expression
	$1 + \mathrm{e}^{\mathrm{j}\theta} = \mathrm{e}^{\frac{1}{2}\mathrm{j}\theta}(\mathrm{e}^{-\frac{1}{2}\mathrm{j}\theta} + \mathrm{e}^{\frac{1}{2}\mathrm{j}\theta})$ $= \mathrm{e}^{\frac{1}{2}\mathrm{j}\theta}(2\cos\frac{1}{2}\theta)$	M1 A1 ag **4**	
	OR $1 + \mathrm{e}^{\mathrm{j}\theta} = 1 + \cos\theta + \mathrm{j}\sin\theta$ $= 2\cos^2\frac{1}{2}\theta + 2\mathrm{j}\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$ M1 $= 2\cos\frac{1}{2}\theta(\cos\frac{1}{2}\theta + \mathrm{j}\sin\frac{1}{2}\theta)$ $= 2\mathrm{e}^{\frac{1}{2}\mathrm{j}\theta}\cos\frac{1}{2}\theta$ A1		
(ii)	$C + \mathrm{j}S = 1 + \binom{n}{1}\mathrm{e}^{\mathrm{j}\theta} + \binom{n}{2}\mathrm{e}^{2\mathrm{j}\theta} + \ldots + \binom{n}{n}\mathrm{e}^{n\mathrm{j}\theta}$ $= (1 + \mathrm{e}^{\mathrm{j}\theta})^n$ $= 2^n \mathrm{e}^{\frac{1}{2}n\theta\mathrm{j}} \cos^n \frac{1}{2}\theta$ $C = 2^n \cos(\frac{1}{2}n\theta)\cos^n\frac{1}{2}\theta$ $S = 2^n \sin(\frac{1}{2}n\theta)\cos^n\frac{1}{2}\theta$ $\frac{S}{C} = \frac{2^n \sin(\frac{1}{2}n\theta)\cos^n\frac{1}{2}\theta}{2^n \cos(\frac{1}{2}n\theta)\cos^n\frac{1}{2}\theta} = \frac{\sin(\frac{1}{2}n\theta)}{\cos(\frac{1}{2}n\theta)} = \tan(\frac{1}{2}n\theta)$	M1 M1A1 M1 A1 A1 B1 ag **7**	Using (i) to obtain a form from which the real and imaginary parts can be written down Allow ft from $C + \mathrm{j}S = \mathrm{e}^{\frac{1}{2}n\theta\,\mathrm{j}} \times$ any real function of *n* and θ

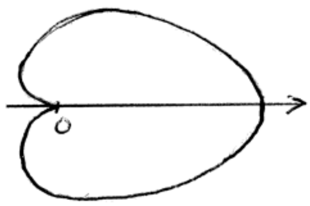
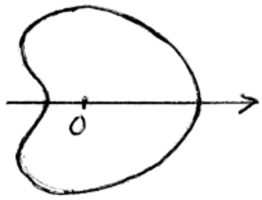
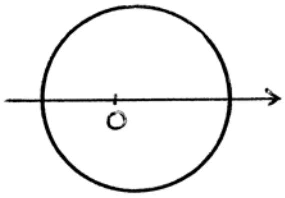
3 (i)	$\det \mathbf{P} = 1(6-k) - 1(4-2)$ $= 4-k$ $\mathbf{P}^{-1} = \frac{1}{4-k}\begin{pmatrix} -1 & 2 & 6-k \\ 4 & -4-k & k-12 \\ -1 & 2 & 2 \end{pmatrix}$ When $k=2$, $\mathbf{P}^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 2 & 4 \\ 4 & -6 & -10 \\ -1 & 2 & 2 \end{pmatrix}$	M1 A1 M1 M1 A1 ft B1 ag **6**	Evaluating at least three cofactors Fully correct method for inverse Ft from wrong determinant Correctly obtained
(ii)	$\mathbf{M}\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$ $\mathbf{M}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ $\mathbf{M}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ Eigenvalues are 0, 1, 2	M1 A1A1A1 **4**	For one evaluation
	OR M1 Eigenvalues are 0, 1, 2 A2 A1		Obtaining an eigenvalue (e.g. by solving $-\lambda^3 + 3\lambda^2 - 2\lambda = 0$) Give A1 for one correct Verifying given eigenvectors, linking with eigenvalues correctly
(iii)	$\mathbf{M}^n = \begin{pmatrix} 4 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix}\frac{1}{2}\begin{pmatrix} -1 & 2 & 4 \\ 4 & -6 & -10 \\ -1 & 2 & 2 \end{pmatrix}$ $= \frac{1}{2}\begin{pmatrix} 0 & 2 & 2^{n+1} \\ 0 & 1 & 3\times 2^n \\ 0 & 0 & -2^n \end{pmatrix}\begin{pmatrix} -1 & 2 & 4 \\ 4 & -6 & -10 \\ -1 & 2 & 2 \end{pmatrix}$ $= \begin{pmatrix} 4-2^n & -6+2^{n+1} & -10+2^{n+1} \\ 2-3\times 2^{n-1} & -3+3\times 2^n & -5+3\times 2^n \\ 2^{n-1} & -2^n & -2^n \end{pmatrix}$ $= \begin{pmatrix} 4 & -6 & -10 \\ 2 & -3 & -5 \\ 0 & 0 & 0 \end{pmatrix} + 2^{n-1}\begin{pmatrix} -2 & 4 & 4 \\ -3 & 6 & 6 \\ 1 & -2 & -2 \end{pmatrix}$	B1B1 M1A1 B1 ft M1 A1 A1 ag **8**	For $\begin{pmatrix} 4 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix}$ seen (for B2, these must be consistent) For $\mathbf{S}\mathbf{D}^n\mathbf{S}^{-1}$ (M1A0 if order wrong) *or* $\frac{1}{2}\begin{pmatrix} 4 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 4 & -6 & -10 \\ -2^n & 2^{n+1} & 2^{n+1} \end{pmatrix}$ Evaluating product of 3 matrices Any correct form

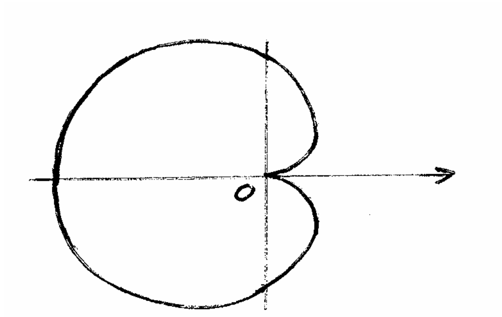
	OR Prove $\mathbf{M}^n = \mathbf{A} + 2^{n-1}\mathbf{B}$ by induction When $n=1$, $\mathbf{A}+\mathbf{B}=\mathbf{M}$	B1		
	Assuming $\mathbf{M}^k = \mathbf{A} + 2^{k-1}\mathbf{B}$, $\mathbf{M}^{k+1} = \mathbf{A}\mathbf{M} + 2^{k-1}\mathbf{B}\mathbf{M}$	M1A2		or $\mathbf{M}^{k+1} = \mathbf{M}\mathbf{A} + 2^{k-1}\mathbf{M}\mathbf{B}$
	$= \mathbf{A} + 2^{k-1}(2\mathbf{B})$	A1A1		
	$= \mathbf{A} + 2^k\mathbf{B}$	A1		
	True for $n=k \Rightarrow$ True for $n=k+1$; hence true for all positive integers *n*	A1		*Dependent on previous 7 marks*

4 (i)	If $y = \text{arcosh}\,x$, $x = \cosh y = \frac{1}{2}(e^y + e^{-y})$ $e^{2y} - 2x e^y + 1 = 0$ $e^y = \dfrac{2x \pm \sqrt{4x^2 - 4}}{2}$ $= x \pm \sqrt{x^2 - 1}$ Since $y \geq 0$, $e^y \geq 1$, so $e^y = x + \sqrt{x^2 - 1}$ $\text{arcosh}\,x = y = \ln(\,x + \sqrt{x^2 - 1}\,)$	M1 M1 M1 A1 A1 ag **5**	$\frac{1}{2}$ and + must be correct
(ii)	$\displaystyle\int_{2.5}^{3.9} \frac{1}{\sqrt{4x^2 - 9}}\,dx = \left[\frac{1}{2}\text{arcosh}\left(\frac{2x}{3}\right)\right]_{2.5}^{3.9}$ $= \frac{1}{2}(\,\text{arcosh}\,2.6 - \text{arcosh}\,\frac{5}{3}\,)$ $= \frac{1}{2}\left(\,\ln(2.6 + \sqrt{2.6^2 - 1}) - \ln(\frac{5}{3} + \sqrt{\frac{25}{9} - 1})\,\right)$ $= \frac{1}{2}(\ln 5 - \ln 3)$ $= \frac{1}{2}\ln\frac{5}{3}$	M1 A1A1 M1 A1 **5**	For arcosh (or any cosh substitution) For $\frac{1}{2}$ and $\frac{2x}{3}$ (or $2x = 3\cosh u$ and $\int \frac{1}{2}du$) (or limits of *u* in logarithmic form)
	OR M2 $\left[\,\frac{1}{2}\ln(2x + \sqrt{4x^2 - 9})\,\right]_{2.5}^{3.9}$ A1A1 $= \frac{1}{2}\ln 15 - \frac{1}{2}\ln 9$ $= \frac{1}{2}\ln\frac{5}{3}$ A1		For $\ln(kx + \sqrt{k^2x^2 - ...})$ Give M1 for $\ln(k_1x + \sqrt{{k_2}^2x^2 - ...})$ For $\frac{1}{2}$ and $\ln(2x + \sqrt{4x^2 - 9})$ (or $\ln(x + \sqrt{x^2 - \frac{9}{4}})$)
(iii)	$\dfrac{dy}{dx} = \dfrac{(2 + \sinh x)\sinh x - (\cosh x)(\cosh x)}{(2 + \sinh x)^2}$ $= \dfrac{2\sinh x - 1}{(2 + \sinh x)^2}$ $\dfrac{dy}{dx} = \dfrac{1}{9}$ when $18\sinh x - 9 = (2 + \sinh x)^2$ $\sinh^2 x - 14\sinh x + 13 = 0$ $\sinh x = 1,\ 13$ When $\sinh x = 1$, $\cosh x = \sqrt{2}$, $x = \ln(1 + \sqrt{2})$ Point is $\left(\ln(1 + \sqrt{2}),\ \dfrac{\sqrt{2}}{3}\right)$ When $\sinh x = 13$, $\cosh x = \sqrt{170}$, $x = \ln(13 + \sqrt{170})$ Point is $\left(\ln(13 + \sqrt{170}),\ \dfrac{\sqrt{170}}{15}\right)$	M1 A1 M1 M1 M1 A1 ag A1A1 **8**	Using quotient rule Any correct form Quadratic in $\sinh x$ (or product of two quadratics in e^x) Solving quadratic to obtain at least one value of $\sinh x$ (or e^x) Obtaining *x* in logarithmic form (must use a correct formula for arsinh) *SR* B1B1 for verifying $y = \frac{1}{3}\sqrt{2}$ and $\dfrac{dy}{dx} = \dfrac{1}{9}$ when $x = \ln(1+\sqrt{2})$

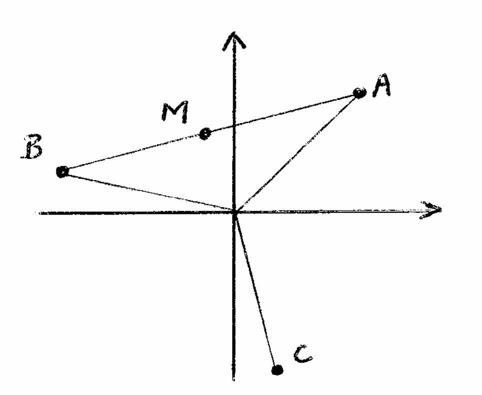
Alternatives for Q4 (i)

	$\cosh \ln(x+\sqrt{x^2-1}) = \frac{1}{2}(e^{\ln(x+\sqrt{x^2-1})} + e^{-\ln(x+\sqrt{x^2-1})})$ $= \frac{1}{2}(x+\sqrt{x^2-1} + \frac{1}{x+\sqrt{x^2-1}})$ $= \frac{1}{2}(x+\sqrt{x^2-1} + x-\sqrt{x^2-1})$ $= x$ Since $\ln(x+\sqrt{x^2-1}) > 0$, $\text{arcosh}\, x = \ln(x+\sqrt{x^2-1})$	M1 M1 M1 A1 A1 **5**	
	If $y = \text{arcosh}\, x$ then $\ln(x+\sqrt{x^2-1}) = \ln(\cosh y + \sqrt{\cosh^2 y - 1})$ $= \ln(\cosh y + \sinh y)$ since $\sinh y > 0$ $= \ln(e^y)$ $= y$	M1 M1 A1 M1 A1 **5**	

5 (i)	$k=1$ 	B1 B1	General shape correct Cusp at O clearly shown
	$k=1.5$ 	B1 B1	General shape correct 'Dimple' correctly shown
	$k=4$ 	B1 **5**	
(ii)	Cusp	B1 **1**	
(iii)	When $k=1$, there are 3 points When $k=1.5$, there are 4 points When $k=4$, there are 2 points	B2 **2**	Give B1 for two cases correct
(iv)	$x = k\cos\theta + \cos^2\theta$ $\frac{dx}{d\theta} = -k\sin\theta - 2\cos\theta\sin\theta$ $= -\sin\theta(k + 2\cos\theta)$ $= 0$ when $\theta = 0,\ \pi,$ or $\cos\theta = -\frac{1}{2}k$ For just two points, $k \geq 2$	B1 B1 M1 A1 **4**	Allow $k > 2$
(v)	$d^2 = r^2 + 1^2 - 2r\cos\theta$ $= (k+\cos\theta)^2 + 1 - 2(k+\cos\theta)\cos\theta$ $= k^2 + 1 - \cos^2\theta \quad (= k^2 + \sin^2\theta)$ Since $0 \leq \cos^2\theta \leq 1$, $k^2 \leq d^2 \leq k^2 + 1$	M1 A1 M1 A1 ag **4**	or $0 \leq \sin^2\theta \leq 1$
(vi)	When k is large, $\sqrt{k^2+1} \approx k$, so $d \approx k$ Curve is very nearly a circle, with centre $(1,\ 0)$ and radius k	M1 A1 **2**	

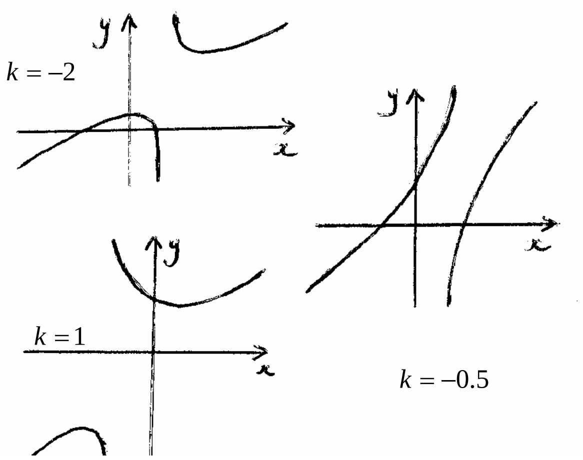
1(a)(i)		B2	2	Must include a sharp point at O and have infinite gradient at $\theta = \pi$ Give B1 for r increasing from zero for $0 < \theta < \pi$, or decreasing to zero for $-\pi < \theta < 0$
(ii)	Area is $\int \frac{1}{2} r^2 \, d\theta = \int_0^{\frac{1}{2}\pi} \frac{1}{2} a^2 (1 - \cos\theta)^2 \, d\theta$	M1 A1		For integral of $(1-\cos\theta)^2$ For a correct integral expression including limits *(may be implied by later work)*
	$= \frac{1}{2} a^2 \int_0^{\frac{1}{2}\pi} \left(1 - 2\cos\theta + \frac{1}{2}(1 + \cos 2\theta) \right) d\theta$	B1		Using $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$
	$= \frac{1}{2} a^2 \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta) \right]_0^{\frac{1}{2}\pi}$	B1B1 ft		Integrating $a + b\cos\theta$ and $k\cos 2\theta$
	$= \frac{1}{2} a^2 (\frac{3}{4}\pi - 2)$	B1	6	Accept $0.178a^2$
(b)	Put $x = 2\sin\theta$	M1		or $x = 2\cos\theta$
	Integral is $\int_0^{\frac{1}{6}\pi} \frac{1}{(4 - 4\sin^2\theta)^{\frac{3}{2}}} (2\cos\theta) \, d\theta$	A1		Limits not required
	$= \int_0^{\frac{1}{6}\pi} \frac{2\cos\theta}{8\cos^3\theta} \, d\theta = \int_0^{\frac{1}{6}\pi} \frac{1}{4}\sec^2\theta \, d\theta$			
	$= \left[\frac{1}{4}\tan\theta \right]_0^{\frac{1}{6}\pi}$	M1		For $\int \sec^2\theta \, d\theta = \tan\theta$
	$= \frac{1}{4} \times \frac{1}{\sqrt{3}} = \frac{1}{4\sqrt{3}}$	A1 ag	4	*SR* If $x = 2\tanh u$ is used M1 for $\frac{1}{4}\sinh(\frac{1}{2}\ln 3)$ A1 for $\frac{1}{8}(\sqrt{3} - \frac{1}{\sqrt{3}}) = \frac{1}{4\sqrt{3}}$ (max 2 / 4)
(c)(i)	$f'(x) = \frac{-2}{\sqrt{1 - 4x^2}}$	B2	2	Give B1 for any non-zero real multiple of this (or for $\frac{-2}{\sin y}$ etc)
(ii)	$f'(x) = -2(1 - 4x^2)^{-\frac{1}{2}}$	M1		Binomial expansion (3 terms, $n = -\frac{1}{2}$)
	$= -2(1 + 2x^2 + 6x^4 + ...)$	A1		Expansion of $(1 - 4x^2)^{-\frac{1}{2}}$ correct (accept unsimplified form)
	$f(x) = C - 2x - \frac{4}{3}x^3 - \frac{12}{5}x^5 + ...$	M1		Integrating series for $f'(x)$
	$f(0) = \frac{1}{2}\pi \Rightarrow C = \frac{1}{2}\pi$			
	$f(x) = \frac{1}{2}\pi - 2x - \frac{4}{3}x^3 - \frac{12}{5}x^5 + ...$	A1	4	Must obtain a non-zero x^5 term C not required

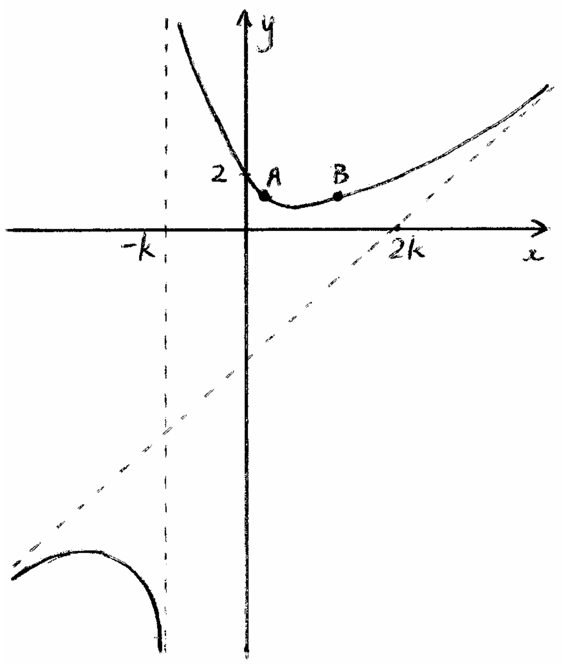
	OR by repeated differentiation Finding $f^{(5)}(x)$ Evaluating $f^{(5)}(0)$ $(=-288)$ $f'(x)=-2-4x^2-12x^4+...$ $f(x)=\frac{1}{2}\pi-2x-\frac{4}{3}x^3-\frac{12}{5}x^5+...$	 M1 M1 A1 ft A1		 Must obtain a non-zero value ft from (c)(i) when B1 given

2 (a)	$(\cos\theta + \mathrm{j}\sin\theta)^5$ $= c^5 + 5\mathrm{j}c^4s - 10c^3s^2 - 10\mathrm{j}c^2s^3 + 5cs^4 + \mathrm{j}s^5$ Equating imaginary parts $\sin 5\theta = 5c^4s - 10c^2s^3 + s^5$ $= 5(1-s^2)^2 s - 10(1-s^2)s^3 + s^5$ $= 5s - 10s^3 + 5s^5 - 10s^3 + 10s^5 + s^5$ $= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$	M1 M1 A1 M1 A1 ag **5**	
(b)(i)	$\lvert -2+2\mathrm{j}\rvert = \sqrt{8}$, $\arg(-2+2\mathrm{j}) = \frac{3}{4}\pi$ $r = \sqrt{2}$ $\theta = \frac{1}{4}\pi$ $\theta = \frac{11}{12}\pi,\ -\frac{5}{12}\pi$	B1B1 B1 ft B1 ft M1 A1 **6**	Accept 2.8; 2.4, 135° (Implies B1 for $\sqrt{8}$) One correct (Implies B1 for $\frac{3}{4}\pi$) Adding or subtracting $\frac{2}{3}\pi$ Accept $\theta = \frac{1}{4}\pi + \frac{2}{3}k\pi,\ k = 0, 1, -1$
(ii)		B2 **2**	Give B1 for two of B, C, M in the correct quadrants Give B1 ft for all four points in the correct quadrants
(iii)	$\lvert w\rvert = \frac{1}{2}\sqrt{2}$ $\arg w = \frac{1}{2}(\frac{1}{4}\pi + \frac{11}{12}\pi) = \frac{7}{12}\pi$	B1 ft B1 **2**	Accept 0.71 Accept 1.8
(iv)	$\lvert w^6\rvert = (\frac{1}{2}\sqrt{2})^6 = \frac{1}{8}$ $\arg(w^6) = 6\times\frac{7}{12}\pi = \frac{7}{2}\pi$ $w^6 = \frac{1}{8}(\cos\frac{7}{2}\pi + \mathrm{j}\sin\frac{7}{2}\pi)$ $= -\frac{1}{8}\mathrm{j}$	M1 A1 ft A1 **3**	Obtaining either modulus or argument Both correct (ft) Allow from $\arg w = \frac{1}{4}\pi$ etc
			SR If B, C interchanged on diagram (ii) B1 (iii) B1 B1 for $-\frac{1}{12}\pi$ (iv) M1A1A1

3 (i)	$\det(\mathbf{M}-\lambda\mathbf{I}) = (3-\lambda)[(3-\lambda)(-4-\lambda)-4]$ $-5[5(-4-\lambda)+4]+2[-10-2(3-\lambda)]$ $=(3-\lambda)(-16+\lambda+\lambda^2)-5(-16-5\lambda)+2(-16+2\lambda)$ $=-48+19\lambda+2\lambda^2-\lambda^3+80+25\lambda-32+4\lambda$ $=48\lambda+2\lambda^2-\lambda^3$ Characteristic equation is $\lambda^3-2\lambda^2-48\lambda=0$	M1 A1 M1 A1 ag **4**	Obtaining $\det(\mathbf{M}-\lambda\mathbf{I})$ Any correct form Simplification
(ii)	$\lambda(\lambda-8)(\lambda+6)=0$ Other eigenvalues are $8,\ -6$ When $\lambda=8$, $3x+5y+2z=8x$ $(\ 5x+3y-2z=8y\ )$ $2x-2y-4z=8z$ $y=x$ and $z=0$; eigenvector is $\begin{pmatrix}1\\1\\0\end{pmatrix}$ When $\lambda=-6$, $3x+5y+2z=-6x$ $5x+3y-2z=-6y$ $y=-x,\ z=-2x$; eigenvector is $\begin{pmatrix}1\\-1\\-2\end{pmatrix}$	M1 A1 M1 M1 A1 M1 M1 A1 **8**	Solving to obtain a non-zero value Two independent equations Obtaining a non-zero eigenvector *($-5x+5y+2z=8x$ etc can earn M0M1)* Two independent equations Obtaining a non-zero eigenvector
(iii)	$\mathbf{P}=\begin{pmatrix}1&1&1\\-1&1&-1\\1&0&-2\end{pmatrix}$ $\mathbf{D}=\begin{pmatrix}0&0&0\\0&8&0\\0&0&-6\end{pmatrix}^2$ $=\begin{pmatrix}0&0&0\\0&64&0\\0&0&36\end{pmatrix}$	B1 ft M1 A1 **3**	B0 if **P** is clearly singular Order must be consistent with **P** when B1 has been earned
(iv)	$\mathbf{M}^3-2\mathbf{M}^2-48\mathbf{M}=\mathbf{0}$ $\mathbf{M}^3=2\mathbf{M}^2+48\mathbf{M}$ $\mathbf{M}^4=2\mathbf{M}^3+48\mathbf{M}^2$ $=2(2\mathbf{M}^2+48\mathbf{M})+48\mathbf{M}^2$ $=52\mathbf{M}^2+96\mathbf{M}$	M1 M1 A1 **3**	

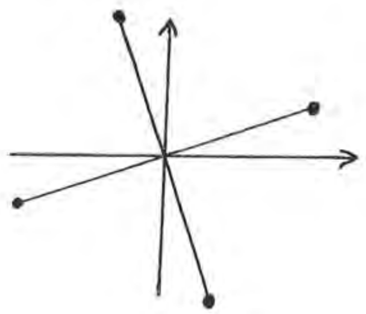
4 (a)	$\int_0^1 \frac{1}{\sqrt{9x^2+16}}\,dx = \left[\frac{1}{3}\text{arsinh}\frac{3x}{4}\right]_0^1$ $= \frac{1}{3}\text{arsinh}\frac{3}{4}$ $= \frac{1}{3}\ln\left(\frac{3}{4}+\sqrt{\frac{9}{16}+1}\right)$ $= \frac{1}{3}\ln 2$	M1 A1 A1 M1 A1 **5**	For arsinh or for any sinh substitution For $\frac{3}{4}x$ or for $3x = 4\sinh u$ For $\frac{1}{3}$ or for $\int \frac{1}{3}du$
	OR M2 $\left[\frac{1}{3}\ln(3x+\sqrt{9x^2+16})\right]_0^1$ A1A1 $= \frac{1}{3}\ln 8 - \frac{1}{3}\ln 4$ $= \frac{1}{3}\ln 2$ A1		For $\ln(kx+\sqrt{k^2x^2+...})$ [Give M1 for $\ln(ax+\sqrt{bx^2+...})$] or $\frac{1}{3}\ln(x+\sqrt{x^2+\frac{16}{9}})$
(b)(i)	$2\sinh x\cosh x = 2\times\frac{1}{2}(e^x-e^{-x})\frac{1}{2}(e^x+e^{-x})$ $= \frac{1}{2}(e^{2x}-e^{-2x})$ $= \sinh 2x$	M1 A1 **2**	$(e^x-e^{-x})(e^x+e^{-x}) = (e^{2x}-e^{-2x})$ For completion
(ii)	$\frac{dy}{dx} = 20\sinh x - 6\sinh 2x$ For stationary points, $20\sinh x - 12\sinh x\cosh x = 0$ $4\sinh x(5-3\cosh x) = 0$ $\sinh x = 0$ or $\cosh x = \frac{5}{3}$ $x = 0,\quad y = 17$ $x = (\pm)\ln\left(\frac{5}{3}+\sqrt{\frac{25}{9}-1}\right) = \ln 3$ $y = 10\left(3+\frac{1}{3}\right) - \frac{3}{2}\left(9+\frac{1}{9}\right) = \frac{59}{3}$ $x = -\ln 3,\quad y = \frac{59}{3}$	B1B1 M1 A1 A1 ag A1 ag B1 **7**	When exponential form used, give B1 for any 2 terms correctly differentiated Solving $\frac{dy}{dx} = 0$ to obtain a value of $\sinh x$, $\cosh x$ or e^x (or $x = 0$ stated) Correctly obtained Correctly obtained *The last A1A1 ag can be replaced by B1B1 ag for a full verification*
(iii)	$\left[20\sinh x - \frac{3}{2}\sinh 2x\right]_{-\ln 3}^{\ln 3}$ $= \left\{10\left(3-\frac{1}{3}\right) - \frac{3}{4}\left(9-\frac{1}{9}\right)\right\}\times 2$ $= \left(\frac{80}{3}-\frac{20}{3}\right)\times 2 = 40$	B1B1 M1 A1 ag **4**	When exponential form used, give B1 for any 2 terms correctly integrated Exact evaluation of $\sinh(\ln 3)$ and $\sinh(2\ln 3)$

5 (i)		B1 B1 B1 B1 B1 B1 **6**	Maximum on LH branch and minimum on RH branch Crossing axes correctly Two branches with positive gradient Crossing axes correctly Maximum on LH branch and minimum on RH branch Crossing positive y-axis and minimum in first quadrant
(ii)	$y=\frac{(x+k)(x-2k)+2k^2+2k}{x+k}$ $=x-2k+\frac{2k(k+1)}{x+k}$ Straight line when $2k(k+1)=0$ $k=0,\quad k=-1$	M1 A1 (ag) B1B1 **4**	Working in either direction For completion
(iii)(*A*)	Hyperbola	B1 **1**	
(*B*)	$x=-k$ $y=x-2k$	B1 B1 **2**	

(iv)		B1 B1 B1 B1 B1 **5**	Asymptotes correctly drawn Curve approaching asymptotes correctly (both branches) Intercept 2 on *y*-axis, and not crossing the *x*-axis Points A and B marked, with minimum point between them Points A and B at the same height $(y=1)$

4756 (FP2) Further Methods for Advanced Mathematics

1(a)	Area is $\int_0^{\pi} \frac{1}{2}a^2(1-\cos 2\theta)^2 \, d\theta$ $= \int_0^{\pi} \frac{1}{2}a^2\left(1-2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right)d\theta$ $= \frac{1}{2}a^2\left[\frac{3}{2}\theta-\sin 2\theta+\frac{1}{8}\sin 4\theta\right]_0^{\pi}$ $= \frac{3}{4}\pi a^2$	M1 A1 B1 B1B1B1 ft A1 **7**	For $\int(1-\cos 2\theta)^2 \, d\theta$ Correct integral expression including limits (may be implied by later work) For $\cos^2 2\theta = \frac{1}{2}(1+\cos 4\theta)$ Integrating $a + b\cos 2\theta + c\cos 4\theta$ *[Max B2 if answer incorrect and no mark has previously been lost]*
(b)(i)	$f'(x) = \dfrac{1}{1+(\sqrt{3}+x)^2}$ $f''(x) = \dfrac{-2(\sqrt{3}+x)}{\left(1+(\sqrt{3}+x)^2\right)^2}$	M1 A1 M1 A1 **4**	Applying $\dfrac{d}{du}\arctan u = \dfrac{1}{1+u^2}$ or $\dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}$ Applying chain (or quotient) rule
(ii)	$f(0) = \frac{1}{3}\pi$ $f'(0) = \frac{1}{4}, \quad f''(0) = -\frac{1}{8}\sqrt{3}$ $\arctan(\sqrt{3}+x) = \frac{1}{3}\pi + \frac{1}{4}x - \frac{1}{16}\sqrt{3}x^2 + \ldots$	B1 M1 A1A1 ft **4**	Stated; or appearing in series *Accept* 1.05 Evaluating $f'(0)$ or $f''(0)$ For $\frac{1}{4}x$ and $-\frac{1}{16}\sqrt{3}x^2$ *ft provided coefficients are non-zero*
(iii)	$\int_{-h}^{h}(\frac{1}{3}\pi x + \frac{1}{4}x^2 - \frac{1}{16}\sqrt{3}x^3 + \ldots)\,dx$ $= \left[\frac{1}{6}\pi x^2 + \frac{1}{12}x^3 - \frac{1}{64}\sqrt{3}x^4 + \ldots\right]_{-h}^{h}$ $\approx (\frac{1}{6}\pi h^2 + \frac{1}{12}h^3 - \frac{1}{64}\sqrt{3}h^4)$ $-(\frac{1}{6}\pi h^2 - \frac{1}{12}h^3 - \frac{1}{64}\sqrt{3}h^4)$ $= \frac{1}{6}h^3$	M1 A1 ft A1 ag **3**	Integrating (award if x is missed) for $\frac{1}{12}x^3$ Allow ft from $a + \frac{1}{4}x + cx^2$ provided that $a \neq 0$ Condone a proof which neglects h^4

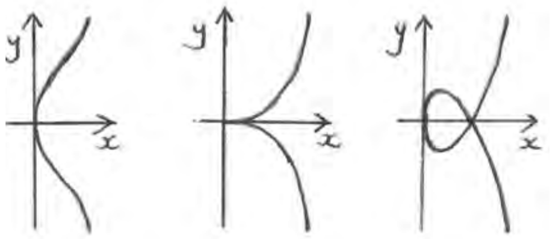
2(a)	4th roots of $16\mathrm{j} = 16\mathrm{e}^{\frac{1}{2}\pi\mathrm{j}}$ are $r\,\mathrm{e}^{\mathrm{j}\theta}$ where $r = 2$ $\theta = \frac{1}{8}\pi$ $\theta = \dfrac{\pi}{8} + \dfrac{2k\pi}{4}$ $\theta = -\frac{7}{8}\pi,\ -\frac{3}{8}\pi,\ \frac{5}{8}\pi$ 	B1 B1 M1 A1 M1 A1 **6**	*Accept* $16^{\frac{1}{4}}$ Implied by at least two correct (ft) further values or stating $k = -2,\ -1,\ (0),\ 1$ Points at vertices of a square centre O *or* 3 correct points (ft) *or* 1 point in each quadrant
(b)(i)	$(1-2\mathrm{e}^{\mathrm{j}\theta})(1-2\mathrm{e}^{-\mathrm{j}\theta}) = 1-2\mathrm{e}^{\mathrm{j}\theta}-2\mathrm{e}^{-\mathrm{j}\theta}+4$ $= 5-2(\mathrm{e}^{\mathrm{j}\theta}+\mathrm{e}^{-\mathrm{j}\theta})$ $= 5-4\cos\theta$	M1 A1 A1 ag **3**	For $\mathrm{e}^{\mathrm{j}\theta}\mathrm{e}^{-\mathrm{j}\theta} = 1$
	OR $(1-2\cos\theta-2\mathrm{j}\sin\theta)(1-2\cos\theta+2\mathrm{j}\sin\theta)$ M1 $= (1-2\cos\theta)^2+4\sin^2\theta$ A1 $= 1-4\cos\theta+4(\cos^2\theta+\sin^2\theta)$ $= 5-4\cos\theta$ A1		
(ii)	$C+\mathrm{j}S = 2\mathrm{e}^{\mathrm{j}\theta}+4\mathrm{e}^{2\mathrm{j}\theta}+8\mathrm{e}^{3\mathrm{j}\theta}+\ldots+2^n\,\mathrm{e}^{n\mathrm{j}\theta}$ $= \dfrac{2\mathrm{e}^{\mathrm{j}\theta}\left(1-(2\mathrm{e}^{\mathrm{j}\theta})^n\right)}{1-2\mathrm{e}^{\mathrm{j}\theta}}$ $= \dfrac{2\mathrm{e}^{\mathrm{j}\theta}(1-2^n\mathrm{e}^{n\mathrm{j}\theta})(1-2\mathrm{e}^{-\mathrm{j}\theta})}{(1-2\mathrm{e}^{\mathrm{j}\theta})(1-2\mathrm{e}^{-\mathrm{j}\theta})}$ $= \dfrac{2\mathrm{e}^{\mathrm{j}\theta}-4-2^{n+1}\mathrm{e}^{(n+1)\mathrm{j}\theta}+2^{n+2}\mathrm{e}^{n\mathrm{j}\theta}}{5-4\cos\theta}$ $C = \dfrac{2\cos\theta-4-2^{n+1}\cos(n+1)\theta+2^{n+2}\cos n\theta}{5-4\cos\theta}$ $S = \dfrac{2\sin\theta-2^{n+1}\sin(n+1)\theta+2^{n+2}\sin n\theta}{5-4\cos\theta}$	M1 M1 A1 M1 A2 M1 A1 ag A1 **9**	Obtaining a geometric series Summing (M0 for sum to infinity) Give A1 for two correct terms in numerator Equating real (or imaginary) parts

3 (i)	Characteristic equation is $(7-\lambda)(-1-\lambda)+12=0$ $\lambda^2-6\lambda+5=0$ $\lambda=1,\ 5$ When $\lambda=1$, $\begin{pmatrix}7 & 3\\ -4 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$ $7x+3y=x$ $-4x-y=y$ $y=-2x$, eigenvector is $\begin{pmatrix}1\\ -2\end{pmatrix}$ When $\lambda=5$, $\begin{pmatrix}7 & 3\\ -4 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=5\begin{pmatrix}x\\ y\end{pmatrix}$ $7x+3y=5x$ $-4x-y=5y$ $y=-\frac{2}{3}x$, eigenvector is $\begin{pmatrix}3\\ -2\end{pmatrix}$	M1 A1A1 M1 M1 A1 M1 A1 **8**	or $\begin{pmatrix}6 & 3\\ -4 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$ *can be awarded for either eigenvalue* Equation relating *x* and *y* or any (non-zero) multiple *SR* $(\mathbf{M}-\lambda\mathbf{I})\mathbf{x}=\lambda\mathbf{x}$ can earn M1A1A1M0M1A0M1A0
(ii)	$\mathbf{P}=\begin{pmatrix}1 & 3\\ -2 & -2\end{pmatrix}$ $\mathbf{D}=\begin{pmatrix}1 & 0\\ 0 & 5\end{pmatrix}$	B1 ft B1 ft **2**	B0 if **P** is singular For B2, the order must be consistent

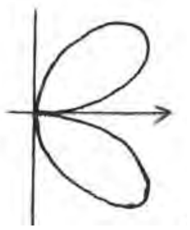
(iii)	$\mathbf{M} = \mathbf{P}\,\mathbf{D}\,\mathbf{P}^{-1}$	M1	*May be implied*
	$\mathbf{M}^n = \mathbf{P}\,\mathbf{D}^n\,\mathbf{P}^{-1}$	M1	
	$= \mathbf{P}\begin{pmatrix}1 & 0\\ 0 & 5^n\end{pmatrix}\mathbf{P}^{-1}$	A1 ft	*Dependent on M1M1*
	$= \begin{pmatrix}1 & 3\\ -2 & -2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 5^n\end{pmatrix}\frac{1}{4}\begin{pmatrix}-2 & -3\\ 2 & 1\end{pmatrix}$	B1 ft	For $\mathbf{P}^{-1}$
	$= \begin{pmatrix}1 & 3\times 5^n\\ -2 & -2\times 5^n\end{pmatrix}\frac{1}{4}\begin{pmatrix}-2 & -3\\ 2 & 1\end{pmatrix}$		or $\begin{pmatrix}1 & 3\\ -2 & -2\end{pmatrix}\frac{1}{4}\begin{pmatrix}-2 & -3\\ 2\times 5^n & 5^n\end{pmatrix}$
	$= \frac{1}{4}\begin{pmatrix}-2+6\times 5^n & -3+3\times 5^n\\ 4-4\times 5^n & 6-2\times 5^n\end{pmatrix}$	M1	Obtaining at least one element in a product of three matrices
	$a = -\frac{1}{2}+\frac{3}{2}\times 5^n$		
	$b = -\frac{3}{4}+\frac{3}{4}\times 5^n$	A1 ag	
	$c = 1-5^n$ $\quad d = \frac{3}{2}-\frac{1}{2}\times 5^n$	A2 **8**	Give A1 for one of *b, c, d* correct *SR* If $\mathbf{M}^n = \mathbf{P}^{-1}\,\mathbf{D}^n\,\mathbf{P}$ is used, max marks are M0M1A0B1M1A0A1 (*d* should be correct) *SR* If their **P** is singular, max marks are M1M1A1B0M0

4 (i)	$\frac{1}{2}(e^x + e^{-x}) = k$ $e^{2x} - 2k\,e^x + 1 = 0$ $e^x = \dfrac{2k \pm \sqrt{4k^2 - 4}}{2} = k \pm \sqrt{k^2 - 1}$ $x = \ln(k + \sqrt{k^2 - 1})$ or $\ln(k - \sqrt{k^2 - 1})$ $(k + \sqrt{k^2 - 1})(k - \sqrt{k^2 - 1}) = k^2 - (k^2 - 1) = 1$ $\ln(k - \sqrt{k^2 - 1}) = \ln(\dfrac{1}{k + \sqrt{k^2 - 1}}) = -\ln(k + \sqrt{k^2 - 1})$ $x = \pm\ln(k + \sqrt{k^2 - 1})$	M1 M1 A1 M1 A1 ag **5**	or $\cosh x + \sinh x = e^x$ or $k \pm \sqrt{k^2 - 1} = e^x$ One value sufficient or $\cosh x$ is an even function *(or equivalent)*
(ii)	$\displaystyle\int_1^2 \frac{1}{\sqrt{4x^2 - 1}}\,dx = \left[\frac{1}{2}\text{arcosh}\,2x\right]_1^2$ $= \frac{1}{2}(\text{arcosh}\,4 - \text{arcosh}\,2)$ $= \frac{1}{2}\left(\ln(4 + \sqrt{15}) - \ln(2 + \sqrt{3})\right)$	M1 A1 A1 M1 A1 **5**	For arcosh *or* $\ln(\lambda x + \sqrt{\lambda^2 x^2 - \ldots})$ or any cosh substitution For $\text{arcosh}\,2x$ or $2x = \cosh u$ or $\ln(2x + \sqrt{4x^2 - 1})$ or $\ln(x + \sqrt{x^2 - \frac{1}{4}})$ For $\frac{1}{2}$ or $\int \frac{1}{2}\,du$ Exact numerical logarithmic form
(iii)	$6\sinh x - 2\sinh x\cosh x = 0$ $\cosh x = 3$ (or $\sinh x = 0$) $x = 0$ $x = \pm\ln(3 + \sqrt{8})$	M1 M1 B1 A1 **4**	 Obtaining a value for $\cosh x$ or $x = \ln(3 \pm \sqrt{8})$
	OR $e^{4x} - 6e^{3x} + 6e^x - 1 = 0$ $(e^{2x} - 1)(e^{2x} - 6e^x + 1) = 0$ M2 $x = 0$ B1 $x = \ln(3 \pm \sqrt{8})$ A1		or $(e^x - e^{-x})(e^x + e^{-x} - 6) = 0$
(iv)	$\dfrac{dy}{dx} = 6\cosh x - 2\cosh 2x$ If $\dfrac{dy}{dx} = 5$ then $6\cosh x - 2(2\cosh^2 x - 1) = 5$ $4\cosh^2 x - 6\cosh x + 3 = 0$ Discriminant $D = 6^2 - 4\times4\times3 = -12$ Since $D < 0$ there are no solutions	B1 M1 M1 A1 **4**	 Using $\cosh 2x = 2\cosh^2 x - 1$ Considering *D*, or completing square, or considering turning point

	OR Gradient $g = 6\cosh x - 2\cosh 2x$ $g' = 6\sinh x - 4\sinh 2x = 2\sinh x(3 - 4\cosh x)$ $= 0$ when $x = 0$ (only) $g'' = 6\cosh x - 8\cosh 2x = -2$ when $x = 0$ Max value $g = 4$ when $x = 0$ So *g* is never equal to 5	B1 M1 M1 A1		Final A1 requires a complete proof showing this is the only turning point

5 (i)	$\lambda = -1$ $\lambda = 0$ $\lambda = 1$ 	B1B1B1	
	cusp loop	B1B1 **5**	Two different features (cusp, loop, asymptote) correctly identified
(ii)	$x = 1$	B1 **1**	
(iii)	Intersects itself when $y = 0$ $t = (\pm)\sqrt{\lambda}$ $\left(\dfrac{\lambda}{1+\lambda},\ 0\right)$	M1 A1 A1 **3**	
(iv)	$\dfrac{\mathrm{d}y}{\mathrm{d}t} = 3t^2 - \lambda = 0$ $t = \pm\sqrt{\dfrac{\lambda}{3}}$ $x = \dfrac{\lambda/3}{1+\lambda/3} = \dfrac{\lambda}{3+\lambda}$ $y = \pm\left((\dfrac{\lambda}{3})^{\frac{3}{2}} - \lambda(\dfrac{\lambda}{3})^{\frac{1}{2}} \right)$ $= \pm\lambda^{\frac{3}{2}}\left(\dfrac{1}{3\sqrt{3}} - \dfrac{1}{\sqrt{3}}\right) = \pm\lambda^{\frac{3}{2}}\left(-\dfrac{2}{3\sqrt{3}}\right)$ $= \pm\sqrt{\dfrac{4\lambda^3}{27}}$	M1 A1 ag M1 A1 ag **4**	One value sufficient
(v)	From asymptote, $a = 8$ From intersection point, $\dfrac{a\lambda}{1+\lambda} = 2$ $\lambda = \dfrac{1}{3}$ From maximum point, $b\sqrt{\dfrac{4\lambda^3}{27}} = 2$ $b = 27$	B1 M1 A1 M1 A1 **5**	

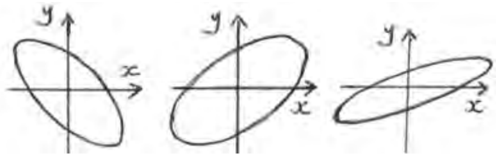
4756 (FP2) Further Methods for Advanced Mathematics

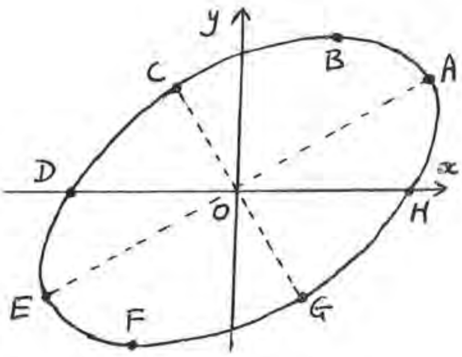
1(a)(i)	$x = r\cos\theta,\ \ y = r\sin\theta$ $(r^2\cos^2\theta + r^2\sin^2\theta)^2 = 3(r\cos\theta)(r\sin\theta)^2$ $r^4 = 3r^3\cos\theta\sin^2\theta$ $r = 3\cos\theta\sin^2\theta$	M1 A1 A1 ag **3**	(M0 for $x = \cos\theta,\ y = \sin\theta$)
(ii)		B1 B1 B1 **3**	Loop in 1st quadrant Loop in 4th quadrant Fully correct curve *Curve may be drawn using continuous or broken lines in any combination*
(b)	$\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,dx = \left[\frac{1}{\sqrt{3}}\arcsin\frac{\sqrt{3}\,x}{2}\right]_0^1$ $= \frac{1}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2}$ $= \frac{\pi}{3\sqrt{3}}$	M1 A1A1 M1 A1 **5**	For arcsin For $\frac{1}{\sqrt{3}}$ and $\frac{\sqrt{3}\,x}{2}$ Exact numerical value *Dependent on first M1* (M1A0 for $60/\sqrt{3}$)
	OR M1 Put $\sqrt{3}\,x = 2\sin\theta$ A1 $\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,dx = \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{3}}\,d\theta$ A1 $= \frac{\pi}{3\sqrt{3}}$ M1A1		Any sine substitution For $\int \frac{1}{\sqrt{3}}\,d\theta$ *M1 dependent on first M1*
(c)(i)	$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \ldots$ $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \ldots$	B1 B1 **2**	*Accept unsimplified forms*
(ii)	$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$ $= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \ldots$	M1 A1 **2**	Obtained from two correct series *Terms need not be added* If M0, then B1 for $2x + \frac{2}{3}x^3 + \frac{2}{5}x^5$

(iii)	$\sum_{r=0}^{\infty} \frac{1}{(2r+1)4^r} = 1 + \frac{1}{3\times4} + \frac{1}{5\times4^2} + ...$ $= 2\times\frac{1}{2} + \frac{2}{3}\times(\frac{1}{2})^3 + \frac{2}{5}\times(\frac{1}{2})^5 + ...$ $= \ln\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right) = \ln 3$	B1 B1 B1 ag **3**	*Terms need not be added* For $x = \frac{1}{2}$ seen or implied Satisfactory completion
2 (i)	$\lvert z \rvert = 8$, $\arg z = \frac{1}{4}\pi$ $\lvert z^* \rvert = 8$, $\arg z^* = -\frac{1}{4}\pi$ $\lvert zw \rvert = 8\times8 = 64$ $\arg(zw) = \frac{1}{4}\pi + \frac{7}{12}\pi = \frac{5}{6}\pi$ $\left\lvert \frac{z}{w} \right\rvert = \frac{8}{8} = 1$ $\arg(\frac{z}{w}) = \frac{1}{4}\pi - \frac{7}{12}\pi = -\frac{1}{3}\pi$	B1B1 B1 ft B1 ft B1 ft B1 ft B1 ft **7**	*Must be given separately* *Remainder may be given in exponential or* $r\,\mathrm{cjs}\,\theta$ *form* (B0 for $\frac{7}{4}\pi$) (B0 if left as $8/8$)
(ii)	$\frac{z}{w} = \cos(-\frac{1}{3}\pi) + \mathrm{j}\sin(-\frac{1}{3}\pi)$ $= \frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{j}$ $a = \frac{1}{2}$, $b = -\frac{1}{2}\sqrt{3}$	M1 A1 **2**	If M0, then B1B1 for $\frac{1}{2}$ and $-\frac{\sqrt{3}}{2}$
(iii)	$r = \sqrt[3]{8} = 2$ $\theta = \frac{1}{12}\pi$ $\theta = \frac{\pi}{12} + \frac{2k\pi}{3}$ $\theta = -\frac{7}{12}\pi$, $\frac{3}{4}\pi$	B1 ft B1 M1 A1 **4**	Accept $\sqrt[3]{8}$ Implied by one further correct (ft) value *Ignore values outside the required range*
(iv)	$w^* = 8\mathrm{e}^{-\frac{7}{12}\pi\mathrm{j}}$, so $2\mathrm{e}^{-\frac{7}{12}\pi\mathrm{j}} = \frac{1}{4}w^*$ $k_1 = \frac{1}{4}$ $z^* = 8\mathrm{e}^{-\frac{1}{4}\pi\mathrm{j}} = -8\mathrm{e}^{\frac{3}{4}\pi\mathrm{j}}$ So $2\mathrm{e}^{\frac{3}{4}\pi\mathrm{j}} = -\frac{1}{4}z^*$ $k_2 = -\frac{1}{4}$ $\mathrm{j}w = 8\mathrm{e}^{(\frac{1}{2}\pi + \frac{7}{12}\pi)\mathrm{j}} = 8\mathrm{e}^{\frac{13}{12}\pi\mathrm{j}}$ $= -8\mathrm{e}^{\frac{1}{12}\pi\mathrm{j}}$, so $2\mathrm{e}^{\frac{1}{12}\pi\mathrm{j}} = -\frac{1}{4}\mathrm{j}w$ $k_3 = -\frac{1}{4}$	B1 ft M1 A1 ft M1 A1 ft **5**	Matching w^* to a cube root with argument $-\frac{7}{12}\pi$ and $k_1 = \frac{1}{4}$ or ft ft is $\frac{r}{8}$ Matching z^* to a cube root with argument $\frac{3}{4}\pi$ *May be implied* ft is $-\frac{r}{\lvert z^* \rvert}$ Matching jw to a cube root with argument $\frac{1}{12}\pi$ *May be implied* OR M1 for $\arg(\mathrm{j}w) = \frac{1}{2}\pi + \arg w$ *(implied by* $\frac{13}{12}\pi$ *or* $-\frac{11}{12}\pi$*)* ft is $-\frac{r}{8}$

3 (i)	$\mathbf{Q}^{-1}=\frac{1}{k-3}\begin{pmatrix}-1 & k+2 & -1\\ 1 & 4-3k & k-2\\ 1 & -5 & 1\end{pmatrix}$ When $k=4$, $\mathbf{Q}^{-1}=\begin{pmatrix}-1 & 6 & -1\\ 1 & -8 & 2\\ 1 & -5 & 1\end{pmatrix}$	M1 A1 M1 A1 M1 A1 **6**	Evaluation of determinant *(must involve k)* For $(k-3)$ Finding at least four cofactors *(including one involving k)* Six signed cofactors correct *(including one involving k)* Transposing and dividing by det *Dependent on previous M1M1* $\mathbf{Q}^{-1}$ correct (in terms of *k*) and result for $k=4$ stated After 0, SC1 for $\mathbf{Q}^{-1}$ when $k=4$ obtained correctly with some working
(ii)	$\mathbf{P}=\begin{pmatrix}2 & -1 & 4\\ 1 & 0 & 1\\ 3 & 1 & 2\end{pmatrix}$, $\mathbf{D}=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 3\end{pmatrix}$ $\mathbf{M}=\mathbf{PDP}^{-1}$ $=\begin{pmatrix}2 & -1 & 4\\ 1 & 0 & 1\\ 3 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 3\end{pmatrix}\begin{pmatrix}-1 & 6 & -1\\ 1 & -8 & 2\\ 1 & -5 & 1\end{pmatrix}$ $=\begin{pmatrix}2 & 1 & 12\\ 1 & 0 & 3\\ 3 & -1 & 6\end{pmatrix}\begin{pmatrix}-1 & 6 & -1\\ 1 & -8 & 2\\ 1 & -5 & 1\end{pmatrix}$ $=\begin{pmatrix}11 & -56 & 12\\ 2 & -9 & 2\\ 2 & -4 & 1\end{pmatrix}$	B1B1 B2 M1 A2 **7**	For B2, order must be consistent Give B1 for $\mathbf{M}=\mathbf{P}^{-1}\mathbf{DP}$ or $\begin{pmatrix}2 & -1 & 4\\ 1 & 0 & 1\\ 3 & 1 & 2\end{pmatrix}\begin{pmatrix}-1 & 6 & -1\\ -1 & 8 & -2\\ 3 & -15 & 3\end{pmatrix}$ Good attempt at multiplying two matrices (no more than 3 errors), leaving third matrix in correct position Give A1 for five elements correct Correct **M** implies B2M1A2 5-8 elements correct implies B2M1A1
(iii)	Characteristic equation is $(\lambda-1)(\lambda+1)(\lambda-3)=0$ $\lambda^3-3\lambda^2-\lambda+3=0$ $\mathbf{M}^3=3\mathbf{M}^2+\mathbf{M}-3\mathbf{I}$ $\mathbf{M}^4=3\mathbf{M}^3+\mathbf{M}^2-3\mathbf{M}$ $=3(3\mathbf{M}^2+\mathbf{M}-3\mathbf{I})+\mathbf{M}^2-3\mathbf{M}$ $=10\mathbf{M}^2-9\mathbf{I}$ $a=10,\ b=0,\ c=-9$	B1 M1 A1 M1 A1 **5**	In any correct form *(Condone omission of =0)* **M** satisfies the characteristic equation Correct expanded form *(Condone omission of* **I** *)*

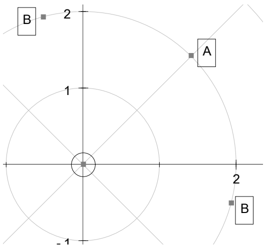
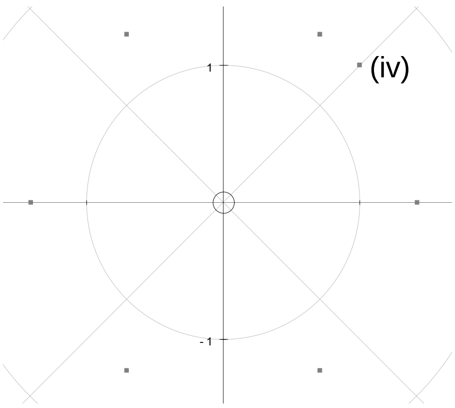
4 (i)	$\cosh^2 x = [\frac{1}{2}(e^x + e^{-x})]^2 = \frac{1}{4}(e^{2x} + 2 + e^{-2x})$ $\sinh^2 x = [\frac{1}{2}(e^x - e^{-x})]^2 = \frac{1}{4}(e^{2x} - 2 + e^{-2x})$ $\cosh^2 x - \sinh^2 x = \frac{1}{4}(2+2) = 1$	B1 B1 B1 ag **3**	For completion
	OR $\cosh x + \sinh x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = e^x$ B1 $\cosh x - \sinh x = \frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x}) = e^{-x}$ B1 $\cosh^2 x - \sinh^2 x = e^x \times e^{-x} = 1$ B1		Completion
(ii)	$4(1+\sinh^2 x) + 9\sinh x = 13$ $4\sinh^2 x + 9\sinh x - 9 = 0$ $\sinh x = \frac{3}{4},\ -3$ $x = \ln 2,\ \ln(-3+\sqrt{10})$	M1 M1 A1A1 A1A1 ft **6**	(M0 for $1-\sinh^2 x$) Obtaining a value for $\sinh x$ Exact logarithmic form *Dep on M1M1* Max A1 if any extra values given
	OR $2e^{4x} + 9e^{3x} - 22e^{2x} - 9e^x + 2 = 0$ $(2e^{2x} - 3e^x - 2)(e^{2x} + 6e^x - 1) = 0$ M1 M1 $e^x = 2,\ -3+\sqrt{10}$ A1A1 $x = \ln 2,\ \ln(-3+\sqrt{10})$ A1A1 ft		Quadratic and / or linear factors Obtaining a value for e^x Ignore extra values *Dependent on M1M1* Max A1 if any extra values given *Just* $x = \ln 2$ *earns M0M1A1A0A0A0*
(iii)	$\frac{dy}{dx} = 8\cosh x \sinh x + 9\cosh x$ $= \cosh x(8\sinh x + 9)$ $= 0$ only when $\sinh x = -\frac{9}{8}$ $\cosh^2 x = 1 + (-\frac{9}{8})^2 = \frac{145}{64}$ $y = 4 \times \frac{145}{64} + 9 \times (-\frac{9}{8}) = -\frac{17}{16}$	B1 B1 M1 A1 **4**	Any correct form *or* $y = (2\sinh x + \frac{9}{4})^2 + \ldots\ (-\frac{17}{16})$ Correctly showing there is only one solution Exact evaluation of *y* or $\cosh^2 x$ or $\cosh 2x$ Give B2 (replacing M1A1) for -1.06 or better
(iv)	$\int_0^{\ln 2} (2 + 2\cosh 2x + 9\sinh x)\,dx$ $= \left[\ 2x + \sinh 2x + 9\cosh x\ \right]_0^{\ln 2}$ $= \left\{ 2\ln 2 + \frac{1}{2}\left(4 - \frac{1}{4}\right) + \frac{9}{2}\left(2 + \frac{1}{2}\right)\right\} - 9$ $= 2\ln 2 + \frac{33}{8}$	M1 A2 M1 A1 ag **5**	Expressing in integrable form Give A1 for two terms correct $\sinh(2\ln 2) = \frac{1}{2}(4 - \frac{1}{4})$ *Must see both terms for M1* *Must also see* $\cosh(\ln 2) = \frac{1}{2}(2 + \frac{1}{2})$ *for A1*

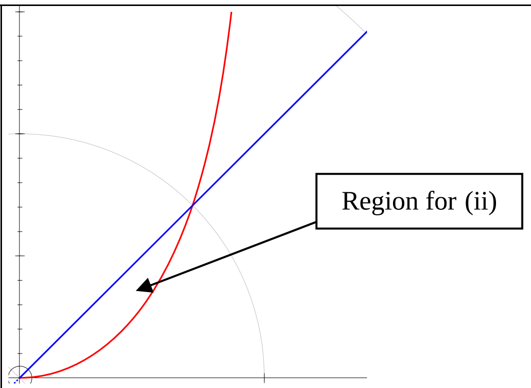
	OR $\int_0^{\ln 2} (e^{2x} + 2 + e^{-2x} + \frac{9}{2}(e^x - e^{-x}))\,dx$ M1 $= \left[\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + \frac{9}{2}e^x + \frac{9}{2}e^{-x} \right]_0^{\ln 2}$ A2 $= \left(2 + 2\ln 2 - \frac{1}{8} + 9 + \frac{9}{4}\right) - \left(\frac{1}{2} - \frac{1}{2} + \frac{9}{2} + \frac{9}{2}\right)$ M1 $= 2\ln 2 + \frac{33}{8}$ A1 ag		Expanded exponential form (M0 if the 2 is omitted) Give A1 for three terms correct $e^{2\ln 2} = 4$ and $e^{-2\ln 2} = \frac{1}{4}$ *both seen* *Must also see* $e^{\ln 2} = 2$ and $e^{-\ln 2} = \frac{1}{2}$ *for A1*
5 (i)	$\lambda = 0.5$ $\lambda = 3$ $\lambda = 5$ 	B1B1B1 **3**	
(ii)	Ellipse	B1 **1**	
(iii)	$y = \sqrt{2}\cos(\theta - \frac{1}{4}\pi)$ Maximum $y = \sqrt{2}$ when $\theta = \frac{1}{4}\pi$	M1 A1 ag **2**	or $\sqrt{2}\sin(\theta + \frac{1}{4}\pi)$
	OR $\frac{dy}{d\theta} = -\sin\theta + \cos\theta = 0$ when $\theta = \frac{1}{4}\pi$ M1 $y = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$ A1		
(iv)	$x^2 + y^2 = \lambda^2\cos^2\theta - 2\cos\theta\sin\theta + \frac{1}{\lambda^2}\sin^2\theta$ $+ \cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta$ $= (\lambda^2 + 1)(1 - \sin^2\theta) + (\frac{1}{\lambda^2} + 1)\sin^2\theta$ $= 1 + \lambda^2 + (\frac{1}{\lambda^2} - \lambda^2)\sin^2\theta$ When $\sin^2\theta = 0$, $x^2 + y^2 = 1 + \lambda^2$ When $\sin^2\theta = 1$, $x^2 + y^2 = 1 + \frac{1}{\lambda^2}$ Since $0 \le \sin^2\theta \le 1$, distance from O, $\sqrt{x^2 + y^2}$, is between $\sqrt{1 + \frac{1}{\lambda^2}}$ and $\sqrt{1 + \lambda^2}$	M1 M1 A1 ag M1 M1 A1 ag **6**	Using $\cos^2\theta = 1 - \sin^2\theta$
(v)	When $\lambda = 1$, $x^2 + y^2 = 2$ Curve is a circle (centre O) with radius $\sqrt{2}$	M1 A1 **2**	

(vi)		B4 **4**	A, E at maximum distance from O C, G at minimum distance from O B, F are stationary points D, H are on the x-axis Give ½ mark for each point, then round down Special properties must be clear from diagram, or stated *Max 3 if curve is not the correct shape*

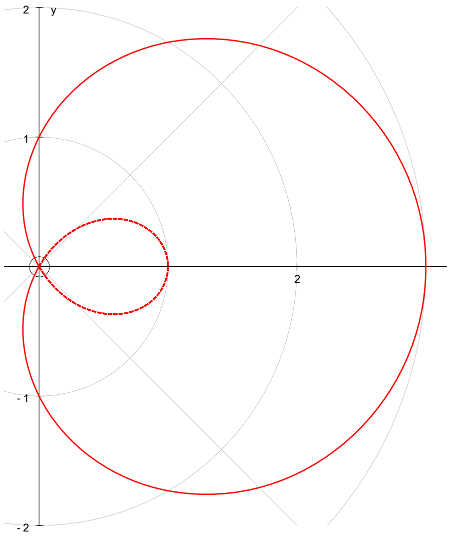
4756 (FP2) Further Methods for Advanced Mathematics

1 (a)(i)	$f(x) = \cos x$ $\quad f(0) = 1$ $f'(x) = -\sin x$ $\quad f'(0) = 0$ $f''(x) = -\cos x$ $\quad f''(0) = -1$ $f'''(x) = \sin x$ $\quad f'''(0) = 0$ $f''''(x) = \cos x$ $\quad f''''(0) = 1$ $\Rightarrow \cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \ldots$	M1 A1 A1 A1 (ag) **4**	Derivatives cos, sin, cos, sin, cos Correct signs Correct values. Dep on previous A1 www
(ii)	$\cos x \times \sec x = 1$ $\Rightarrow \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right)\left(1 + ax^2 + bx^4\right) = 1$ $\Rightarrow 1 + \left(a - \frac{1}{2}\right)x^2 + \left(b - \frac{1}{2}a + \frac{1}{24}\right)x^4 = 1$ $\Rightarrow a - \frac{1}{2} = 0,\ b - \frac{1}{2}a + \frac{1}{24} = 0$ $\Rightarrow a = \frac{1}{2}$ $b = \frac{5}{24}$	E1 M1 A1 B1 B1 **5**	o.e. Multiply to obtain terms in x^2 and x^4 Terms correct in any form (may not be collected) Correctly obtained by any method: must not just be stated Correctly obtained by any method
(b)(i)	$y = \arctan \frac{x}{a}$ $\Rightarrow x = a \tan y$ $\Rightarrow \frac{dx}{dy} = a \sec^2 y$ $\Rightarrow \frac{dx}{dy} = a(1 + \tan^2 y)$ $\Rightarrow \frac{dy}{dx} = \frac{a}{a^2 + x^2}$	M1 A1 A1 A1 (ag) **4**	(a) tan y = and attempt to differentiate both sides Or $\sec^2 y \frac{dy}{dx} = \frac{1}{a}$ Use $\sec^2 y = 1 + \tan^2 y$ o.e. www SC1: Use derivative of arctan x and Chain Rule (properly shown)
(ii)(A)	$\int_{-2}^{2} \frac{1}{4 + x^2}\,dx = \left[\frac{1}{2}\arctan\frac{x}{2}\right]_{-2}^{2}$ $= \frac{\pi}{4}$	M1 A1 A1 **3**	arctan alone, or any tan substitution $\frac{1}{2}$ and $\frac{x}{2}$, or $\int \frac{1}{2}\,d\theta$ without limits Evaluated in terms of π
(ii)(B)	$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{4}{1 + 4x^2}\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\frac{1}{4} + x^2}\,dx$ $= \left[2\arctan(2x)\right]_{-\frac{1}{2}}^{\frac{1}{2}}$ $= \pi$	M1 A1 A1 **3**	arctan alone, or any tan substitution 2 and $2x$, or $\int 2\,d\theta$ without limits Evaluated in terms of π **19**

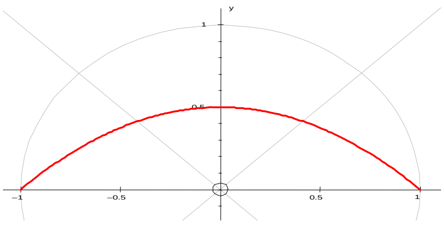
2 (i)	Modulus = 1 Argument = $\frac{\pi}{3}$	B1 B1 **2**	Must be separate Accept 60°, 1.05^c
(ii)	 $a = 2e^{\frac{j\pi}{4}}$ $\arg b = \frac{\pi}{4} \pm \frac{\pi}{3}$ $b = 2e^{-\frac{j\pi}{12}}, 2e^{\frac{7j\pi}{12}}$	G2,1,0 B1 M1 A1ft **5**	G2: A in first quadrant, argument $\approx \frac{\pi}{4}$ B in second quadrant, same mod B′ in fourth quadrant, same mod Symmetry G1: 3 points and at least 2 of above, or B, B′ on axes, or BOB′ straight line, or BOB′ reflex Must be in required form (accept $r = 2$, $\theta = \pi/4$) Rotate by adding (or subtracting) $\pi/3$ to (or from) argument. Must be $\pi/3$ Both. Ft value of r for a. Must be in required form, but don't penalise twice
(iii)	$z_1^6 = \left(\sqrt{2}e^{\frac{j\pi}{3}}\right)^6 = \left(\sqrt{2}\right)^6 e^{2j\pi}$ = 8 Others are $re^{j\theta}$ where $r = \sqrt{2}$ and $\theta = -\frac{2\pi}{3}, -\frac{\pi}{3}, 0, \frac{2\pi}{3}, \pi$ 	M1 A1 (ag) M1 A1 G1 G1 **6**	$\left(\sqrt{2}\right)^6 = 8$ or $\frac{\pi}{3} \times 6 = 2\pi$ seen www "Add" $\frac{\pi}{3}$ to argument more than once Correct constant r and five values of θ. Accept θ in $[0, 2\pi]$ or in degrees 6 points on vertices of regular hexagon Correctly positioned (2 roots on real axis). Ignore scales SC1 if G0 and 5 points correctly plotted
(iv)	$w = z_1 e^{-\frac{j\pi}{12}} = \sqrt{2}e^{\frac{j\pi}{3}} e^{-\frac{j\pi}{12}} = \sqrt{2}e^{\frac{j\pi}{4}}$ $= \sqrt{2}\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)$ $= 1 + j$	M1 A1 G1 **3**	$\arg w = \frac{\pi}{3} - \frac{\pi}{12}$ Or B2 Same modulus as z_1
(v)	$w^6 = \left(\sqrt{2}e^{\frac{j\pi}{4}}\right)^6 = 8e^{\frac{3j\pi}{2}}$ $= -8j$	M1 A1 **2**	Or $z_1^6 e^{-\frac{j\pi}{2}} = 8e^{-\frac{j\pi}{2}}$ cao. Evaluated **18**

3(a)(i)		G1 G1 G1 **3**	r increasing with θ Correct for $0 \le \theta \le \pi/3$ (ignore extra) Gradient less than 1 at O
(ii)	Area = $\int_0^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta = \frac{1}{2}a^2 \int_0^{\frac{\pi}{4}} \tan^2\theta \, d\theta$ $= \frac{1}{2}a^2 \int_0^{\frac{\pi}{4}} \sec^2\theta - 1 \, d\theta$ $= \frac{1}{2}a^2 [\tan\theta - \theta]_0^{\frac{\pi}{4}}$ $= \frac{1}{2}a^2\left(1 - \frac{\pi}{4}\right)$	M1 M1 A1 A1 G1 **5**	Integral expression involving $\tan^2\theta$ Attempt to express $\tan^2\theta$ in terms of $\sec^2\theta$ $\tan\theta - \theta$ and limits 0, $\frac{\pi}{4}$ A0 if e.g. triangle − this answer Mark region on graph
(b)(i)	Characteristic equation is $(0.2 - \lambda)(0.7 - \lambda) - 0.24 = 0$ $\Rightarrow \lambda^2 - 0.9\lambda - 0.1 = 0$ $\Rightarrow \lambda = 1, -0.1$ When $\lambda = 1$, $\begin{pmatrix} -0.8 & 0.8 \\ 0.3 & -0.3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Rightarrow -0.8x + 0.8y = 0,\ 0.3x - 0.3y = 0$ $\Rightarrow x - y = 0$, eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ o.e. When $\lambda = -0.1$, $\begin{pmatrix} 0.3 & 0.8 \\ 0.3 & 0.8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Rightarrow 0.3x + 0.8y = 0$ $\Rightarrow$ eigenvector is $\begin{pmatrix} 8 \\ -3 \end{pmatrix}$ o.e.	M1 A1 M1 A1 M1 A1 **6**	$(\mathbf{M} - \lambda\mathbf{I})\mathbf{x} = \mathbf{x}$ M0 below At least one equation relating x and y At least one equation relating x and y
(ii)	**Q** = $\begin{pmatrix} 1 & 8 \\ 1 & -3 \end{pmatrix}$ **D** = $\begin{pmatrix} 1 & 0 \\ 0 & -0.1 \end{pmatrix}$	B1ft B1ft B1 **3**	B0 if **Q** is singular. Must label correctly If order consistent. Dep on B1B1 earned **17**

4 (a)(i)	$\cosh^2 x = \left[\tfrac{1}{2}\left(e^x + e^{-x}\right)\right]^2 = \tfrac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right)$ $\sinh^2 x = \left[\tfrac{1}{2}\left(e^x - e^{-x}\right)\right]^2 = \tfrac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right)$ $\cosh^2 x - \sinh^2 x = \tfrac{1}{4}(2+2) = 1$	M1 A1 (ag) **2**	Both expressions (M0 if no "middle" term) and subtraction www
	OR $\cosh x + \sinh x = e^x$ $\cosh x - \sinh x = e^{-x}$ $\cosh^2 x - \sinh^2 x = e^x \times e^{-x} = 1$ A1		Both, and multiplication Completion
(ii)(A)	$\cosh x = \sqrt{1+\sinh^2 x} = \sqrt{1+\tan^2 y}$ $= \sec y$ $\Rightarrow \tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{\tan y}{\sec y} = \sin y$	M1 A1 A1 (ag) **3**	Use of $\cosh^2 x = 1 + \sinh^2 x$ and $\sinh x = \tan y$ www
(ii)(B)	$\operatorname{arsinh} x = \ln(x + \sqrt{1+x^2})$ $\Rightarrow \operatorname{arsinh}(\tan y) = \ln(\tan y + \sqrt{1+\tan^2 y})$ $\Rightarrow x = \ln(\tan y + \sec y)$	M1 A1 A1 (ag) **3**	Attempt to use ln form of arsinh www
	OR $\sinh x = \tan y \Rightarrow \dfrac{e^x - e^{-x}}{2} = \tan y$ $\Rightarrow e^{2x} - 2e^x \tan y - 1 = 0$ M1 $\Rightarrow e^x = \tan y \pm \sqrt{\tan^2 y + 1}$ A1 $\Rightarrow x = \ln(\tan y + \sec y)$ A1		Arrange as quadratic and solve for e^x o.e. www
(b)(i)	$y = \operatorname{artanh} x \Rightarrow x = \tanh y$ $\Rightarrow \dfrac{dx}{dy} = \operatorname{sech}^2 y$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\operatorname{sech}^2 y} = \dfrac{1}{1-\tanh^2 y} = \dfrac{1}{1-x^2}$ Integral $= \left[\operatorname{artanh} x\right]_{-\frac{1}{2}}^{\frac{1}{2}}$ $= 2\operatorname{artanh}\dfrac{1}{2}$	M1 A1 M1 A1 (ag) **4**	tanh y = and attempt to differentiate Or $\operatorname{sech}^2 y \dfrac{dy}{dx} = 1$ Or B2 for $\dfrac{1}{1-x^2}$ www artanh or any tanh substitution www
(ii)	$\dfrac{1}{1-x^2} = \dfrac{1}{(1-x)(1+x)} = \dfrac{A}{1-x} + \dfrac{B}{1+x}$ $\Rightarrow 1 = A(1+x) + B(1-x)$ $\Rightarrow A = ½, B = ½$ $\Rightarrow \int \dfrac{1}{1-x^2}\,dx = \int \dfrac{\frac{1}{2}}{1-x} + \dfrac{\frac{1}{2}}{1+x}\,dx$ $= -\tfrac{1}{2}\ln\lvert 1-x\rvert + \tfrac{1}{2}\ln\lvert 1+x\rvert + c$ or $\tfrac{1}{2}\ln\left\lvert\tfrac{1+x}{1-x}\right\rvert + c$ o.e.	 M1 A1 M1 A1 **4**	Correct form of partial fractions and attempt to evaluate constants Log integrals www. Condone omitted modulus signs and constant After 0 scored, SC1 for correct answer
(iii)	$\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \dfrac{1}{1-x^2}\,dx = \left[-\tfrac{1}{2}\ln\lvert 1-x\rvert + \tfrac{1}{2}\ln\lvert 1+x\rvert\right]_{-\frac{1}{2}}^{\frac{1}{2}} = \ln 3$ $\Rightarrow 2\operatorname{artanh}\dfrac{1}{2} = \ln 3 \Rightarrow \operatorname{artanh}\dfrac{1}{2} = \dfrac{1}{2}\ln 3$	M1 A1 (ag) **2**	Substitution of ½ **and** –½ seen anywhere (or correct use of 0, ½) www **18**

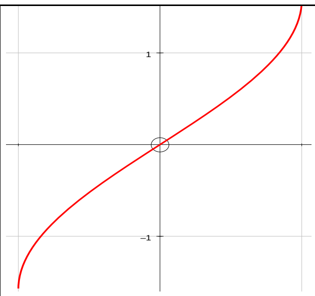
5 (i)		G1 G1 G1 **3**	Symmetry in horizontal axis (3, 0) to (0, 0) (0, 0) to (0, 1)
(ii)(A)	$a > 0.5$ $a < -0.5$	B1 B1	
(ii)(B)	Circle: r is constant	B1	Shape and reason
(ii)(C)	The two loops get closer together The shape becomes more nearly circular	B1 B1	
(ii)(D)	Cusp $a = -0.5$	B1 B1 **7**	
(iii)	$1 + 2a\cos\theta = 0 \Rightarrow \cos\theta = -\frac{1}{2a}$	B1	Equation
	If $a > 0.5$, $-1 < -\frac{1}{2a} < 0$ and there are two values of θ in $[0, 2\pi]$, $\pi - \arccos\left(\frac{1}{2a}\right)$ and $\pi + \arccos\left(\frac{1}{2a}\right)$	M1	
	These differ by $2\arccos\left(\frac{1}{2a}\right)$	A1 (ag)	
	$\arccos\left(\frac{1}{2a}\right) = \arctan\sqrt{4a^2-1}$	M1 A1	Relating arccos to arctan by triangle or $\tan^2\theta = \sec^2\theta - 1$
	Tangents are $y = x\sqrt{4a^2-1}$	A1	
	and $y = -x\sqrt{4a^2-1}$	A1ft	Negative of above
	$\sqrt{4a^2-1}$ is real for $a > 0.5$ if $a > 0$	E1 **8**	**18**

4756 (FP2) Further Methods for Advanced Mathematics

1 (a)(i)	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \ldots$ $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \ldots$ $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$ $= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} \ldots$ Valid for $-1 < x < 1$	B1 M1 A1 B1 **4**	Series for $\ln(1-x)$ as far as x^5 s.o.i. Seeing series subtracted Inequalities must be strict
(ii)	$\frac{1+x}{1-x} = 3$ $\Rightarrow$ $1 + x = 3(1 - x)$ $\Rightarrow$ $1 + x = 3 - 3x$ $\Rightarrow$ $4x = 2$ $\Rightarrow$ $x = \frac{1}{2}$ $\ln 3 \approx 2\times\frac{1}{2} + \frac{2}{3}\times\left(\frac{1}{2}\right)^3 + \frac{2}{5}\times\left(\frac{1}{2}\right)^5$ $= 1 + \frac{1}{12} + \frac{1}{80}$ $= 1.096$ (3 d.p.)	M1 A1 M1 A1 **4**	Correct method of solution B2 for $x = ½$ stated Substituting their x into their series in (a) (i), even if outside range of validity. Series must have at least two terms *SR*: if >3 correct terms seen in (i), allow a better answer to 3 d.p. Must be 3 decimal places
(b)(i)		G1 G1 G1 **3**	$r(0) = a$, $r(\pi/2) = a/2$ indicated Symmetry in $\theta = \pi/2$ Correct basic shape: flat at $\theta = \pi/2$, not vertical or horizontal at ends, no dimple Ignore beyond $0 \le \theta \le \pi$
(ii)	$r + y = r + r \sin\theta$ $= r(1 + \sin\theta) = \frac{a}{1+\sin\theta} \times (1 + \sin\theta)$ $= a$ $\Rightarrow$ $r = a - y$ $\Rightarrow$ $x^2 + y^2 = (a - y)^2$ $\Rightarrow$ $x^2 + y^2 = a^2 - 2ay + y^2$ $\Rightarrow$ $2ay = a^2 - x^2$ $\Rightarrow$ $y = \frac{a^2 - x^2}{2a}$	M1 A1 (ag) M1 A1 A1 **5**	Using $y = r\sin\theta$ Using $r^2 = x^2 + y^2$ in $r + y = a$ Unsimplified A correct final answer, not spoiled **16**

2 (i)	$\mathbf{M}-\lambda\mathbf{I}=\begin{pmatrix}3-\lambda & 1 & -2\\ 0 & -1-\lambda & 0\\ 2 & 0 & 1-\lambda\end{pmatrix}$		
	det(**M** – *λ***I**) = (3 – *λ*)[(–1 – *λ*)(1 – *λ*)] + 2[2(–1 – *λ*)]	M1	Attempt at det(**M** – *λ***I**) with all elements present. Allow sign errors
	= (3 – *λ*)(λ^2 – 1) + 4(–1 – *λ*)	A1	Unsimplified. Allow signs reversed. Condone omission of = 0
	⇒ $\lambda^3 - 3\lambda^2 + 3\lambda + 7 = 0$ det **M** = –7	B1 **3**	
(ii)	$f(\lambda) = \lambda^3 - 3\lambda^2 + 3\lambda + 7$		
	f (–1) = –1 – 3 – 3 + 7 = 0 ⇒ –1 eigenvalue	B1	Showing –1 satisfies a correct characteristic equation
	$f(\lambda) = (\lambda + 1)(\lambda^2 - 4\lambda + 7)$ $\lambda^2 - 4\lambda + 7 = (\lambda - 2)^2 + 3 \geq 3$ so no real roots (**M** – *λ***I**)**s** = **0**, *λ* = –1	M1 A1	Obtaining quadratic factor www (**M** – *λ***I**)**s** = (*λ*)**s** M0 below
	⇒ $\begin{pmatrix}4 & 1 & -2\\ 0 & 0 & 0\\ 2 & 0 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$		
	⇒ 4*x* + *y* – 2*z* = 0 2*x* + 2*z* = 0	M1	Obtaining equations relating *x*, *y* and *z*
	⇒ *x* = –*z* *y* = 2*z* – 4*x* = 2*z* + 4*z* = 6*z*	M1	Obtaining equations relating two variables to a third. Dep. on first M1
	⇒ **s** = $\begin{pmatrix}-1\\ 6\\ 1\end{pmatrix}$	A1	Or any non-zero multiple
	$\begin{pmatrix}3 & 1 & -2\\ 0 & -1 & 0\\ 2 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}-0.1\\ 0.6\\ 0.1\end{pmatrix}$	M1	Solution by any method, e.g. use of multiple of **s**, but M0 if **s** itself quoted without further work
	⇒ *x* = 0.1, *y* = –0.6, *z* = –0.1	A2 **9**	Give A1 if any two correct
(iii)	C-H: a matrix satisfies its own characteristic equation ⇒ $\mathbf{M}^3 - 3\mathbf{M}^2 + 3\mathbf{M} + 7\mathbf{I} = \mathbf{0}$	B1	Idea of λ ↔ **M**
	⇒ $\mathbf{M}^3 = 3\mathbf{M}^2 - 3\mathbf{M} - 7\mathbf{I}$	B1 (ag)	Must be derived www. Condone omitted **I**
	⇒ $\mathbf{M}^2 = 3\mathbf{M} - 3\mathbf{I} - 7\mathbf{M}^{-1}$ ⇒ $\mathbf{M}^{-1} = -\frac{1}{7}\mathbf{M}^2 + \frac{3}{7}\mathbf{M} - \frac{3}{7}\mathbf{I}$	M1 A1 **4**	Multiplying by $\mathbf{M}^{-1}$ o.e.
(iv)	$\mathbf{M}^2 = \begin{pmatrix}3 & 1 & -2\\ 0 & -1 & 0\\ 2 & 0 & 1\end{pmatrix}\begin{pmatrix}3 & 1 & -2\\ 0 & -1 & 0\\ 2 & 0 & 1\end{pmatrix}=\begin{pmatrix}5 & 2 & -8\\ 0 & 1 & 0\\ 8 & 2 & -3\end{pmatrix}$	M1	Correct attempt to find $\mathbf{M}^2$
	$-\frac{1}{7}\begin{pmatrix}5 & 2 & -8\\ 0 & 1 & 0\\ 8 & 2 & -3\end{pmatrix}+\frac{3}{7}\begin{pmatrix}3 & 1 & -2\\ 0 & -1 & 0\\ 2 & 0 & 1\end{pmatrix}-\frac{3}{7}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$	M1	Using their (iii)
	$=\begin{pmatrix}\frac{1}{7} & \frac{1}{7} & \frac{2}{7}\\ 0 & -1 & 0\\ -\frac{2}{7} & -\frac{2}{7} & \frac{3}{7}\end{pmatrix}$ or $\frac{1}{7}\begin{pmatrix}1 & 1 & 2\\ 0 & -7 & 0\\ -2 & -2 & 3\end{pmatrix}$	A1	SC1 for answer without working

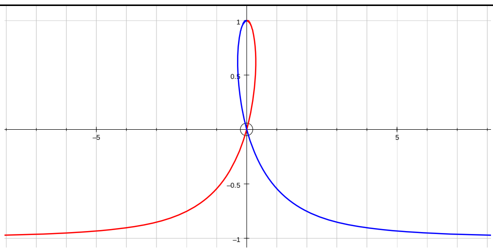
	OR Matrix of cofactors: $\begin{pmatrix} -1 & 0 & 2 \\ -1 & 7 & 2 \\ -2 & 0 & -3 \end{pmatrix}$	M1		Finding at least four cofactors
	Adjugate matrix $\begin{pmatrix} -1 & -1 & -2 \\ 0 & 7 & 0 \\ 2 & 2 & -3 \end{pmatrix}$: det **M** = –7	M1	3	Transposing and dividing by determinant. Dep. on M1 above 19

3(a)(i)		G1	1	Correct basic shape (positive gradient, through (0, 0))
	$y = \arcsin x \Rightarrow \sin y = x$	M1		sin y = and attempt to diff. both sides
	$\Rightarrow \frac{dx}{dy} = \cos y$	A1		Or $\cos y \frac{dy}{dx} = 1$
	$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$	A1		www. SC1 if quoted without working
	Positive square root because gradient positive	B1	4	Dep. on graph of an increasing function
(ii)	$\int_0^1 \frac{1}{\sqrt{2-x^2}} dx = \left[\arcsin \frac{x}{\sqrt{2}}\right]_0^1$	M1		arcsin function alone, or any sine substitution
		A1		$\frac{x}{\sqrt{2}}$, or $\int 1\, d\theta$ www without limits
	$= \frac{\pi}{4}$	A1	3	Evaluated in terms of π
(b)	$C + jS = e^{j\theta} + \frac{1}{3}e^{3j\theta} + \frac{1}{9}e^{5j\theta} + \ldots$	M1		Forming $C + jS$ as a series of powers
	This is a geometric series	M1		Identifying geometric series and attempting sum to infinity or to n terms
	with first term $a = e^{j\theta}$, common ratio $r = \frac{1}{3}e^{2j\theta}$	A1		Correct a and r
	Sum to infinity $= \frac{a}{1-r} = \frac{e^{j\theta}}{1-\frac{1}{3}e^{2j\theta}}$ $\left(= \frac{3e^{j\theta}}{3-e^{2j\theta}}\right)$	A1		Sum to infinity
	$= \frac{3e^{j\theta}}{3-e^{2j\theta}} \times \frac{3-e^{-2j\theta}}{3-e^{-2j\theta}}$	M1*		Multiplying numerator and denominator by $1-\frac{1}{3}e^{-2j\theta}$ o.e. Or writing in terms of trig functions and realising the denominator
	$= \frac{9e^{j\theta} - 3e^{-j\theta}}{9-3e^{-2j\theta} - 3e^{2j\theta} + 1}$	M1		Multiplying out numerator and denominator. Dep. on M1*
	$= \frac{9(\cos\theta + j\sin\theta) - 3(\cos\theta - j\sin\theta)}{10 - 3(\cos 2\theta - j\sin 2\theta) - 3(\cos 2\theta + j\sin 2\theta)}$	M1		Valid attempt to express in terms of trig functions. If trig functions used from start, M1 for using the compound angle formulae and Pythagoras Dep. on M1*
	$= \frac{6\cos\theta + 12j\sin\theta}{10 - 6\cos 2\theta}$	A1		
	$\Rightarrow C = \frac{6\cos\theta}{10-6\cos 2\theta}$	M1		Equating real and imaginary parts. Dep. on M1*

	$= \dfrac{3\cos\theta}{5-3\cos 2\theta}$	A1 (ag)	
	$S = \dfrac{6\sin\theta}{5-3\cos 2\theta}$	A1	o.e.
		11	**19**

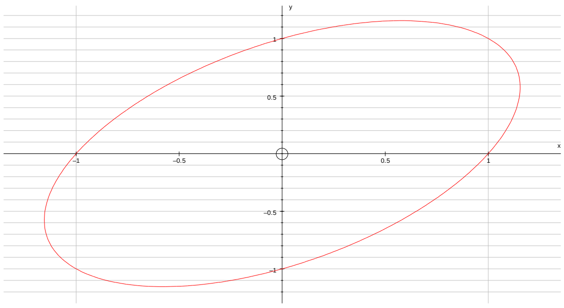
4 (i)	cosh u = $\frac{e^u + e^{-u}}{2}$ $\Rightarrow$ 2 cosh^2u = $\frac{e^{2u} + 2 + e^{-2u}}{2}$ $\Rightarrow$ 2 cosh^2u – 1 = $\frac{e^{2u} + e^{-2u}}{2}$ = cosh 2u	 B1 B1 B1 (ag) **3**	 $\left(e^u + e^{-u}\right)^2 = e^{2u} + 2 + e^{-2u}$ cosh 2u = $\frac{e^{2u} + e^{-2u}}{2}$ Completion www
(ii)	x = arsinh y $\Rightarrow$ sinh x = y $\Rightarrow$ y = $\frac{e^x - e^{-x}}{2}$ $\Rightarrow$ e^{2x} – 2ye^x – 1 = 0 $\Rightarrow$ $(e^x - y)^2 - y^2 - 1 = 0$ $\Rightarrow$ $(e^x - y)^2 = y^2 + 1$ $\Rightarrow$ $e^x - y = \pm\sqrt{y^2+1}$ $\Rightarrow$ $e^x = y \pm \sqrt{y^2+1}$ Take + because $e^x > 0$ $\Rightarrow$ $x = \ln(y + \sqrt{y^2+1})$	 M1 M1 B1 A1 (ag) **4**	 Expressing y in exponential form ($\frac{1}{2}$, – must be correct) Reaching e^x by quadratic formula or completing the square. Condone no ± Or argument of ln must be positive Completion www but independent of B1
(iii)	x = 2 sinh $u \Rightarrow \frac{dx}{du}$ = 2 cosh u $\int \sqrt{x^2+4}\,dx = \int \sqrt{4\sinh^2 u + 4} \times 2\cosh u\,du$ = $\int 4\cosh^2 u\,du$ = $\int 2\cosh 2u + 2\,du$ = sinh 2u + 2u + c = 2 sinh u cosh u + 2u + c = $x\sqrt{1+\frac{x^2}{4}}$ + 2 arsinh $\frac{x}{2}$ + c = $\frac{1}{2}x\sqrt{4+x^2}$ + 2 arsinh $\frac{x}{2}$ + c	M1 A1 M1 A1 M1 A1 (ag) **6**	$\frac{dx}{du}$ and substituting for all elements Substituting for all elements correctly Simplifying to an integrable form Any form, e.g. $\frac{1}{2}e^{2u} - \frac{1}{2}e^{-2u} + 2u$ Condone omission of + c throughout Using double "angle" formula and attempt to express cosh u in terms of x Completion www
(iv)	$t^2 + 2t + 5 = (t+1)^2 + 4$ $\int_{-1}^{1} \sqrt{t^2+2t+5}\,dt = \int_{-1}^{1} \sqrt{(t+1)^2+4}\,dt$ = $\int_0^2 \sqrt{x^2+4}\,dx$ = $\left[\frac{1}{2}x\sqrt{4+x^2} + 2\text{arsinh}\frac{x}{2}\right]_0^2$	B1 M1 A1	Completing the square Simplifying to an integrable form, by substituting $x = t + 1$ s.o.i. or complete alternative method Correct limits consistent with their method seen anywhere

	$= \sqrt{8} + 2$ arsinh 1 $= 2\sqrt{2} + 2\ln(1 + \sqrt{2})$ $= 2(\ln(1 + \sqrt{2}) + \sqrt{2})$	M1 A1 (ag) **5**	Using (iii) or otherwise reaching the result of integration, and using limits Completion www. Condone $\sqrt{8}$ etc. **18**

5 (i)	If $a = 1$, angle OCP = 45° so P is (1 – cos 45°, sin 45°) $\Rightarrow$ P $\left(1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	M1 A1 (ag)	Completion www
	OR Circle $(x-1)^2 + y^2 = 1$, line $y = -x + 1$ $(x-1)^2 + (-x+1)^2 = 1$ M1 $\Rightarrow$ $x = 1 \pm \frac{1}{\sqrt{2}}$ and hence P A1		Complete algebraic method to find x
	Q $\left(1 + \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	B1 **3**	
(ii)	cos OCP = $\frac{a}{\sqrt{a^2+1}}$ sin OCP = $\frac{1}{\sqrt{a^2+1}}$ P is (a – a cos OCP, a sin OCP) $\Rightarrow$ P $\left(a - \frac{a^2}{\sqrt{a^2+1}}, \frac{a}{\sqrt{a^2+1}}\right)$	M1 A1 A1 (ag)	Attempt to find cos OCP and sin OCP in terms of a Both correct Completion www
	OR Circle $(x-a)^2 + y^2 = a^2$, line $y = -\frac{1}{a}x + 1$ $(x-a)^2 + \left(-\frac{1}{a}x+1\right)^2 = a^2$ M1 $\Rightarrow$ $x = \dfrac{2a + \frac{2}{a} \pm \sqrt{\left(2a+\frac{2}{a}\right)^2 - 4\left(1+\frac{1}{a^2}\right)}}{2\left(1+\frac{1}{a^2}\right)}$ A1 $\Rightarrow$ $x = a \pm \frac{a^2}{\sqrt{a^2+1}}$ and hence P A1		Complete algebraic method to find x Unsimplified
	Q $\left(a + \frac{a^2}{\sqrt{a^2+1}}, -\frac{a}{\sqrt{a^2+1}}\right)$	B1 **4**	
(iii)	 As $a \to \infty$, P $\to$ (0, 1) As $a \to -\infty$, y co-ordinate of P $\to -1$ $\frac{a}{\sqrt{a^2+1}} \to \frac{a}{-a} = -1$ as $a \to -\infty$	G1 G1 G1 G1ft B1 B1 M1 A1 **8**	Locus of P (1st & 3rd quadrants) through (0, 0) Locus of P terminates at (0, 1) Locus of P: fully correct shape Locus of Q (2nd & 4th quadrants: dotted) reflection of locus of P in y-axis Stated separately Stated Attempt to consider y as $a \to -\infty$ Completion www
(iv)	POQ = 90° Angle in semicircle Loci cross at 90°	B1 B1 B1 **3**	o.e. **18**

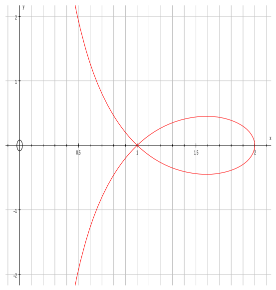
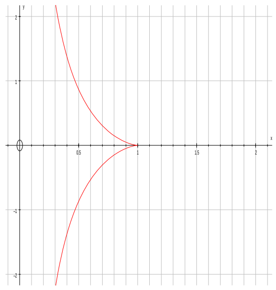
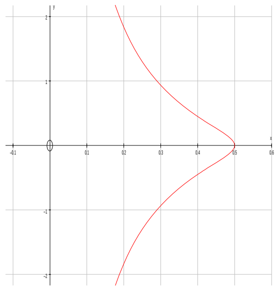
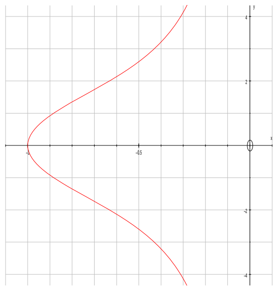
4756 (FP2) Further Methods for Advanced Mathematics

1 (a)	$y = \arctan\sqrt{x}$ $u = \sqrt{x}$, $y = \arctan u$ $\Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}}, \frac{dy}{du} = \frac{1}{1+u^2}$ $\Rightarrow \frac{dy}{dx} = \frac{1}{1+u^2} \times \frac{1}{2\sqrt{x}}$ $= \frac{1}{1+x} \times \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}(x+1)}$	M1 A1 A1	Using Chain Rule Correct derivative in any form Correct derivative in terms of x
	OR $\tan y = \sqrt{x}$ $\Rightarrow \sec^2 y \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ M1A1 $\sec^2 y = 1 + \tan^2 y = 1 + x$ $\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}(x+1)}$ A1		Rearranging for $\sqrt{x}$ or x and differentiating implicitly
	$\Rightarrow \int_0^1 \frac{1}{\sqrt{x}(x+1)} dx = \left[2\arctan\sqrt{x}\right]_0^1$ $= 2 \arctan 1 - 2 \arctan 0$ $= 2 \times \frac{\pi}{4} = \frac{\pi}{2}$	M1 A1 A1 (ag)	Integral in form $k \arctan\sqrt{x}$ $k = 2$
(b)(i)	$x = r\cos\theta, y = r\sin\theta, x^2 + y^2 = r^2$ $x^2 + y^2 = xy + 1$ $\Rightarrow r^2 = r^2 \cos\theta \sin\theta + 1$ $\Rightarrow r^2 = ½r^2 \sin 2\theta + 1$ $\Rightarrow 2r^2 = r^2 \sin 2\theta + 2$ $\Rightarrow r^2(2 - \sin 2\theta) = 2$ $\Rightarrow r^2 = \frac{2}{2-\sin 2\theta}$	M1 A1 A1 A1 (ag) **4**	Using at least one of these LHS RHS Clearly obtained *SR*: $x = r\sin\theta$, $y = r\cos\theta$ used M1A1A0A0 max.
(ii)	Max r is $\sqrt{2}$ Occurs when $\sin 2\theta = 1$ $\Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$ Min $r = \sqrt{\frac{2}{3}}$ Occurs when $\sin 2\theta = -1$ $\Rightarrow \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$	B1 M1 A1 B1 M1 A1 **6**	 Attempting to solve Both. Accept degrees. A0 if extras in range $\frac{\sqrt{6}}{3}$ Attempting to solve (must be –1) Both. Accept degrees. A0 if extras in range

(iii)		G1 G1 **2**	Closed curve, roughly elliptical, with no points or dents Major axis along $y = x$ **18**
2 (a)	$\cos 5\theta + \mathrm{j}\sin 5\theta = (\cos\theta + \mathrm{j}\sin\theta)^5$ $= \cos^5\theta + 5\cos^4\theta\,\mathrm{j}\sin\theta + 10\cos^3\theta\,\mathrm{j}^2\sin^2\theta$ $+ 10\cos^2\theta\,\mathrm{j}^3\sin^3\theta + 5\cos\theta\,\mathrm{j}^4\sin^4\theta + \mathrm{j}^5\sin^5\theta$ $= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta + \mathrm{j}(\ldots)$ $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ $= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$ $= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$	M1 M1 A1 M1 M1 A1 **6**	Using de Moivre Using binomial theorem appropriately Correct real part. Must evaluate powers of *j* Equating real parts Replacing $\sin^2\theta$ by $1 - \cos^2\theta$ $a = 16$, $b = -20$, $c = 5$
(b)	$C + \mathrm{j}S$ $= e^{\mathrm{j}\theta} + e^{\mathrm{j}\left(\theta + \frac{2\pi}{n}\right)} + \ldots + e^{\mathrm{j}\left(\theta + \frac{(2n-2)\pi}{n}\right)}$ This is a G.P. $a = e^{\mathrm{j}\theta}$, $r = e^{\mathrm{j}\frac{2\pi}{n}}$ Sum $= \dfrac{e^{\mathrm{j}\theta}\left(1 - \left(e^{\mathrm{j}\frac{2\pi}{n}}\right)^n\right)}{1 - e^{\mathrm{j}\frac{2\pi}{n}}}$ Numerator $= e^{\mathrm{j}\theta}\left(1 - e^{2\pi\mathrm{j}}\right)$ and $e^{2\pi\mathrm{j}} = 1$ so sum = 0 $\Rightarrow$ $C = 0$ and $S = 0$	M1 A1 M1 A1 A1 E1 E1 **7**	Forming series $C + \mathrm{j}S$ as exponentials Need not see whole series Attempting to sum finite or infinite G.P. Correct *a*, *r* used or stated, and *n* terms Must see *j* Convincing explanation that sum = 0 $C = S = 0$. Dep. on previous E1 Both E marks dep. on 5 marks above
(c)	$e^t \approx 1 + t + \frac{1}{2}t^2$ $\dfrac{t}{e^t - 1} \approx \dfrac{t}{t + \frac{1}{2}t^2}$ $\dfrac{t}{t + \frac{1}{2}t^2} = \dfrac{1}{1 + \frac{1}{2}t} = \left(1 + \frac{1}{2}t\right)^{-1} = 1 - \frac{1}{2}t + \ldots$	B1 M1 A1 M1	Ignore terms in higher powers Substituting Maclaurin series Suitable manipulation and use of binomial theorem
	OR $\dfrac{1}{1 + \frac{1}{2}t} = \dfrac{1}{1 + \frac{1}{2}t} \times \dfrac{1 - \frac{1}{2}t}{1 - \frac{1}{2}t} = \dfrac{1 - \frac{1}{2}t}{1 - \frac{1}{4}t^2}$ M1		
	Hence $\dfrac{t}{e^t - 1} \approx 1 - \frac{1}{2}t$	A1 (ag)	
	OR $\left(e^t - 1\right)\left(1 - \frac{1}{2}t\right) = \left(t + \frac{1}{2}t^2 + \ldots\right)\left(1 - \frac{1}{2}t\right)$ M1 A1 $\approx t$ + terms in t^3 M1 $\Rightarrow$ $\dfrac{t}{e^t - 1} \approx 1 - \frac{1}{2}t$ A1		Substituting Maclaurin series Correct expression Multiplying out Convincing explanation
		5	**18**

3 (i)	$\mathbf{M}^{-1} = \frac{1}{4-a}\begin{pmatrix} 2 & -2-2a & 2+a \\ 2 & 2-3a & 2a-2 \\ -1 & 5 & -3 \end{pmatrix}$ When $a = -1$, $\mathbf{M}^{-1} = \frac{1}{5}\begin{pmatrix} 2 & 0 & 1 \\ 2 & 5 & -4 \\ -1 & 5 & -3 \end{pmatrix}$	M1 A1 M1 A1 M1 A1 **6**	Evaluating determinant $4 - a$ Finding at least four cofactors Six signed cofactors correct Transposing and dividing by det $\mathbf{M}^{-1}$ correct (in terms of a) and result for $a = -1$ stated *SR*: After 0 scored, SC1 for $\mathbf{M}^{-1}$ when $a = -1$, obtained correctly with some working
(ii)	$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & 0 & 1 \\ 2 & 5 & -4 \\ -1 & 5 & -3 \end{pmatrix}\begin{pmatrix} -2 \\ b \\ 1 \end{pmatrix}$ $\Rightarrow \quad x = -\frac{3}{5},\ y = b - \frac{8}{5},\ z = b - \frac{1}{5}$	M2 M1 A2	Attempting to multiply $(-2\ \ b\ \ 1)^T$ by given matrix (M0 if wrong order) Multiplying out A1 for one correct
	OR $4x + y = b - 4$ $x - y = 1 - b$ o.e. M1 M1 $\Rightarrow \quad x = -\frac{3}{5}$ A1 M1 $\Rightarrow \quad y = b - \frac{8}{5},\ z = b - \frac{1}{5}$ A1		Eliminating one unknown in 2 ways Or e.g. $3x + z = b - 2$, $5x = -3$ Or e.g. $3y - 4z = -b - 4$, $5y - 5z = -7$ Solve to obtain one value. Dep. on M1 above One unknown correct After M0, SC1 for value of x Finding the other two unknowns Both correct
		5	
(iii)	e.g. $3x - 3y = 2b + 2$ $5x - 5y = 4$ Consistent if $\frac{2b+2}{3} = \frac{4}{5}$ $\Rightarrow \quad b = \frac{1}{5}$ Solution is a line	M1 A1A1 M1 A1 B2 **7**	Eliminating one unknown in 2 ways Two correct equations Or e.g. $3x + 6z = b - 2$, $5x + 10z = -3$ Or e.g. $3y + 6z = -b - 4$, $5y + 10z = -7$ Attempting to find b **18**

4 (i)	$\sinh x = \frac{e^x - e^{-x}}{2} \Rightarrow \sinh^2 x = \frac{(e^x - e^{-x})^2}{4}$		
	$= \frac{e^{2x} - 2 + e^{-2x}}{4}$	B1	$e^{2x} - 2 + e^{-2x}$
	$\Rightarrow 2\sinh^2 x + 1 = \frac{e^{2x} - 2 + e^{-2x}}{2} + 1$		
	$= \frac{e^{2x} + e^{-2x}}{2} = \cosh 2x$	B1	Correct completion
	$\Rightarrow 2\sinh 2x = 4\sinh x\cosh x$	B1	Both correct derivatives
	$\Rightarrow \sinh 2x = 2\sinh x\cosh x$	B1	Correct completion
		4	
(ii)	$2\cosh 2x + 3\sinh x = 3$		
	$\Rightarrow 2(1 + 2\sinh^2 x) + 3\sinh x = 3$	M1	Using identity
	$\Rightarrow 4\sinh^2 x + 3\sinh x - 1 = 0$	A1	Correct quadratic
	$\Rightarrow (4\sinh x - 1)(\sinh x + 1) = 0$	M1	Solving quadratic
	$\Rightarrow \sinh x = \frac{1}{4}, -1$	A1	Both
		M1	Use of arsinh $x = \ln(x + \sqrt{x^2+1})$ o.e. Must obtain at least one value of x
	$\Rightarrow x = \text{arsinh}(\frac{1}{4}) = \ln(\frac{1+\sqrt{17}}{4})$	A1	Must evaluate $\sqrt{x^2+1}$
	$x = \text{arsinh}(-1) = \ln(-1+\sqrt{2})$	A1	
	OR $2e^{4x} + 3e^{3x} - 6e^{2x} - 3e^x + 2 = 0$		
	$\Rightarrow (2e^{2x} - e^x - 2)(e^{2x} + 2e^x - 1) = 0$ M1A1		Factorising quartic
	$\Rightarrow e^x = \frac{1 \pm \sqrt{17}}{4}$ or $-1 \pm \sqrt{2}$ M1A1		Solving either quadratic
	$\Rightarrow x = \ln(\frac{1+\sqrt{17}}{4})$ or $\ln(-1+\sqrt{2})$ M1A1A1		Using ln (dependent on first M1)
		7	
(iii)	$\cosh t = \frac{5}{4} \Rightarrow \frac{e^t + e^{-t}}{2} = \frac{5}{4}$		
	$\Rightarrow 2e^{2t} - 5e^t + 2 = 0$	M1	Forming quadratic in e^t
	$\Rightarrow (2e^t - 1)(e^t - 2) = 0$	M1	Solving quadratic
	$\Rightarrow e^t = \frac{1}{2}, 2$	A1	
	$\Rightarrow t = \pm\ln 2$	A1 (ag)	Convincing working
	$\int_4^5 \frac{1}{\sqrt{x^2-16}}\,dx = \left[\text{arcosh}\frac{x}{4}\right]_4^5$	B1	
	$= \text{arcosh}\frac{5}{4} - \text{arcosh } 1$	M1	Substituting limits
	$= \ln 2$	A1	A0 for $\pm\ln 2$
	OR $\int_4^5 \frac{1}{\sqrt{x^2-16}}\,dx = \left[\ln\left(x + \sqrt{x^2-16}\right)\right]_4^5$ B1		
	$= \ln 8 - \ln 4$ M1		Substituting limits
	$= \ln 2$ A1		
		7	**18**

5 (i)	Horz. projection of QP = $k\cos\theta$ Vert. projection of QP = $k\sin\theta$ Subtract OQ = $\tan\theta$	B1 B1 B1 **3**	 Clearly obtained
(ii)	$k = 2$  $k = 1$  $k = ½$  $k = -1$ 	G1 G1 G1 G1 **4**	Loop Cusp
(iii)(A) **(B)** **(C)**	for all k, y axis is an asymptote $k = 1$ $k > 1$	B1 B1 B1 **3**	Both
(iv)	Crosses itself at (1, 0) $k = 2 \Rightarrow \cos\theta = ½ \Rightarrow \theta = 60°$ $\Rightarrow$ curve crosses itself at 120°	 M1 A1 **2**	 Obtaining a value of θ Accept 240°
(v)	$y = 8\sin\theta - \tan\theta$ $\Rightarrow \frac{dy}{d\theta} = 8\cos\theta - \sec^2\theta$ $\Rightarrow 8\cos\theta - \frac{1}{\cos^2\theta} = 0$ at highest point $\Rightarrow \cos^3\theta = \frac{1}{8} \Rightarrow \cos\theta = \pm\frac{1}{2} \Rightarrow \theta = 60°$ at top $\Rightarrow x = 4$ $y = 3\sqrt{3}$	 M1 A1 A1 **3**	 Complete method giving θ Both
(vi)	RHS = $\frac{(k\cos\theta - 1)^2}{k^2\cos^2\theta}(k^2 - k^2\cos^2\theta)$ $= \frac{(k\cos\theta - 1)^2}{k^2\cos^2\theta} \times k^2\sin^2\theta$ $= (k\cos\theta - 1)^2\tan^2\theta$ $= ((k\cos\theta - 1)\tan\theta)^2$ $= (k\sin\theta - \tan\theta)^2$ = LHS	M1 M1 E1 **3**	Expressing one side in terms of θ Using trig identities **18**

GCE

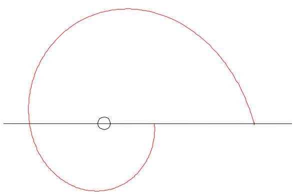
Mathematics (MEI)

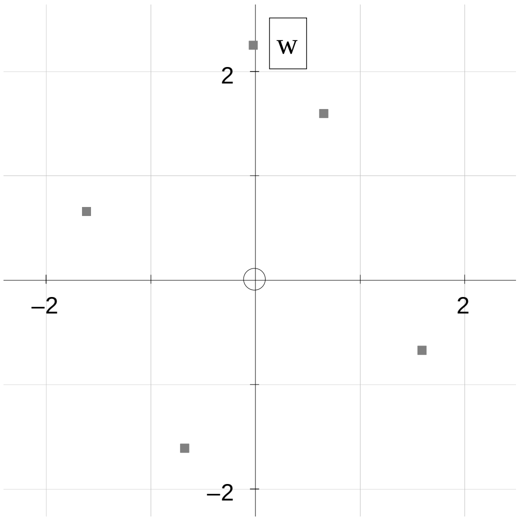
Advanced GCE **4756**

Further Methods for Advanced Mathematics (FP2)

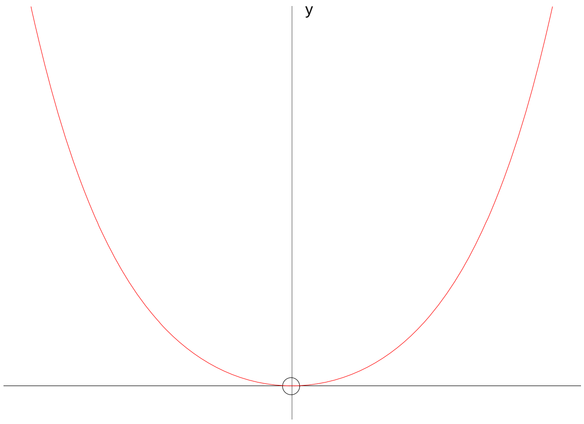
Mark Scheme for June 2010

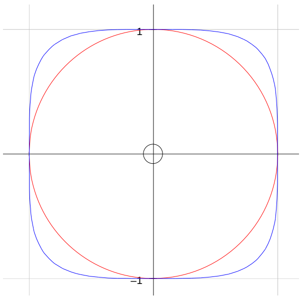
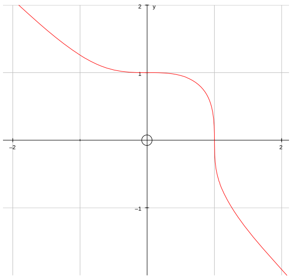
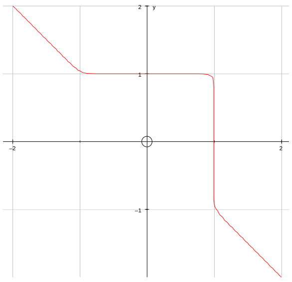
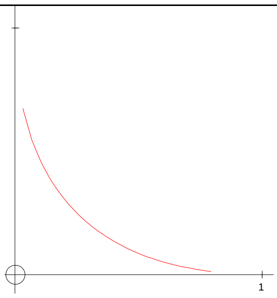
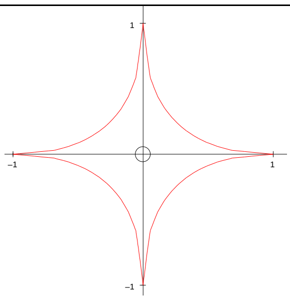
Oxford Cambridge and RSA Examinations

1 (a)(i)	$f(t) = \arcsin t$ $\Rightarrow \quad f'(t) = \dfrac{1}{\sqrt{1-t^2}} = \left(1-t^2\right)^{-\frac{1}{2}}$ $\Rightarrow \quad f''(t) = -\tfrac{1}{2}\left(1-t^2\right)^{-\frac{3}{2}} \times -2t$ $= \dfrac{t}{\left(1-t^2\right)^{\frac{3}{2}}}$	B1 M1 A1 (ag) **3**	Any form Using Chain Rule
(ii)	$f(x) = \arcsin(x + ½)$ $\Rightarrow \quad f(0) = \arcsin(½) = \dfrac{\pi}{6}$ $f'(0) = \left(1-\left(\tfrac{1}{2}\right)^2\right)^{-\frac{1}{2}} = \dfrac{2}{\sqrt{3}}$ and $f''(0) = \dfrac{\frac{1}{2}}{\left(1-\left(\frac{1}{2}\right)^2\right)^{\frac{3}{2}}} = \dfrac{4\sqrt{3}}{9}$ $f(x) = f(0) + x f'(0) + \dfrac{x^2}{2} f''(0) + \ldots$ $\Rightarrow$ term in x^2 is $\dfrac{2\sqrt{3}}{9}x^2$	B1 (ag) M1 A1 (ag) M1 A1 **5**	$\dfrac{\pi}{6}$ obtained clearly from $f(0)$ www Clear substitution of $x = 0$ or $t = ½$ Evaluating $f''(0)$ and dividing by 2 Accept $0.385x^2$ or better
(b)	 Area $= \displaystyle\int_0^{\pi} \tfrac{1}{2} r^2 \, d\theta$ $= \displaystyle\int_0^{\pi} \frac{\pi^2 a^2}{2(\pi+\theta)^2} d\theta = \frac{\pi^2 a^2}{2}\int_0^{\pi} \frac{1}{(\pi+\theta)^2} d\theta$ $= \dfrac{\pi^2 a^2}{2}\left[\dfrac{-1}{\pi+\theta}\right]_0^{\pi}$ $= \dfrac{\pi^2 a^2}{2}\left(\dfrac{-1}{2\pi} + \dfrac{1}{\pi}\right)$ $= \tfrac{1}{4}\pi a^2$	G1 G1 M1 A1 M1 A1 **6**	Complete spiral with $r(2\pi) < r(0)$ $r(0) = a$, $r(2\pi) = a/3$ indicated or $r(0) > r(\pi/2) > r(\pi) > r(3\pi/2) > r(2\pi)$ Dep. on G1 above Max. G1 if not fully correct Integral expression involving r^2 Correct result of integration with correct limits Substituting limits into an expression of the form $\dfrac{k}{\pi+\theta}$. Dep. on M1 above
(c)	$\displaystyle\int_0^{\frac{3}{2}} \frac{1}{9+4x^2} dx = \frac{1}{4}\int_0^{\frac{3}{2}} \frac{1}{\frac{9}{4}+x^2} dx = \frac{1}{4} \times \left[\frac{2}{3}\arctan\frac{2x}{3}\right]_0^{\frac{3}{2}}$ $= \dfrac{1}{6}\arctan 1$ $= \dfrac{\pi}{24}$	M1 A1A1 M1 A1 **5**	arctan $\dfrac{1}{4} \times \dfrac{2}{3}$ and $\dfrac{2x}{3}$ Substituting limits. Dep. on M1 above Evaluated in terms of π **19**

2 (a)	$z^n + \frac{1}{z^n} = 2\cos n\theta$, $z^n - \frac{1}{z^n} = 2j\sin n\theta$	B1	Both
	$\left(z - \frac{1}{z}\right)^5 = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$	M1	Expanding $\left(z - \frac{1}{z}\right)^5$
	$= z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$		
	$\Rightarrow$ $32j\sin^5\theta = 2j\sin 5\theta - 10j\sin 3\theta + 20j\sin\theta$	M1 A1	Introducing sines (and possibly cosines) of multiple angles RHS
	$\Rightarrow$ $\sin^5\theta = \frac{1}{16}\sin 5\theta - \frac{5}{16}\sin 3\theta + \frac{5}{8}\sin\theta$	A1ft	Division by 32(j)
	$A = \frac{5}{8}$, $B = -\frac{5}{16}$, $C = \frac{1}{16}$	**5**	
(b)(i)	4th roots of $-9j = 9e^{\frac{3}{2}\pi j}$ are $re^{j\theta}$ where		
	$r = \sqrt{3}$	B1	Accept $9^{\frac{1}{4}}$
	$\theta = \frac{3\pi}{8}$	B1	
	$\Rightarrow$ $\theta = \frac{3\pi}{8} + \frac{2k\pi}{4}$	M1	Implied by at least two correct (ft) further values
	$\Rightarrow$ $\theta = \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$	A1	Or stating k = (0), 1, 2, 3 Allow arguments in range $-\pi \le \theta \le \pi$
		M1 A1 **6**	Points at vertices of a square centre O or 3 correct points (ft) or 1 point in each quadrant
(ii)	Mid-point of SP has argument $\frac{\pi}{8}$	B1	
	and modulus of $\sqrt{\frac{3}{2}}$	B1	
	Argument of $w = 4 \times \frac{\pi}{8} = \frac{\pi}{2}$		
	and modulus = $\left(\sqrt{\frac{3}{2}}\right)^4 = \frac{9}{4}$	M1 A1 G1 **5**	Multiplying argument by 4 and modulus raised to power of 4 Both correct w plotted on imag. axis above level of P **16**

3 (a)(i)	$2\lambda^3+\lambda^2-13\lambda+6=0 \Rightarrow (\lambda-2)(2\lambda^2+5\lambda-3)=0$ $\Rightarrow \quad \lambda=2$ or $2\lambda^2+5\lambda-3=0$ $\Rightarrow \quad (2\lambda-1)(\lambda+3)=0$ $\Rightarrow \quad \lambda=\frac{1}{2}, \lambda=-3$	B1 M1 A1A1 **4**	Substituting $\lambda = 2$ or factorising Obtaining and solving a quadratic
(ii)	$\mathbf{M}\begin{pmatrix}3\\-3\\1\end{pmatrix}=2\begin{pmatrix}3\\-3\\1\end{pmatrix}=\begin{pmatrix}6\\-6\\2\end{pmatrix}$ $\mathbf{M}^2\mathbf{v}=2^2\mathbf{v}=4\begin{pmatrix}1\\-1\\\frac{1}{3}\end{pmatrix}=\begin{pmatrix}4\\-4\\\frac{4}{3}\end{pmatrix}$ $\mathbf{M}\begin{pmatrix}\frac{3}{2}\\-\frac{3}{2}\\\frac{1}{2}\end{pmatrix}=2\begin{pmatrix}\frac{3}{2}\\-\frac{3}{2}\\\frac{1}{2}\end{pmatrix}=\begin{pmatrix}3\\-3\\1\end{pmatrix}$ $\Rightarrow \quad x=\frac{3}{2}, y=-\frac{3}{2}, z=\frac{1}{2}$	B1 B2 M1 A1 **5**	 Give B1 for one component with the wrong sign Recognising that the solution is a multiple of the given RHS Correct multiple
(iii)	$2\lambda^3+\lambda^2-13\lambda+6=0$ $\Rightarrow \quad 2\mathbf{M}^3+\mathbf{M}^2-13\mathbf{M}+6\mathbf{I}=\mathbf{0}$ $\Rightarrow \quad \mathbf{M}^3=-\frac{1}{2}\mathbf{M}^2+\frac{13}{2}\mathbf{M}-3\mathbf{I}$ $\Rightarrow \quad \mathbf{M}^4=-\frac{1}{2}\mathbf{M}^3+\frac{13}{2}\mathbf{M}^2-3\mathbf{M}$ $\Rightarrow \quad \mathbf{M}^4=-\frac{1}{2}(-\frac{1}{2}\mathbf{M}^2+\frac{13}{2}\mathbf{M}-3\mathbf{I})+\frac{13}{2}\mathbf{M}^2-3\mathbf{M}$ $\Rightarrow \quad \mathbf{M}^4=\frac{27}{4}\mathbf{M}^2-\frac{25}{4}\mathbf{M}+\frac{3}{2}\mathbf{I}$ $A=\frac{27}{4}, B=-\frac{25}{4}, C=\frac{3}{2}$	 M1 M1 M1 A1 **4**	 Using Cayley-Hamilton Theorem Multiplying by **M** Substituting for $\mathbf{M}^3$
(b)	$\mathbf{N}=\mathbf{PDP}^{-1}$ where $\mathbf{D}=\begin{pmatrix}-1&0\\0&2\end{pmatrix}$ and $\mathbf{P}=\begin{pmatrix}1&-1\\2&1\end{pmatrix}$ $\Rightarrow \quad \mathbf{P}^{-1}=\frac{1}{3}\begin{pmatrix}1&1\\-2&1\end{pmatrix}$ $\Rightarrow \quad \mathbf{N}=\frac{1}{3}\begin{pmatrix}1&-1\\2&1\end{pmatrix}\begin{pmatrix}-1&0\\0&2\end{pmatrix}\begin{pmatrix}1&1\\-2&1\end{pmatrix}$ $=\frac{1}{3}\begin{pmatrix}-1&-2\\-2&2\end{pmatrix}\begin{pmatrix}1&1\\-2&1\end{pmatrix}$ $=\frac{1}{3}\begin{pmatrix}3&-3\\-6&0\end{pmatrix}=\begin{pmatrix}1&-1\\-2&0\end{pmatrix}$	B1 B1 B1 B1ft M1 A1	Order must be correct For B1B1, order must be consistent Ft their **P** Attempting matrix product
	OR Let $\mathbf{N}=\begin{pmatrix}a&c\\b&d\end{pmatrix}$ $\begin{pmatrix}a&c\\b&d\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}=-1\begin{pmatrix}1\\2\end{pmatrix}$ B1 $\begin{pmatrix}a&c\\b&d\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}=2\begin{pmatrix}-1\\1\end{pmatrix}$ B1 $\Rightarrow \quad a+2c=-1, -a+c=-2$ B1 $b+2d=-2, -b+d=2$ B1 $\Rightarrow \quad a=1, c=-1; b=-2, d=0$ M1A1		 Or $\begin{pmatrix}a+1&c\\b&d+1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ Or $\begin{pmatrix}a-2&c\\b&d-2\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ Solving both pairs of equations
		6	**19**

4 (i)	$2\sinh x\cosh x$ $= 2\times\dfrac{e^x+e^{-x}}{2}\times\dfrac{e^x-e^{-x}}{2}$ $=\dfrac{e^{2x}-e^{-2x}}{2}$ $=\sinh 2x$ Differentiating, $2\cosh 2x = 2\cosh^2 x + 2\sinh^2 x$ $\Rightarrow\quad \cosh 2x = \cosh^2 x + \sinh^2 x$	M1 A1 (ag) B1 B1 **4**	Using exponential definitions and multiplying or factorising One side correct Correct completion
(ii)	 Volume $=\pi\displaystyle\int_0^2(\cosh x-1)^2\,dx$ $=\pi\displaystyle\int_0^2\cosh^2 x-2\cosh x+1\,dx$ $=\pi\displaystyle\int_0^2\tfrac{1}{2}\cosh 2x-2\cosh x+\tfrac{3}{2}\,dx$ $=\pi\left[\tfrac{1}{4}\sinh 2x-2\sinh x+\tfrac{3}{2}x\right]_0^2$ $=\pi\left[\tfrac{1}{4}\sinh 4-2\sinh 2+3\right]$ $= 8.070$	G1 M1 A1 M1 A2 A1 **7**	Correct shape and through origin $\int(\cosh x-1)^2\,dx$ A correct expanded integral expression including limits 0, 2 (may be implied by later work) Attempting to obtain an integrable form Dep. on M1 above Give A1 for two terms correct 3 d.p. required. Condone 8.07
(iii)	$y=\cosh 2x+\sinh x$ $\Rightarrow\quad \dfrac{dy}{dx}=2\sinh 2x+\cosh x$ At S.P. $2\sinh 2x+\cosh x=0$ $\Rightarrow\quad 4\sinh x\cosh x+\cosh x=0$ $\Rightarrow\quad \cosh x(4\sinh x+1)=0$ $\Rightarrow\quad \cosh x=0$ (rejected) $\Rightarrow\quad \sinh x=-\tfrac{1}{4}$ $\Rightarrow\quad x=\ln\left(-\dfrac{1}{4}+\dfrac{\sqrt{17}}{4}\right)$	B1 M1 M1 A1 A1 M1 A1 **7**	Any correct form Setting derivative equal to zero and using identity Solving $\dfrac{dy}{dx}=0$ to obtain value of $\sinh x$ Repudiating $\cosh x = 0$ Using log form of arsinh, or setting up and solving quadratic in e^x A0 if extra "roots" quoted **18**

5(i)(A)	Circle	B1	
(B)		G1 G1	Sketch of circle, centre (0, 0) Sketch of "squarer" circle on same axes
(C)	Square	B1	
(D)	$-1 \le x \le 1$	B1	Give B1B0 for not all non-strict or
	$-1 \le y \le 1$	B1	unclear
		6	
(ii)(A)	Odd roots exist for all real numbers	B1	Any equivalent explanation
(B)	Line	B1	Sketch insufficient
(C)		G1	
	Asymptote: $x + y = 0$	B1	
(D)		G1 G1	Line $x + y = 0$ outside unit square Lines $y = 1$ and $x = 1$ on unit square
		6	
(iii)		G1	G0 if curve beyond (1, 0) or (0, 1)
	$0 \le x, y \le 1$	B1	Accept strict, or indication on graph
		2	
(iv)(A)		G2ft	Give G1 for a partial attempt. Ft from (iii) on shape
(B)	Limit is a "plus sign"	B1	
	where $x \to 0$ for $-1 \le y \le 1$ and vice versa	B1	
		4	**18**

OCR
RECOGNISING ACHIEVEMENT

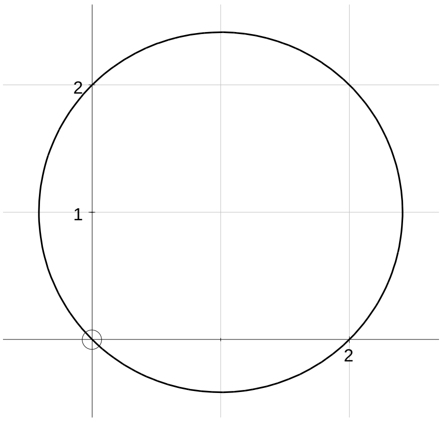
GCE

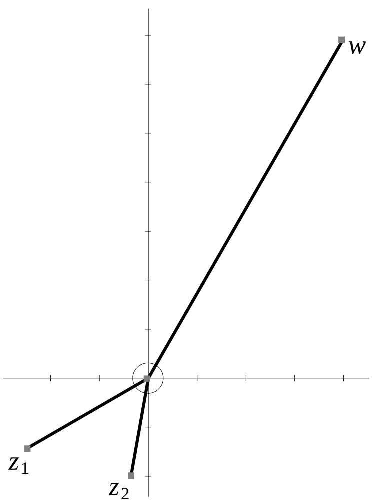
Mathematics (MEI)

Advanced GCE

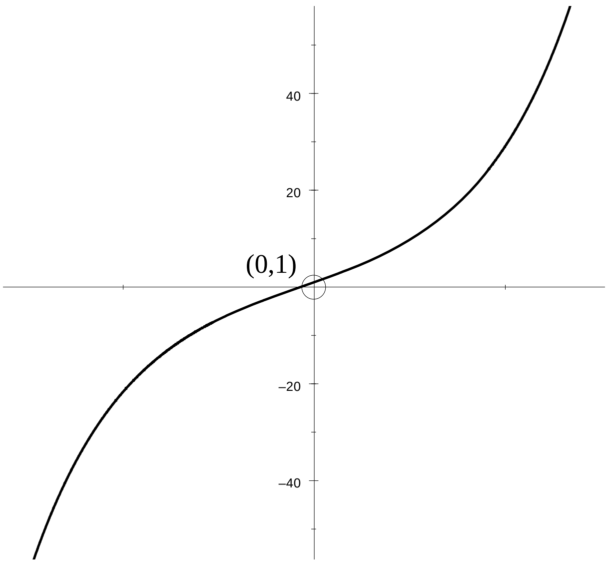
Unit **4756:** Further Methods for Advanced Mathematics

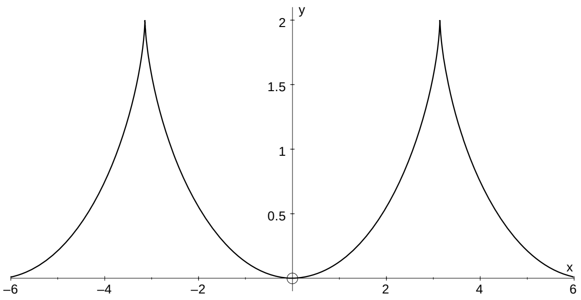
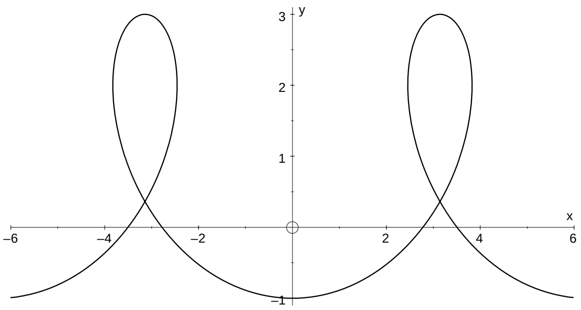
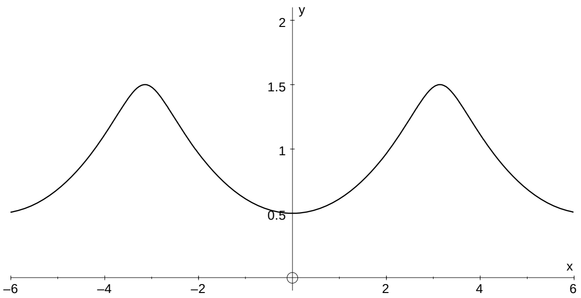
Mark Scheme for January 2011

1 (a)(i)	$x = r\cos\theta,\ y = r\sin\theta,\ x^2 + y^2 = r^2$ $r = 2(\cos\theta + \sin\theta)$ $\Rightarrow r^2 = 2r(\cos\theta + \sin\theta)$ $\Rightarrow x^2 + y^2 = 2x + 2y$ $\Rightarrow x^2 - 2x + y^2 - 2y = 0$ $\Rightarrow (x-1)^2 + (y-1)^2 = 2$ which is a circle centre (1, 1) radius $\sqrt{2}$ 	M1 A1 (ag) M1 G1 G1 **5**	Using at least one of these Working must be convincing Recognise as circle or appropriate algebra leading to $(x-a)^2 + (y-b)^2 = r^2$ Attempt at complete circle with centre in first quadrant A circle with centre and radius indicated, or centre (1, 1) indicated and passing through (0, 0), or (2, 0) and (0, 2) indicated and passing through (0, 0)
(ii)	Area $= \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 d\theta$ $= 2\int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta)^2\,\mathrm{d}\theta$ $= 2\int_0^{\frac{\pi}{2}} (\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta)\mathrm{d}\theta$ $= 2\int_0^{\frac{\pi}{2}} (1 + 2\sin\theta\cos\theta)\mathrm{d}\theta$ $= 2\left[\theta - \frac{1}{2}\cos 2\theta\right]_0^{\frac{\pi}{2}}$ or $2\left[\theta + \sin^2\theta\right]_0^{\frac{\pi}{2}}$ etc. $= 2\left(\left(\frac{\pi}{2} + \frac{1}{2}\right) - \left(0 - \frac{1}{2}\right)\right)$ $= \pi + 2$	M1 M1 A1 A2 M1 A1 **7**	Integral expression involving r^2 in terms of θ Multiplying out $\cos^2\theta + \sin^2\theta = 1$ used Correct result of integration with correct limits. Give A1 for one error Substituting limits. Dep. on both M1s Mark final answer
(b)(i)	$f'(x) = \frac{1}{2}\frac{1}{\left(1 + \frac{1}{4}x^2\right)} = \frac{2}{4 + x^2}$	M1 A1 **2**	Using Chain Rule Correct derivative in any form
(ii)	$f'(x) = \frac{1}{2}\left(1 + \frac{1}{4}x^2\right)^{-1} = \frac{1}{2}\left(1 - \frac{1}{4}x^2 + \frac{1}{16}x^4 - ...\right)$ $= \frac{1}{2} - \frac{1}{8}x^2 + \frac{1}{32}x^4 - ...$ $\Rightarrow f(x) = \frac{1}{2}x - \frac{1}{24}x^3 + \frac{1}{160}x^5 - ... + c$ But $c = 0$ because arctan(0) = 0	M1 A1 M1 A1 A1 **5**	Correctly using binomial expansion Correct expansion Integrating at least two terms Independent **19**

2 (a)(i)	$z^n + z^{-n} = 2\cos n\theta$ $z^n - z^{-n} = 2\text{j}\sin n\theta$	B1 B1 **2**	
(ii)	$(z + z^{-1})^6 = z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6}$ $= z^6 + z^{-6} + 6(z^4 + z^{-4}) + 15(z^2 + z^{-2}) + 20$ $\Rightarrow 64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$ $\Rightarrow \cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$ $\Rightarrow \cos^6\theta = \frac{1}{32}\left(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right)$	M1 M1 A1 (ag) **3**	Expanding $(z + z^{-1})^6$ Using $z^n + z^{-n} = 2\cos n\theta$ with $n = 2$, 4 or 6. Allow M1 if 2 omitted, etc.
(iii)	$(z - z^{-1})^6 = z^6 + z^{-6} - 6(z^4 + z^{-4}) + 15(z^2 + z^{-2}) - 20$ $\Rightarrow -64\sin^6\theta = 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$ $\Rightarrow -\sin^6\theta = \frac{1}{32}\cos 6\theta - \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta - \frac{5}{16}$ $\Rightarrow \cos^6\theta - \sin^6\theta = \frac{1}{16}\cos 6\theta + \frac{15}{16}\cos 2\theta$	B1 M1 A1 M1 A1	 Using (i) as in part (ii) Correct expression in any form Attempting to add or subtract
	OR $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$ B1 $16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$ M1 $\Rightarrow \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ A1 $\cos^6\theta - \sin^6\theta = 2\cos^6\theta - 3\cos^4\theta + 3\cos^2\theta - 1$ $\Rightarrow = \frac{1}{16}\cos 6\theta + \frac{15}{16}\cos 2\theta$ M1A1	 **5**	This used Obtaining an expression for $\cos^4\theta$ Correct expression in any form Attempting to add or subtract
(b)(i)	$z_1^{\ 2} = 8e^{\frac{j\pi}{3}} \Rightarrow z_1 = 2\sqrt{2}e^{j\left(\frac{\pi}{6}+\pi\right)}$ $= 2\sqrt{2}e^{\frac{j7\pi}{6}}$ $z_2^{\ 3} = 8e^{\frac{j\pi}{3}} \Rightarrow z_2 = 2e^{j\left(\frac{\pi}{9}+\frac{4\pi}{3}\right)}$ $= 2e^{\frac{j13\pi}{9}}$ 	M1 A1 M1 A1 G1 G1 **6**	Correctly manipulating modulus and argument $\sqrt{8}$, $\frac{7\pi}{6}$ or $-\frac{5\pi}{6}$. Condone $r(c + js)$ Correctly manipulating modulus and argument 2, $\frac{13\pi}{9}$ or $-\frac{5\pi}{9}$. Condone $r(c + js)$ Moduli approximately correct Arguments approximately correct Give G1G0 for two points approximately correct
(ii)	$z_1 z_2 = 2\sqrt{2}e^{\frac{j7\pi}{6}} \times 2e^{\frac{j13\pi}{9}}$ $= 4\sqrt{2}e^{j\left(\frac{7\pi}{6}+\frac{13\pi}{9}\right)}$ $= 4\sqrt{2}e^{\frac{j11\pi}{18}}$ Lies in second quadrant	 M1 A1 A1 **3**	 Correctly manipulating modulus and argument Accept any equivalent form **19**

3 (i)	$\det(\mathbf{M}-\lambda\mathbf{I}) = (1-\lambda)[(3-\lambda)(1-\lambda)+8]$ $+ 4[2(1-\lambda)-2] + 5[8+(3-\lambda)]$ $= (1-\lambda)(\lambda^2-4\lambda+11) + 4(-2\lambda) + 5(11-\lambda)$ $= -\lambda^3+5\lambda^2-15\lambda+11-8\lambda+55-5\lambda = 0$ $\Rightarrow \quad \lambda^3-5\lambda^2+28\lambda-66=0$	M1 A1 M1 A1 (ag) **4**	Obtaining $\det(\mathbf{M}-\lambda\mathbf{I})$ Any correct form Simplification www, but condone omission of = 0
(ii)	$\lambda^3-5\lambda^2+28\lambda-66=0$ $\Rightarrow (\lambda-3)(\lambda^2-2\lambda+22)=0$ $\lambda^2-2\lambda+22=0 \Rightarrow b^2-4ac=-84$ so no other real eigenvalues	M1 A1 M1 A1 **4**	Factorising and obtaining a quadratic. If M0, give B1 for substituting $\lambda = 3$ Correct quadratic Considering discriminant o.e. Conclusion from correct evidence www
(iii)	$\lambda = 3 \Rightarrow \begin{pmatrix} -2 & -4 & 5 \\ 2 & 0 & -2 \\ -1 & 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\Rightarrow \quad -2x-4y+5z=0$ $2x-2z=0$ $-x+4y-2z=0$ $\Rightarrow \quad x = z = k,\ y = \frac{3}{4}k$ $\Rightarrow$ eigenvector is $\begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix}$ $\Rightarrow$ eigenvector with unit length is $\mathbf{v} = \frac{1}{\sqrt{41}}\begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix}$ Magnitude of $\mathbf{M}^n\mathbf{v}$ is 3^n	 M1 M1 A1 B1 B1 **5**	 Two independent equations Obtaining a non-zero eigenvector Must be a magnitude
(iv)	$\lambda^3-5\lambda^2+28\lambda-66=0$ $\Rightarrow \mathbf{M}^3-5\mathbf{M}^2+28\mathbf{M}-66\mathbf{I}=\mathbf{0}$ $\Rightarrow \mathbf{M}^2-5\mathbf{M}+28\mathbf{I}-66\mathbf{M}^{-1}=\mathbf{0}$ $\Rightarrow \mathbf{M}^{-1} = \frac{1}{66}(\mathbf{M}^2-5\mathbf{M}+28\mathbf{I})$	 M1 M1 A1 **3**	 Use of Cayley-Hamilton Theorem Multiplying by $\mathbf{M}^{-1}$ and rearranging Must contain **I** **16**

4 (i)	$\sinh t + 7\cosh t = 8$ $\Rightarrow \frac{1}{2}(e^t - e^{-t}) + 7 \times \frac{1}{2}(e^t + e^{-t}) = 8$ $\Rightarrow 4e^t + 3e^{-t} = 8$ $\Rightarrow 4e^{2t} - 8e^t + 3 = 0$ $\Rightarrow (2e^t - 1)(2e^t - 3) = 0$ $\Rightarrow e^t = \frac{1}{2}$ or $\frac{3}{2}$ $\Rightarrow t = \ln(\frac{1}{2})$ or $\ln(\frac{3}{2})$	 M1 M1 M1 A1A1 A1 **6**	 Substituting correct exponential forms Obtaining quadratic in e^t Solving to obtain at least one value of e^t Condone extra values These two values o.e. only. Exact form
(ii)	$\frac{dy}{dx} = 2\sinh 2x + 14\cosh 2x$ or $8e^{2x} + 6e^{-2x}$ $2\sinh 2x + 14\cosh 2x = 16 \Rightarrow \sinh 2x + 7\cosh 2x = 8$ $\Rightarrow 2x = \ln(\frac{1}{2})$ or $\ln(\frac{3}{2}) \Rightarrow x = \frac{1}{2}\ln(\frac{1}{2})$ or $\frac{1}{2}\ln(\frac{3}{2})$ $x = \frac{1}{2}\ln(\frac{1}{2}) \Rightarrow y = -4$ $(\frac{1}{2}\ln(\frac{1}{2}), -4)$ $x = \frac{1}{2}\ln(\frac{3}{2}) \Rightarrow y = 4$ $(\frac{1}{2}\ln(\frac{3}{2}), 4)$ $\frac{dy}{dx} = 0 \Rightarrow 2\sinh 2x + 14\cosh 2x = 0$ $\Rightarrow \tanh 2x = -7$ or $e^{4x} = -\frac{3}{4}$ etc. No solutions because $-1 < \tanh 2x < 1$ or $e^x > 0$ etc. 	B1 M1 A1 B1 M1 A1 (ag) G1 G1 **8**	 Complete method to obtain an x value Both x co-ordinates in any exact form Both y co-ordinates Any complete method www Curve (not st. line) with correct general shape (positive gradient throughout) Curve through (0, 1). Dependent on last G1
(iii)	$\int_0^a (\cosh 2x + 7\sinh 2x)\,dx = \frac{1}{2}$ $\Rightarrow \left[\frac{1}{2}\sinh 2x + \frac{7}{2}\cosh 2x\right]_0^a = \frac{1}{2}$ $\Rightarrow \left(\frac{1}{2}\sinh 2a + \frac{7}{2}\cosh 2a\right) - \frac{7}{2} = \frac{1}{2}$ $\Rightarrow \sinh 2a + 7\cosh 2a = 8$ $\Rightarrow 2a = \ln(\frac{1}{2})$ or $\ln(\frac{3}{2}) \Rightarrow a = \frac{1}{2}\ln(\frac{1}{2})$ or $\frac{1}{2}\ln(\frac{3}{2})$ $\Rightarrow a = \frac{1}{2}\ln(\frac{3}{2})$ $(\frac{1}{2}\ln(\frac{1}{2}) < 0)$	M1 A1 M1 A1 **4**	Attempting integration Correct result of integration Using both limits and a complete method to obtain a value of a Must reject $\frac{1}{2}\ln(\frac{1}{2})$, but reason need not be given **18**

5 (i)	$a = 1$ 	G1	
	$a = 2$ 	G1	
	$a = 0.5$ 	G1	
		M2	Evidence s.o.i. of further investigation
(A)	Loops when $a > 1$	A1	
(B)	Cusps when $a = 1$	A1	
		7	
(ii)	If $x \to -x$, $t \to -t$	M1	Considering effect on t
	but $y(-t) = y(t)$	A1 (ag)	Effect on y
	Curve is symmetrical in the y-axis	B1	
		3	
(iii)	$\frac{dy}{dx} = \frac{a\sin t}{1 + a\cos t}$	M1 A1	Using Chain Rule
	$\frac{dy}{dx} = 0 \Rightarrow a \sin t = 0 \Rightarrow t = 0$ and $\pm\pi$	A1	Values of t
	$t = 0 \Rightarrow$ T.P. is $(0, 1 - a)$	A1	
	$t = \pm\pi \Rightarrow$ T.P. are $(\pm\pi, 1 + a)$	A1	Both, in any form
		5	
(iv)	$a = \frac{\pi}{2}$: both $t = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ give the point $(\pi, 1)$	B1 (ag)	Verification
	Gradients are a and $-a$ (or $\frac{\pi}{2}$ and $-\frac{\pi}{2}$)		
	Hence angle is 2 arctan($\frac{\pi}{2}$) ≈ 2.01 radians	M1	Complete method for angle
		A1	Accept 115° (or 65°)
		3	**18**

GCE

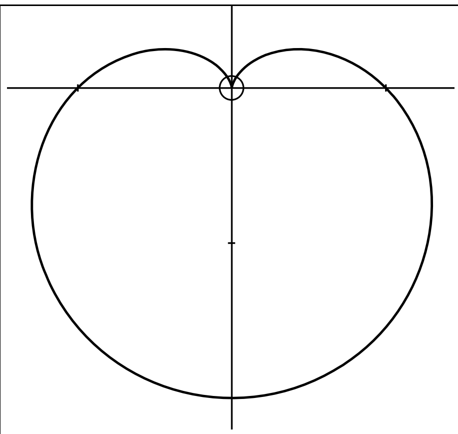
Mathematics (MEI)

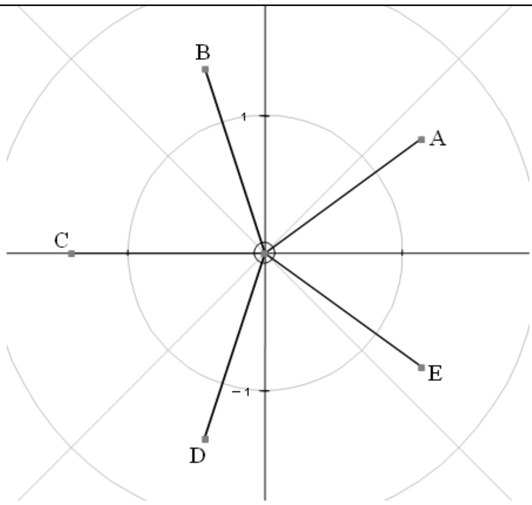
Advanced GCE

Unit **4756:** Further Methods for Advanced Mathematics

Mark Scheme for June 2011

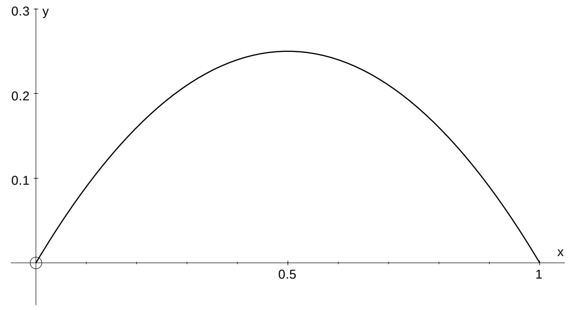
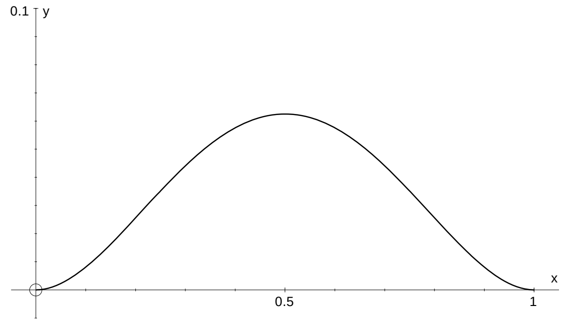
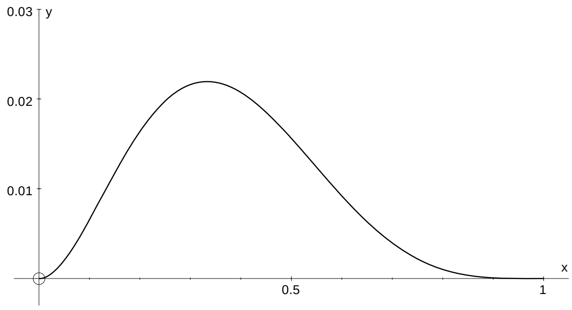
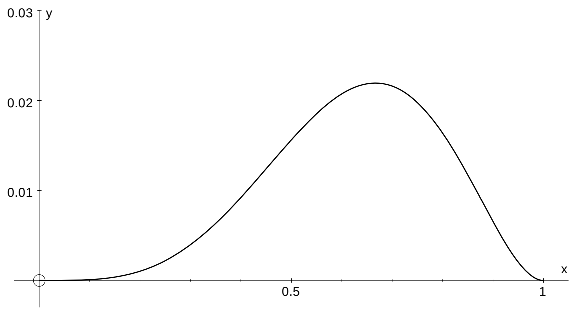
4756 (FP2) Further Methods for Advanced Mathematics

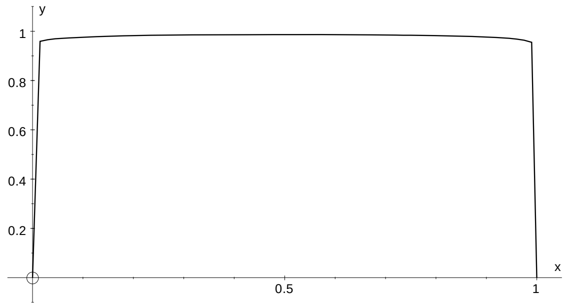
1 (a)(i)		G1 G1 **2**	Correct general shape including symmetry in vertical axis Correct form at O and no extra sections. Dependent on first G1 For an otherwise correct curve with a sharp point at the bottom, award G1G0
(ii)	Area $=\frac{1}{2}a^2\int_0^{2\pi}(1-\sin\theta)^2\,d\theta$ $=\frac{1}{2}a^2\int_0^{2\pi}\left(1-2\sin\theta+\sin^2\theta\right)d\theta$ $=\frac{1}{2}a^2\int_0^{2\pi}\left(\frac{3}{2}-2\sin\theta-\frac{1}{2}\cos 2\theta\right)d\theta$ $=\frac{1}{2}a^2\left[\frac{3}{2}\theta+2\cos\theta-\frac{1}{4}\sin 2\theta\right]_0^{2\pi}$ $=\frac{3}{2}\pi a^2$	M1 M1 A1 M1 A2 A1 **7**	Integral expression involving $(1-\sin\theta)^2$ Expanding Correct integral expression, incl. limits (which may be implied by later work) Using $\sin^2\theta=\frac{1}{2}-\frac{1}{2}\cos 2\theta$ Correct result of integration. Give A1 for one error Dependent on previous A2
(b)(i)	$\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{1+4x^2}dx=\frac{1}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{1}{\frac{1}{4}+x^2}dx=\frac{1}{4}\left[2\arctan 2x\right]_{-\frac{1}{2}}^{\frac{1}{2}}$ $=\frac{1}{2}\left(\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right)$ $=\frac{\pi}{4}$	M1 A1 A1 **3**	arctan alone, or any tan substitution $\frac{1}{4}\times 2$ and $2x$ Evaluated in terms of π
(ii)	$x=\frac{1}{2}\tan\theta$ $\Rightarrow dx=\frac{1}{2}\sec^2\theta\,d\theta$ $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{\left(\sec^2\theta\right)^{\frac{3}{2}}}\times\frac{\sec^2\theta}{2}d\theta$ $=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{2}\cos\theta\,d\theta$ $=\left[\frac{1}{2}\sin\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$ $=\frac{1}{2}\left(\frac{1}{\sqrt{2}}-\left(-\frac{1}{\sqrt{2}}\right)\right)$ $=\frac{1}{\sqrt{2}}$	M1 A1A1 M1 A1ft A1 **6**	Any tan substitution $\frac{1}{\left(\sec^2\theta\right)^{\frac{3}{2}}}$, $\frac{\sec^2\theta}{2}$ Integrating $a\cos b\theta$ and using consistent limits. Dependent on M1 above $\frac{a}{b}\sin b\theta$ **18**

2 (a)	$\cos 5\theta + j\sin 5\theta = (\cos\theta + j\sin\theta)^5$ $= c^5 + 5c^4js - 10c^3s^2 - 10c^2js^3 + 5cs^4 + js^5$ $\Rightarrow \cos 5\theta = c^5 - 10c^3s^2 + 5cs^4$ $\sin 5\theta = 5c^4s - 10c^2s^3 + s^5$ $\Rightarrow \tan 5\theta = \dfrac{5c^4s - 10c^2s^3 + s^5}{c^5 - 10c^3s^2 + 5cs^4}$ $= \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ $= \dfrac{t(t^4 - 10t^2 + 5)}{5t^4 - 10t^2 + 1}$	M1 M1 A1 A1 M1 A1 (ag) **6**	Expanding Separating real and imaginary parts. Dependent on first M1 Alternative: $16c^5 - 20c^3 + 5c$ Alternative: $16s^5 - 20s^3 + 5s$ Using $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and simplifying
(b)(i)	$\arg(-4\sqrt{2}) = \pi$ $\Rightarrow$ fifth roots have $r = \sqrt{2}$ and $\theta = \dfrac{\pi}{5}$ $\Rightarrow z = \sqrt{2}e^{\frac{1}{5}j\pi}, \sqrt{2}e^{\frac{3}{5}j\pi}, \sqrt{2}e^{j\pi}, \sqrt{2}e^{\frac{7}{5}j\pi}, \sqrt{2}e^{\frac{9}{5}j\pi}$	B1 B1 M1 A1 **4**	No credit for arguments in degrees Adding (or subtracting) $\dfrac{2\pi}{5}$ All correct. Allow $-\pi \le \theta < \pi$
(ii)		G1 G1 **2**	Points at vertices of "regular" pentagon, with one on negative real axis Points correctly labelled
(iii)	$\arg(w) = \dfrac{1}{2}\left(\dfrac{\pi}{5} + \dfrac{3\pi}{5}\right) = \dfrac{2\pi}{5}$ $\lvert w\rvert = \sqrt{2}\cos\dfrac{\pi}{5}$	B1 M1 A1ft **3**	 Attempting to find length F.t. (positive) r from (i)
(iv)	$w = \sqrt{2}\cos\dfrac{\pi}{5}\, e^{\frac{2}{5}\pi j} \Rightarrow w^n = \left(\sqrt{2}\cos\dfrac{\pi}{5}\right)^n e^{\frac{2}{5}\pi nj}$ which is real if $\sin\dfrac{2\pi n}{5} = 0 \Rightarrow n = 5$ $\lvert w^5\rvert = \left(\sqrt{2}\cos\dfrac{\pi}{5}\right)^5$ $\Rightarrow a = 2^{\frac{5}{2}}\cos^5\dfrac{\pi}{5}$	 B1 M1 A1 **3**	 Attempting the nth power of his modulus in (iii), or attempting the modulus of the nth power here Accept 1.96 or better **18**

3 (i)	det(**M**) = 1(16 − 12) + 1(20 − 18) + k(10 − 12) = 6 − 2k ⇒ no inverse if k = 3 $\mathbf{M}^{-1} = \frac{1}{6-2k}\begin{pmatrix} 4 & 4+2k & -6-4k \\ -2 & 4-3k & 5k-6 \\ -2 & -5 & 9 \end{pmatrix}$	M1 A1 A1 M1 A1 M1 A1 **7**	Obtaining det(**M**) in terms of k Accept $k \neq 3$ after correct determinant Evaluating at least four cofactors (including one involving k) Six signed cofactors correct (including one involving k) Transposing and dividing by det(**M**). Dependent on previous M1M1
(ii)	$\begin{pmatrix} 1 & -1 & 3 \\ 5 & 4 & 6 \\ 3 & 2 & 4 \end{pmatrix}\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$	M1 A1 **2**	Setting k = 3 and multiplying
(iii)	$\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to an eigenvalue of 1	B1 B1 **2**	For credit here, 2/2 scored in (ii) Accept "invariant point"
(iv)	$3x + 6y = 1 - 2t$, $x + 2y = 2$, $5x + 10y = -4t$ (or $9x + 18z = 4t + 1$, $5x + 10z = 2t$, $x + 2z = -1$) (or $9y - 9z = 1 - 5t$, $5y - 5z = -3t$, $2y - 2z = 3$) For solutions, $1 - 2t = 3 \times 2$ $\Rightarrow t = -\frac{5}{2}$ $x = \lambda$, $y = 1 - \frac{1}{2}\lambda$, $z = -\frac{1}{2} - \frac{1}{2}\lambda$ Straight line	M1 A1 M1 A1 M1 A1 B1 **7**	Eliminating one variable in two different ways Two correct equations Validly obtaining a value of t Obtaining general solution by setting one unknown = λ and finding other two in terms of λ (accept unknown instead of λ) Accept "sheaf". Independent of all previous marks **18**

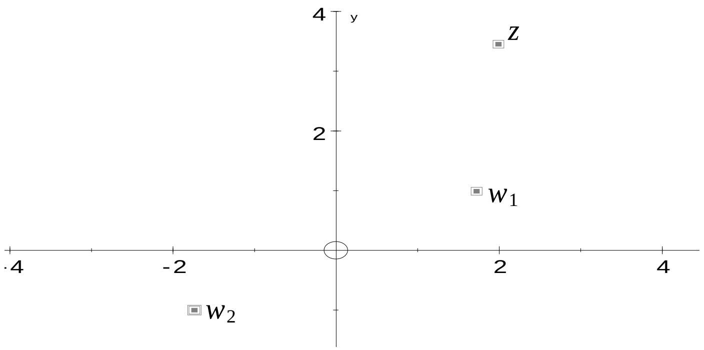
4 (i)	$\cosh y = x \Rightarrow x = \frac{1}{2}\left(e^{y} + e^{-y}\right)$	B1	Using correct exponential definition
	$\Rightarrow 2x = e^{y} + e^{-y}$		
	$\Rightarrow \left(e^{y}\right)^{2} - 2xe^{y} + 1 = 0$	M1	Obtaining quadratic in e^{y}
	$\Rightarrow e^{y} = \frac{2x \pm \sqrt{4x^{2}-4}}{2} = x \pm \sqrt{x^{2}-1}$	M1 A1	Solving quadratic $x \pm \sqrt{x^{2}-1}$
	$\Rightarrow y = \ln(x \pm \sqrt{x^{2}-1})$		
	$\left(x+\sqrt{x^{2}-1}\right)\left(x-\sqrt{x^{2}-1}\right) = 1$	M1	Validly attempting to justify ± in printed answer
	$\Rightarrow y = \pm\ln(x + \sqrt{x^{2}-1})$	A1 (ag)	
	arcosh(x) = $\ln(x + \sqrt{x^{2}-1})$ because this is the principal value	B1	Reference to arcosh as a function, or correctly to domains/ranges
		7	
(ii)	$\int_{\frac{4}{5}}^{1} \frac{1}{\sqrt{25x^{2}-16}}\,dx = \frac{1}{5}\int_{\frac{4}{5}}^{1} \frac{1}{\sqrt{x^{2}-\frac{16}{25}}}\,dx$		
	$= \frac{1}{5}\left[\text{arcosh}\left(\frac{5\text{x}}{4}\right)\right]_{\frac{4}{5}}^{1}$	M1 A1A1	arcosh alone, or any cosh substitution $\frac{1}{5}, \frac{5x}{4}$
	$= \frac{1}{5}\left(\text{arcosh}\left(\frac{5}{4}\right) - \text{arcosh}(1)\right)$		
	$= \frac{1}{5}\ln\left(\frac{5}{4} + \sqrt{\left(\frac{5}{4}\right)^{2} - 1}\right) - 0$	M1	Substituting limits and using (i) correctly at any stage (or using limits of u in logarithmic form). Dep. on first M1
	$= \frac{1}{5}\ln 2$	A1	
	OR $= \frac{1}{5}\left[\ln\left(x + \sqrt{x^{2} - \frac{16}{25}}\right)\right]_{\frac{4}{5}}^{1}$ M A1A $= \frac{1}{5}\ln\frac{8}{5} - \frac{1}{5}\ln\frac{4}{5}$ $= \frac{1}{5}\ln 2$ A		$\ln\left(kx + \sqrt{k^{2}x^{2} + ...}\right)$ Give M1 for $\ln\left(k_{1}x + \sqrt{k_{2}{}^{2}x^{2} + ...}\right)$ $\frac{1}{5}$, $\ln\left(x + \sqrt{x^{2} - \frac{16}{25}}\right)$ o.e.
		5	
(iii)	5 cosh x − cosh 2x = 3		
	⇒ 5 cosh x − (2 cosh²x − 1) = 3	M1	Attempting to express cosh 2x in terms of cosh x
	⇒ 2 cosh²x − 5 cosh x + 2 = 0		
	⇒ (2 cosh x − 1)(cosh x − 2) = 0	M1	Solving quadratic to obtain at least one real value of cosh x
	⇒ cosh x = $\frac{1}{2}$ (rejected)	A1	Or factor 2 cosh x − 1
	or cosh x = 2	A1	
	$\Rightarrow x = \ln\left(2+\sqrt{3}\right)$	A1ft	F.t. cosh x = k, k > 1
	$x = -\ln\left(2+\sqrt{3}\right)$ or $\ln\left(2-\sqrt{3}\right)$	A1ft	F.t. other value. Max. A1A0 if additional real values quoted
		6	**18**

5 (i)	**(A)** $m = 1, n = 1$ 	G1	Negative parabola from (0,0) to (1,0), symmetrical about $x = 0.5$
	(B) $m = 2, n = 2$ 	G1	Bell-shape from (0,0) to (1,0), symmetrical about $x = 0.5$; flat ends, and obviously different to (A)
	(C) $m = 2, n = 4$ 	G1	Skewed curve from (0,0) to (1,0), maximum to left of $x = 0.5$
	(D) $m = 4, n = 2$ 	G1 **4**	Skewed curve from (0,0) to (1,0), maximum to right of $x = 0.5$
(ii)	When $m = n$, the curve is symmetrical Exchanging m and n reflects the curve	B1 B1 **2**	
(iii)	If $m > n$, the maximum is to the right of $x = 0.5$ As m increases relative to n, the maximum point moves further to the right	B1 B1	o.e. Give B1B0 if the idea is correct but vaguely expressed
	$y = x^m(1-x)^n \Rightarrow \frac{dy}{dx} = mx^{m-1}(1-x)^n - nx^m(1-x)^{n-1}$ $= x^{m-1}(1-x)^{n-1}\left[m(1-x) - nx\right]$	M1 A1	Using product rule Any correct form
	$\frac{dy}{dx} = 0 \Rightarrow$ maximum at $x = \frac{m}{m+n}$	M1 A1 **6**	Setting derivative = 0 and solving to find a value of x other than 0 or 1

(iv)	$y'(0) = 0$ provided $m > 1$ $y'(1) = 0$ provided $n > 1$	B1 B1 **2**	
(v)	For large m and n, the curve approaches the x-axis $\Rightarrow \int_0^1 x^m (1-x)^n dx \to 0$ as $m, n \to \infty$	B1 B1 **2**	Comment on shape Independent
(vi)	e.g. $m = 0.01$, $n = 0.01$  The curve tends to $y = 1$	M1 A1 **2**	Evidence of investigation s.o.i. Accept "three sides of (unit) square" **18**

Question			Answer	Marks	Guidance	
1	(a)	(i)	$\sin y = x \Rightarrow \cos y \dfrac{dy}{dx} = 1$	M1	Differentiating w.r.t. x or y	$\dfrac{dy}{dx} = \cos y$
			$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\cos y}$	A1		
			$\Rightarrow \dfrac{dy}{dx} = (\pm)\dfrac{1}{\sqrt{1-x^2}}$	A1(ag)	Completion www, but independent of B1	$\dfrac{dy}{dx} = \pm\dfrac{1}{\sqrt{1-x^2}}$ or ± not considered scores max. 3
			Taking + sign because gradient is positive	B1	Validly rejecting − sign. Dependent on A1 above	Or $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2} \Rightarrow 0 \le \cos y \le 1$
				[4]		
1	(a)	(ii)	(A) $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{2-x^2}}\,dx = \left[\arcsin\frac{x}{\sqrt{2}}\right]_{-1}^{1}$	M1	arcsin alone, or any appropriate substitution	
				A1	$\arcsin\dfrac{x}{\sqrt{2}}$ or $\int 1\,d\theta$ www	Condone omitted or incorrect limits
			$= \dfrac{\pi}{2}$	A1		
				[3]		
			(B) $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-2x^2}}\,dx = \frac{1}{\sqrt{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{\frac{1}{2}-x^2}}\,dx$			
				M1	arcsin alone, or any appropriate substitution	
			$= \dfrac{1}{\sqrt{2}}\left[\arcsin\sqrt{2}x\right]_{-\frac{1}{2}}^{\frac{1}{2}}$	A1	$\dfrac{1}{\sqrt{2}}$ and $\sqrt{2}x$ or $\int \dfrac{1}{\sqrt{2}}\,d\theta$ www	
				M1	Using consistent limits in order and evaluating in terms of π. Dependent on M1 above	e.g. $\pm\dfrac{\pi}{4}$ with sub. $x = \dfrac{1}{\sqrt{2}}\sin\theta$
			$= \dfrac{\pi}{2\sqrt{2}}$	A1		
				[4]		

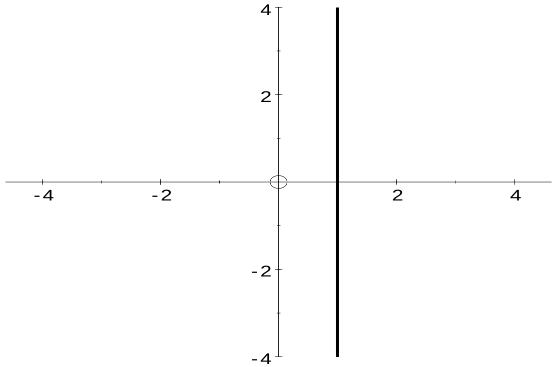
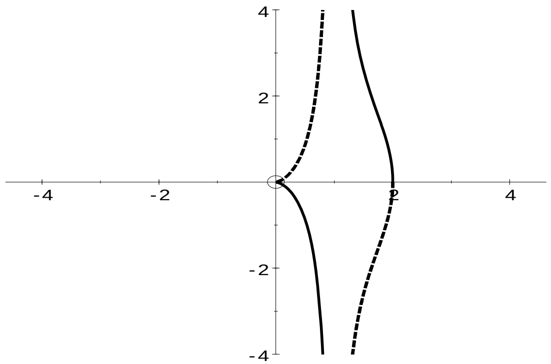
Question			Answer	Marks	Guidance	
1	(b)		$r = \tan\theta$			
			$\Rightarrow x = r\cos\theta = \dfrac{\sin\theta}{\cos\theta} \times \cos\theta = \sin\theta$	M1 A1(ag)	Using $x = r\cos\theta$ o.e.	
			$\Rightarrow r^2 = \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{\sin^2\theta}{1-\sin^2\theta} = \dfrac{x^2}{1-x^2}$	M1 A1(ag)	Obtaining r^2 in terms of x	
			$r^2 = x^2 + y^2 \Rightarrow x^2 + y^2 = \dfrac{x^2}{1-x^2}$			
			$\Rightarrow y^2 = \dfrac{x^2}{1-x^2} - x^2$	M1	Obtaining y^2 in terms of x	
			$\Rightarrow y^2 = \dfrac{x^2 - x^2(1-x^2)}{1-x^2} = \dfrac{x^4}{1-x^2}$			
			$\Rightarrow y = \dfrac{x^2}{\sqrt{1-x^2}}$	A1(ag)		Ignore discussion of ±
			Asymptote $x = 1$	B1 **[7]**	Condone $x = \pm 1$	$x \neq 1$, $x^2 = 1$ B0
2	(a)	(i)	$z^n + \dfrac{1}{z^n} = 2\cos n\theta$	B1	Mark final answer	
			$z^n - \dfrac{1}{z^n} = 2\mathrm{j}\sin n\theta$	B1 **[2]**	Mark final answer	
2	(a)	(ii)	$\left(z + \dfrac{1}{z}\right)^4 = z^4 + 4z^2 + 6 + \dfrac{4}{z^2} + \dfrac{1}{z^4} = z^4 + \dfrac{1}{z^4} + 4\left(z^2 + \dfrac{1}{z^2}\right) + 6$	M1	Expanding by Binomial or complete equivalent	
			$\Rightarrow (2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$	M1 A1	Introducing cosines of multiple angles RHS correct	Condone lost 2s Both As depend on both Ms
			$\Rightarrow \cos^4\theta = \dfrac{3}{8} + \dfrac{1}{2}\cos 2\theta + \dfrac{1}{8}\cos 4\theta$	A1ft **[4]**	Dividing both sides by 16. F.t. line above	$A = \dfrac{3}{8}$, $B = \dfrac{1}{2}$, $C = \dfrac{1}{8}$ Give SC2 for fully correct answer found "otherwise"

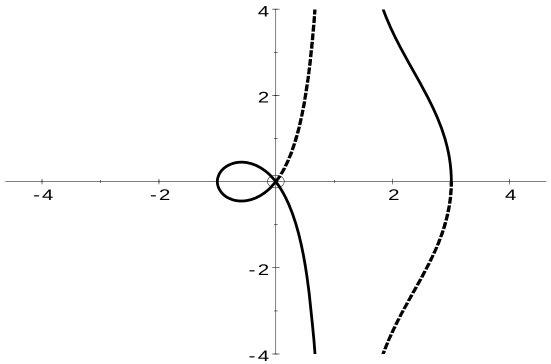
Question			Answer	Marks	Guidance	
2	(a)	(iii)	$\cos^4\theta = \frac{3}{8}+\frac{1}{2}\left(2\cos^2\theta-1\right)+\frac{1}{8}\cos 4\theta$	M1	Using (ii), obtaining cos 4θ and expressing cos 2θ in terms of $\cos^2\theta$	Condone cos $2\theta = \pm1 \pm 2\cos^2\theta$
			$\Rightarrow \cos^4\theta = \cos^2\theta - \frac{1}{8}+\frac{1}{8}\cos 4\theta$			
			$\Rightarrow \cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$	A1	c.a.o.	
				[2]		
2	(b)	(i)	$z = 4e^{\frac{j\pi}{3}}$ and $w^2 = z$: let $w = re^{j\theta} \Rightarrow w^2 = r^2e^{2j\theta}$			
			$\Rightarrow r^2 = 4 \Rightarrow r = 2$	B1		Condone $r = \pm2$
			and $\theta = \frac{\pi}{6}, \frac{7\pi}{6}$	B1B1	Or $-\frac{5\pi}{6}$	Award B2 for $\pi\left(k+\frac{1}{6}\right)$
						Ignore annotations and scales
				B1	Roots with approx. equal moduli and approx. correct argument	$\le \pi/4$
				B1	Dependent on first B1	
					z in correct position	Modulus and argument bigger
				[5]		
2	(b)	(ii)	$z = 4e^{\frac{j\pi}{3}} \Rightarrow z^n = 4^n e^{\frac{j\pi n}{3}}$ so real if $\frac{\pi n}{3} = \pi \Rightarrow n = 3$	B1		Ignore other larger values
			Imaginary if $\frac{\pi n}{3} = \frac{\pi}{2} + k\pi \Rightarrow n = \frac{3}{2} + 3k$	M1	cos $\frac{\pi n}{3} = 0$ or $\frac{\pi n}{3} = \frac{\pi}{2}$...	
			which is not an integer for any k	A1(ag)	An argument which covers the positive and negative im. axis	
			$w_1 = 2e^{\frac{j\pi}{6}} \Rightarrow w_1^3 = 8e^{\frac{j\pi}{2}} = 8j$	M1	Attempting their w^3 in any form	Must deal with mod and arg
			$w_2 = 2e^{\frac{7j\pi}{6}} \Rightarrow w_2^3 = 8e^{\frac{7j\pi}{2}} = -8j$	A1	8j, −8j	
				[5]		

Question			Answer	Marks	Guidance	
3	(i)		det ($\mathbf{M}$) = 1(2a + 8) − 2(−2 − 12) + 3(2 − 3a) = 42 − 7a	M1A1	Obtaining det($\mathbf{M}$) in terms of a	Accept unsimplified
			$\Rightarrow$ no inverse if $a = 6$	A1		Accept $a \neq 6$ after correct det
			$\mathbf{M}^{-1} = \frac{1}{42-7a}\begin{pmatrix} 2a+8 & -10 & 8-3a \\ 14 & -7 & -7 \\ 2-3a & 8 & a+2 \end{pmatrix}$	M1 A1 M1 A1 **[7]**	At least 4 cofactors correct (including one involving a) Six signed cofactors correct Transposing and ÷ by det($\mathbf{M}$). Dependent on previous M1M1	M0 if more than 1 is multiplied by the corresponding element Mark final answer
3	(ii)		$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{42}\begin{pmatrix} 8 & -10 & 8 \\ 14 & -7 & -7 \\ 2 & 8 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$	M1 M1	Substituting $a = 0$ Correct use of inverse	One correct element. Condone missing determinant
			$\Rightarrow x = \frac{6}{7}, y = \frac{1}{2}, z = -\frac{2}{7}$	A2 **[4]**	Dependent on both M marks. Give A1 for one correct SC1 for $x = 6, y = 3.5, z = -2$	After M0, give SC2 for correct solution and SC1 for one correct Answers unsupported score 0
3	(iii)		e.g. $7x - 10y = 10$, $7x - 10y = 3b - 2$	M1	Eliminating one variable in two different ways	Or $7x - 10y = 2b + 2$
			(or e.g. $4x + 5z = 5$, $4x + 5z = b + 1$) (or e.g. $8y + 7z = -1$, $8y + 7z = 3 - b$) For solutions, $10 = 3b - 2$ $\Rightarrow b = 4$	A1 M1 A1	Two correct equations Validly obtaining a value of b	Or $8x + 10z = 3b - 2$ Or $16y + 14z = b - 6$
			OR M2 A1 $b = 4$ A1		A method leading to an equation from which b could be found A correct equation	E.g. setting $z = 0$, augmented matrix, adjoint matrix, etc.
			$x = \lambda, y = 0.7\lambda - 1, z = 1 - 0.8\lambda$	M1 A1	Obtaining general soln. by e.g. setting one unknown = λ and finding other two in terms of λ Any correct form	Accept unknown instead of λ $x = \frac{10}{7}\lambda + \frac{10}{7}, y = \lambda, z = -\frac{8}{7}\lambda - \frac{1}{7}$ $x = \frac{5}{4} - \frac{5}{4}\lambda, y = -\frac{7}{8}\lambda - \frac{1}{8}, z = \lambda$
			Straight line	B1 **[7]**	Accept "sheaf", "pages of a book", etc.	Independent of all previous marks. Ignore other comments

Question			Answer	Marks	Guidance	
4	(i)		$\sinh u = \frac{e^u - e^{-u}}{2} \Rightarrow \sinh^2 u = \frac{e^{2u} - 2 + e^{-2u}}{4}$	B1	$\left(e^u - e^{-u}\right)^2 = e^{2u} - 2 + e^{-2u}$	Accept other or mixed variables
			$\Rightarrow 2\sinh^2 u + 1 = \frac{e^{2u} - 2 + e^{-2u}}{2} + 1 = \frac{e^{2u} + e^{-2u}}{2}$	B1	$\cosh 2u = \frac{e^{2u} + e^{-2u}}{2}$	
			$= \cosh 2u$	B1	Completion www	
				[3]		
4	(ii)		If $\cosh y = u$, $u = \frac{e^y + e^{-y}}{2}$	M1	Expressing u in exponential form	$\frac{1}{2}$, + must be correct
			$\Rightarrow e^y + e^{-y} = 2u \Rightarrow e^{2y} - 2ue^y + 1 = 0$			
			$\Rightarrow \left(e^y - u\right)^2 - u^2 + 1 = 0$			
			$\Rightarrow e^y = u \pm \sqrt{u^2 - 1}$	M1	Reaching e^y	Condone omitted ±
			$\Rightarrow y = \ln\left(u + \sqrt{u^2 - 1}\right)$	A1(ag)	Completion www; indep. of B1	$y = \ln\left(u \pm \sqrt{u^2 - 1}\right)$ or ± not considered scores max. 3
			$y \geq 0 \Rightarrow e^y = u + \sqrt{u^2 - 1}$	B1	Validly rejecting − sign Dependent on A1 above	
			OR $\ln\left(u + \sqrt{u^2 - 1}\right) = \ln\left(\cosh y + \sqrt{\cosh^2 y - 1}\right)$ M1		Substituting $u = \cosh y$	
			$= \ln\left(\cosh y + \sinh y\right)$			
			since $\sinh y > 0$ B1		Rejecting −ve square root	Dependent on A1
			$= \ln\left(e^y\right)$ M1		Reaching e^y	
			$= y$ A1		Completion www; indep. of B1	
				[4]		

Question			Answer	Marks	Guidance	
4	(iii)		$x=\frac{1}{2}\cosh u \Rightarrow \frac{dx}{du}=\frac{1}{2}\sinh u$	M1	Reaching integrand equivalent to $k\sinh^2 u$	
			$\int\sqrt{4x^2-1}\,dx=\int\sqrt{\cosh^2 u-1}\times\frac{1}{2}\sinh u\,du$			
			$=\int\frac{1}{2}\sinh^2 u\,du$	A1		
			$=\int\frac{1}{4}\cosh 2u-\frac{1}{4}\,du$	M1	Simplifying to integrable form. Dependent on M1 above	Or $\frac{1}{8}e^{2u}-\frac{1}{4}+\frac{1}{8}e^{-2u}$
			$=\frac{1}{8}\sinh 2u-\frac{1}{4}u+c$	A1A1	For $\frac{1}{8}\sinh 2u$ o.e. and $-\frac{1}{4}u$ seen	Or $\frac{1}{16}e^{2u}-\frac{1}{4}u-\frac{1}{16}e^{-2u}+c$
			$=\frac{1}{4}\sinh u\cosh u-\frac{1}{4}u+c$			Condone omission of + c throughout
			$=\frac{1}{4}\sqrt{4x^2-1}\times 2x-\frac{1}{4}\operatorname{arcosh} 2x+c$	M1	Clear use of $\sinh 2u = 2\sinh u\cosh u$ Dependent on M1M1 above	
			$=\frac{1}{2}x\sqrt{4x^2-1}-\frac{1}{4}\operatorname{arcosh} 2x+c$			
			$a=\frac{1}{2}$	A1		a, b need not be written separately
				[7]		
4	(iv)		$\int_{\frac{1}{2}}^{1}\sqrt{4x^2-1}\,dx=\left[\frac{1}{2}x\sqrt{4x^2-1}-\frac{1}{4}\operatorname{arcosh} 2x\right]_{\frac{1}{2}}^{1}$	M1	Using their (iii) and using limits correctly	
			$=\frac{\sqrt{3}}{2}-\frac{1}{4}\operatorname{arcosh} 2+\frac{1}{4}\operatorname{arcosh} 1$	A1ft	May be implied F.t. values of a and b in (iii)	$a\sqrt{3}-b\operatorname{arcosh} 2$. No decimals. Must have obtained values for a and b
			$=\frac{\sqrt{3}}{2}-\frac{1}{4}\ln\left(2+\sqrt{3}\right)+\frac{1}{4}\ln 1$	M1	Using (ii) accurately Dependent on M1 above	
			$=\frac{\sqrt{3}}{2}-\frac{1}{4}\ln\left(2+\sqrt{3}\right)$	A1	c.a.o. A0 if ln 1 retained Mark final answer	Correct answer www scores 4/4
				[4]		

Question			Answer	Marks	Guidance	
5	(i)		Undefined for $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$	B1B1 **[2]**		
5	(ii)		 $r = \sec\theta \Rightarrow r\cos\theta = 1$ $\Rightarrow x = 1$	B1 M1 A1 **[3]**	Vertical line through (1, 0) (indicated, e.g. by scale) Use of $x = r\cos\theta$	
5	(iii)		$a = 1$:  $a = -1$ gives same curve $a = 1$, $0 < \theta < \pi$ corresponds to $a = -1$, $\pi < \theta < 2\pi$ $a = -1$, $0 < \theta < \pi$ corresponds to $a = 1$, $\pi < \theta < 2\pi$	B1 B2 B1 B1 B1 **[6]**	Section through (2, 0) (indicated) Section through (0, 0) (give B1 for one error)	If asymptote included max. 2/3

Question			Answer	Marks	Guidance	
5	(iv)		Loop e.g. $a = 2$  	B1 B2 **[3]**	 Give B1 for one error	
5	(v)		$r = \sec\theta + a$ $\Rightarrow \quad r = \frac{r}{x} + a$ $\Rightarrow \quad r\left(1 - \frac{1}{x}\right) = a$ $\Rightarrow \quad \sqrt{x^2 + y^2}\left(\frac{x-1}{x}\right) = a$ $\Rightarrow \quad \sqrt{x^2 + y^2}\,(x-1) = ax$ $\Rightarrow \quad (x^2 + y^2)(x-1)^2 = a^2x^2$	 M1 M1 M1 A1(ag) **[4]**	 Use of $x = r\cos\theta$ Use of $r = \sqrt{x^2 + y^2}$ Correct manipulation	

OCR
RECOGNISING ACHIEVEMENT

GCE

Mathematics (MEI)

Advanced GCE

Unit **4756:** Further Methods for Advanced Mathematics

Mark Scheme for January 2013

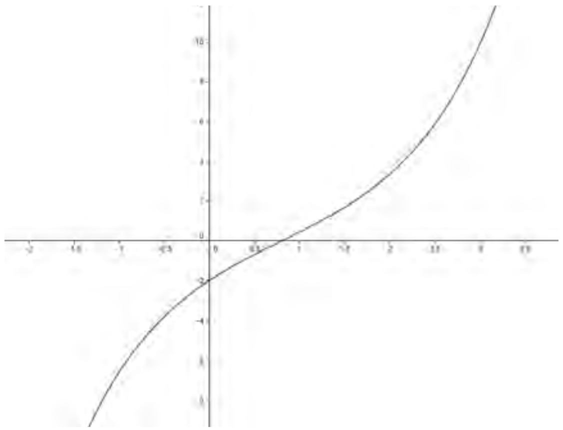
Question			Answer	Marks	Guidance	
1	(a)	(i)	$a\tan y = x \Rightarrow a\sec^2 y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$	M1	Differentiating with respect to x or y	$\dfrac{\mathrm{d}x}{\mathrm{d}y} = a\sec^2 y$
			$\Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{a\sec^2 y}$	A1	For $\dfrac{\mathrm{d}y}{\mathrm{d}x}$	Or $a\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\sec^2 y}$
			$\Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{a\left(1+\dfrac{x^2}{a^2}\right)}$			
			$\Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{a}{a^2+x^2}$	A1(ag)	Completion www with sufficient detail	
				[3]		
1	(a)	(ii)	$x^2 - 4x + 8 = (x-2)^2 + 4$	B1		
			$\int_0^4 \dfrac{1}{x^2-4x+8}\mathrm{d}x = \dfrac{1}{2}\left[\arctan\dfrac{x-2}{2}\right]_0^4$	M1	Integral of form a arctan bu or any appropriate substitution	$\dfrac{1}{2}\left[\arctan\dfrac{u}{2}\right]_{-2}^{2}$
				A1	Correct integral with consistent limits	
			$= \dfrac{1}{2}\left(\arctan(1) - \arctan(-1)\right)$			
			$= \dfrac{\pi}{4}$	A1	Evaluated in terms of π	
				[4]		
1	(a)	(iii)	$\int 1\times\arctan x\,\mathrm{d}x$	M1	Using parts with u = arctan x and v' = 1	Allow one other error
			$= x\arctan x - \int\dfrac{x}{1+x^2}\mathrm{d}x$	A1		
				M1	$\int\dfrac{x}{1+x^2}\mathrm{d}x = a\ln\left(1+x^2\right)$	
			$= x\arctan x - \dfrac{1}{2}\ln\left(1+x^2\right) + c$	A1	$a = \dfrac{1}{2}$. Condone omitted c	
				[4]		

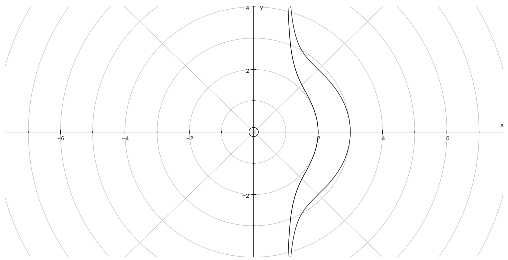
Question			Answer	Marks	Guidance	
1	(b)	(i)	$r = 2\cos\theta \Rightarrow r^2 = 2r\cos\theta$ $\Rightarrow x^2 + y^2 = 2x$ $\Rightarrow (x-1)^2 + y^2 = 1$	M1 A1 A1(ag)	Using $r^2 = x^2 + y^2$ and $x = r\cos\theta$ A correct cartesian equation in any form Explaining that the curve is a circle	 e.g. writing as $(x-\alpha)^2 + (y-\beta)^2 = r^2$
			OR $x = r\cos\theta \Rightarrow x = 2\cos^2\theta$ $y = r\sin\theta \Rightarrow y = 2\cos\theta\sin\theta = \sin 2\theta$ M1 $\cos 2\theta = 2\cos^2\theta - 1 \Rightarrow x = \cos 2\theta + 1$ A1 $\Rightarrow (x-1)^2 + y^2 = 1$ A1(ag)		Using $x = r\cos\theta$, $y = r\sin\theta$ and linking x in terms of $\cos 2\theta$ Explaining that the curve is a circle	 e.g. writing as $(x-\alpha)^2 + (y-\beta)^2 = r^2$
			Centre (1, 0) Radius 1	B1 B1 **[5]**	Independent Independent	
1	(b)	(ii)	$x^2 + (y-2)^2 = 4 \Rightarrow x^2 + y^2 = 4y$ $\Rightarrow r^2 = 4r\sin\theta$ $\Rightarrow r = 4\sin\theta$	 M1 A1 **[2]**	Using $r^2 = x^2 + y^2$ and $y = r\sin\theta$	For answer alone www: B1 for $r = k\sin\theta$, B1 for $k = 4$
2	(a)	(i)	$1 + e^{j2\theta} = 1 + \cos 2\theta + j\sin 2\theta$ $= 1 + (2\cos^2\theta - 1) + 2j\sin\theta\cos\theta$ $= 2\cos^2\theta + 2j\sin\theta\cos\theta$ $= 2\cos\theta(\cos\theta + j\sin\theta)$	 M1 A1(ag)	Using $e^{2j\theta} = \cos 2\theta + j\sin 2\theta$ and double angle formulae Completion www	Allow one error
			OR $1 + e^{j2\theta} = e^{j\theta}(e^{-j\theta} + e^{j\theta})$ M1 $= (\cos\theta + j\sin\theta) \times 2\cos\theta$ A1(ag)		"Factorising" and complete replacement by trigonometric functions Completion www	
			OR $1 + e^{j2\theta} = 1 + (\cos\theta + j\sin\theta)^2$ $= 1 + \cos^2\theta - \sin^2\theta + 2j\sin\theta\cos\theta$ $= 2\cos^2\theta + 2j\sin\theta\cos\theta$ M1 $= 2\cos\theta(\cos\theta + j\sin\theta)$ A1(ag)		Using $e^{j\theta} = \cos\theta + j\sin\theta$ and $1 - \sin^2\theta = \cos^2\theta$ Completion www	
				[2]		

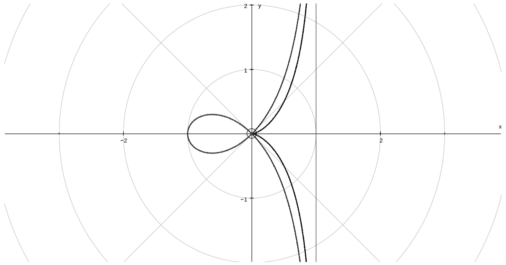
Question			Answer	Marks	Guidance	
2	(a)	(ii)	$C+\mathrm{j}S=1+\binom{n}{1}e^{\mathrm{j}2\theta}+\binom{n}{2}e^{\mathrm{j}4\theta}+\ldots+e^{\mathrm{j}2n\theta}$	M1	Forming $C+\mathrm{j}S$	
			$=\left(1+e^{\mathrm{j}2\theta}\right)^n$	M1 A1	Recognising as binomial expansion	
			$=2^n\cos^n\theta\left(\cos\theta+\mathrm{j}\sin\theta\right)^n$			
			$=2^n\cos^n\theta\left(\cos n\theta+\mathrm{j}\sin n\theta\right)$	M1 A1	Applying (i) and De Moivre o.e.	Dependent on M1M1 above
			$\Rightarrow C=2^n\cos^n\theta\cos n\theta$	A1(ag)	Completion www	Need to see $e^{\mathrm{j}n\theta}=\cos n\theta+\mathrm{j}\sin n\theta$ o.e.
			and $S=2^n\cos^n\theta\sin n\theta$	A1 **[7]**		
2	(b)	(i)	$e^{\mathrm{j}\frac{2\pi}{3}}=\cos\frac{2\pi}{3}+\mathrm{j}\sin\frac{2\pi}{3}=-\frac{1}{2}+\mathrm{j}\frac{\sqrt{3}}{2}$	B1 **[1]**	Must evaluate trigonometric functions	
2	(b)	(ii)	Other two vertices are $(2+4\mathrm{j})e^{\mathrm{j}\frac{2\pi}{3}}$	M1	Award for idea of rotation by $\frac{2\pi}{3}$	e.g. use of $\arctan 2+\frac{2\pi}{3}$ (3.202 rad) (must be 2)
			$=(2+4\mathrm{j})\left(-\frac{1}{2}+\mathrm{j}\frac{\sqrt{3}}{2}\right)$			
			$=\left(-1-2\sqrt{3}\right)+\mathrm{j}\left(-2+\sqrt{3}\right)$	A1A1	May be given as co-ordinates	
			and $(2+4\mathrm{j})e^{\mathrm{j}\frac{4\pi}{3}}=(2+4\mathrm{j})e^{-\mathrm{j}\frac{2\pi}{3}}$	M1	Award for idea of rotation by $-\frac{2\pi}{3}$	e.g. use of $\arctan 2+\frac{4\pi}{3}$ (5.296 rad) (must be 2)
			$=(2+4\mathrm{j})\left(-\frac{1}{2}-\mathrm{j}\frac{\sqrt{3}}{2}\right)$			
			$=\left(-1+2\sqrt{3}\right)+\mathrm{j}\left(-2-\sqrt{3}\right)$	A1A1 **[6]**	May be given as co-ordinates	If A0A0A0A0 award SC1 for awrt −4.46 − 0.27j and 2.46 − 3.73j

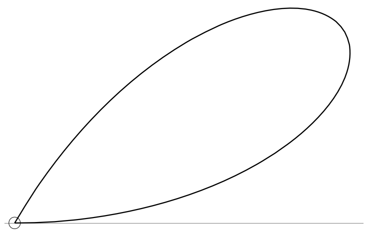
Question			Answer	Marks	Guidance	
2	(b)	(iii)	Length of (2 + 4j) = $\sqrt{20}$ So length of side = $2\sqrt{20}\cos\frac{\pi}{6} = 2\sqrt{20}\times\frac{\sqrt{3}}{2}$ $= 2\sqrt{15}$	M1 A1(ag) **[2]**	Complete method Completion www	Alternative: finding distance between (2, 4) and $\left(-1-2\sqrt{3}, -2+\sqrt{3}\right)$ o.e.
3	(i)		$\mathbf{M}-\lambda\mathbf{I}=\begin{pmatrix}1-\lambda & 3 & 0\\ 3 & -2-\lambda & -1\\ 0 & -1 & 1-\lambda\end{pmatrix}$ $\det(\mathbf{M}-\lambda\mathbf{I})$ $=(1-\lambda)\left[(-2-\lambda)(1-\lambda)-1\right]-3\left[3(1-\lambda)\right]$ $=(1-\lambda)\left(\lambda^2+\lambda-3\right)-9(1-\lambda)$ $\Rightarrow \lambda^3-13\lambda+12=0$	M1 A1 A1(ag) **[3]**	Forming $\det(\mathbf{M}-\lambda\mathbf{I})$ Any correct form Condone omission of 0	Sarrus: $(1-\lambda)^2(-2-\lambda)-10(1-\lambda)$ or e.g. $\lambda-1+(1-\lambda)\left(\lambda^2+\lambda-11\right)$
3	(ii)		$(\lambda-1)\left(\lambda^2+\lambda-12\right)=0$ $\Rightarrow(\lambda-1)(\lambda-3)(\lambda+4)=0$ $\Rightarrow$ eigenvalues are 1, 3, −4 $\lambda=1: \begin{pmatrix}0 & 3 & 0\\ 3 & -3 & -1\\ 0 & -1 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$ $\Rightarrow y=0, 3x-z=0$ $\Rightarrow$ eigenvector is $\begin{pmatrix}1\\ 0\\ 3\end{pmatrix}$ $\lambda=3: \begin{pmatrix}-2 & 3 & 0\\ 3 & -5 & -1\\ 0 & -1 & -2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$ $\Rightarrow -2x+3y=0, -y-2z=0$	M1 A1 A1 M2 M1 A1 A1 A1	Factorising as far as quadratic <u>For any one of λ = 1, 3, −4</u> Obtaining two independent equations Obtaining a non-zero eigenvector o.e. o.e.	Allow one error From which an eigenvector could be found Allow e.g. $3y=0, 3x-3y-z=0$

Question			Answer	Marks	Guidance	
			$\Rightarrow y=-2z,\ x=-3z$ $\Rightarrow$ eigenvector is $\begin{pmatrix}-3\\-2\\1\end{pmatrix}$	A1		
			$\lambda=-4:\ \begin{pmatrix}5&3&0\\3&2&-1\\0&-1&5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\Rightarrow 5x+3y=0,\ -y+5z=0$	A1	o.e.	
			$\Rightarrow y=5z,\ x=-3z$ $\Rightarrow$ eigenvector is $\begin{pmatrix}-3\\5\\1\end{pmatrix}$	A1		
				[12]		
3	(iii)		E.g. $\mathbf{P}=\begin{pmatrix}1&-3&-3\\0&-2&5\\3&1&1\end{pmatrix}$	B1	Use of eigenvectors (ft) as columns	
			$\mathbf{D}=\begin{pmatrix}1&0&0\\0&3^n&0\\0&0&(-4)^n\end{pmatrix}$	M1	Use of 1, 3, −4 (ft) in correct order	n not required for M1
				A1	Power n	-4^n A0
				[3]		

Question			Answer	Marks	Guidance	
4	(i)		$y = 3\sinh x - 2\cosh x$			$\frac{1}{2}e^x - \frac{5}{2}e^{-x}$
			$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 3\cosh x - 2\sinh x$	B1		$\frac{1}{2}e^x + \frac{5}{2}e^{-x}$
			At TPs, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \tanh x = \frac{3}{2}$	M1	Considering $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$	$e^{2x} = -5$; $e^x > 0$ and $e^{-x} > 0$
			which has no (real) solutions	A1(ag)	Showing no real roots www	
			$y = 0 \Rightarrow \tanh x = \frac{2}{3}$	M1	Solving $y = 0$ as far as e^{2x} or tanh x etc.	$e^{2x} = 5$; $\cosh x = \frac{3}{\sqrt{5}}$; $\sinh x = \frac{2}{\sqrt{5}}$
			$\Rightarrow x = \frac{1}{2}\ln\frac{1+\frac{2}{3}}{1-\frac{2}{3}}$	M1	Solving as far as x	Attempt to verify Award M1 for substituting $x = \frac{1}{2}\ln 5$
			$\Rightarrow x = \frac{1}{2}\ln 5$	A1(ag)	Completion www	and M1 for clearly attempting to evaluate exactly
			$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 3\sinh x - 2\cosh x = y$			
			so $y = 0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$	B1(ag)		$3\sinh\left(\frac{1}{2}\ln 5\right) - 2\cosh\left(\frac{1}{2}\ln 5\right) = 0$ must be explained, e.g. connected with $y = 0$
				[7]		
4	(ii)			B2	For a curve with the following features: • increasing • intersecting the positive x-axis • $(0, -2)$ indicated • gradient increasing with large $\|x\|$ • one point of inflection	Award B1 for a curve lacking one of these features
				[2]		

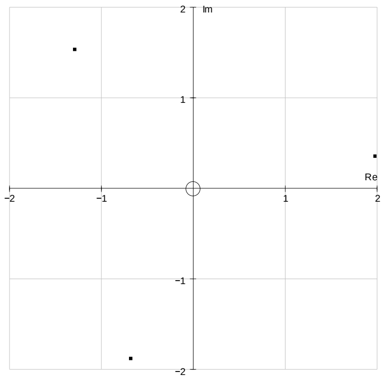
Question			Answer	Marks	Guidance	
4	(iii)		$(3\sinh x - 2\cosh x)^2$			
			$= 9\sinh^2 x - 12\sinh x\cosh x + 4\cosh^2 x$	B1		
			$= \frac{9}{2}(\cosh 2x - 1) - 6\sinh 2x + 2(\cosh 2x + 1)$	M1	Using double "angle" formulae or complete alternative	Condone sign errors but need $\frac{1}{2}$ s
			$= \frac{13}{2}\cosh 2x - 6\sinh 2x - \frac{5}{2}$	A1	Accept unsimplified	$\frac{1}{4}e^{2x} + \frac{25}{4}e^{-2x} - \frac{5}{2}$
			$V = \pi\int_0^{\frac{1}{2}\ln 5} y^2\,dx$	M1	Attempting to integrate their y^2 (ignore limits)	
			$= \pi\left[\frac{13}{4}\sinh 2x - 3\cosh 2x - \frac{5}{2}x\right]_0^{\frac{1}{2}\ln 5}$	A2	Correct results and limits c.a.o. Ignore omitted π	Give A1 for one error, or for all three terms correct and incorrect limits
			$= \pi\left[\frac{13}{4}\times\frac{12}{5} - 3\times\frac{13}{5} - \frac{5}{4}\ln 5 + 3\right]$	M1 M1	Substituting both of their limits Obtaining exact values of $\sinh(\ln 5)$ and $\cosh(\ln 5)$	$\sinh(\ln 5) = \frac{12}{5}$, $\cosh(\ln 5) = \frac{13}{5}$
			OR $= \pi\left[\frac{1}{8}e^{2x} - \frac{25}{8}e^{-2x} - \frac{5}{2}x\right]_0^{\frac{1}{2}\ln 5}$ A2		Correct results and limits	Give A1 for one error, or for all three terms correct and incorrect limits
			$= \pi\left[\frac{5}{8} - \frac{5}{8} - \frac{5}{4}\ln 5 + 3\right]$ M1 M1		Substituting both of their limits Obtaining exact values of e^{2x} and e^{-2x}	$e^{2x} = 5$, $e^{-2x} = \frac{1}{5}$
			$= \pi\left[3 - \frac{5}{4}\ln 5\right]$	A1(ag) **[9]**	Completion www	
5	(i)			B2 B1 **[3]**	Three curves of correct shape Correctly identified	Give B1 for two correct curves $a = 0$, $a = 1$, $a = 2$ from left to right

Question			Answer	Marks	Guidance	
5	(ii)			B1 B1 **[2]**	Curve for $a = -1$ Curve for $a = -2$	Curve with cusp Curve with loop
5	(iii)		Asymptote	B1 **[1]**		
5	(iv)		$a = -1$: cusp $a = -2$: loop	B1 B1 **[2]**		
5	(v)		$r = \sec\theta + a\cos\theta \Rightarrow r\cos\theta = 1 + a\cos^2\theta$ $\Rightarrow x = 1 + a\left(\dfrac{x^2}{r^2}\right)$ $\Rightarrow x - 1 = a\left(\dfrac{x^2}{x^2+y^2}\right)$ $\Rightarrow x^2 + y^2 = a\left(\dfrac{x^2}{x-1}\right) \Rightarrow y^2 = a\left(\dfrac{x^2}{x-1}\right) - x^2$ Hence asymptote at $x = 1$	 M1 M1 M1 A1(ag) B1 **[5]**	 Using $x = r\cos\theta$ Using $r^2 = x^2 + y^2$ Making y^2 subject	
5	(vi)		Curve exists for $y^2 \geq 0$ $\Rightarrow a\left(\dfrac{1}{x-1}\right) - 1 \geq 0$ If $a > 0$ then $x - 1 > 0$ and so $a \geq x - 1$ i.e. $1 < x \leq 1 + a$ If $a < 0$ then $x - 1 < 0$ and so $a \leq x - 1$ i.e. $1 + a \leq x < 1$	 M1 M1 A1(ag) M1 A1 **[5]**	 Considering $y^2 \geq 0$	

Question			Answer	Marks	Guidance	
1	(a)		$f(x)=(1-2x)^{-2}$ $\Rightarrow f'(x)=-2(1-2x)^{-3}\times -2=4(1-2x)^{-3}$ $\Rightarrow f''(x)=24(1-2x)^{-4}$ $\Rightarrow f'''(x)=192(1-2x)^{-5}$ $\Rightarrow f(0)=1, f'(0)=4,$ $f''(0)=24, f'''(0)=192$ $\Rightarrow f(x)=1+4x+\frac{24}{2!}x^2+\frac{192}{3!}x^3+\ldots$ $\Rightarrow f(x)=1+4x+12x^2+32x^3+\ldots$ Valid for $-1<2x<1$ $\Rightarrow -\frac{1}{2}<x<\frac{1}{2}$	M1 A1 A1 A1 M1 A1 B1 **[7]**	Derivative in the form $k(1-2x)^{-3}$ o.e. Any correct form www Any correct form www Any correct form www Using Maclaurin series with derivatives evaluated at $x = 0$ Strict inequalities	For first derivative Must have $r!$ in denominator SR: after M0M0 B2 for correct binomial
1	(b)	(i)		B2 **[2]**	For a complete loop correct at the origin and at the extremity	Ignore beyond $0 \le \theta \le \pi/3$. Incomplete loop B0. Give B1 for wrong shape at one of origin or extremity
1	(b)	(ii)	$\theta=\frac{\pi}{6}$ $r=a$ $\Rightarrow x=a\cos\frac{\pi}{6}=\frac{a\sqrt{3}}{2}$ and $y=a\sin\frac{\pi}{6}=\frac{a}{2}$	B1 B1 M1 A1 **[4]**	s.o.i. s.o.i. Using $x=r\cos\theta$ and $y=r\sin\theta$ with a value of θ Both. Condone $0.87a$	

Question			Answer	Marks	Guidance	
1	(b)	(iii)	$A=\int_0^{\frac{\pi}{3}}\frac{1}{2}a^2\sin^2 3\theta\,\mathrm{d}\theta$	M1 A1	An integral expression including $\sin^2 3\theta$ Correct integral expression with limits	 Limits may be inserted below
			$=\frac{1}{4}a^2\int_0^{\frac{\pi}{3}}1-\cos 6\theta\,\mathrm{d}\theta$	M1	Using $\sin^2 3\theta=\frac{1}{2}-\frac{1}{2}\cos 6\theta$ and attempting integration. Dep. on 1st M1	Allow sign and factor errors, but must be cos 6θ
			$=\frac{1}{4}a^2\left[\theta-\frac{1}{6}\sin 6\theta\right]_0^{\frac{\pi}{3}}$	A1	Correct result of integration	i.e. $\int\sin^2 3\theta\,\mathrm{d}\theta=\frac{1}{2}\theta-\frac{1}{12}\sin 6\theta$
			$=\frac{1}{12}\pi a^2$	A1	Dependent on previous A1	Allow awrt $0.26a^2$
				[5]		

Question			Answer	Marks	Guidance	
2	(a)	(i)	$\cos 5\theta + \text{j}\sin 5\theta = (\cos\theta + \text{j}\sin\theta)^5$			
			$= c^5 + 5c^4\text{j}s + 10c^3\text{j}^2s^2 + 10c^2\text{j}^3s^3 + 5c\text{j}^4s^4 + \text{j}^5s^5$	M1	Expanding $(c + \text{j}s)^5$ (real terms only)	Allow one error. Must get beyond ${}^5C_2$. Must collect terms
			$= c^5 - 10c^3s^2 + 5cs^4 + \text{j}(5c^4s - 10c^2s^3 + s^5)$			
			$\Rightarrow \cos 5\theta = c^5 - 10c^3(1-c^2) + 5c(1-c^2)^2$	M1	Separating real part and replacing s^2 with $1-c^2$	Independent of M1
			$= c^5 - 10c^3 + 10c^5 + 5c(1 - 2c^2 + c^4)$			
			$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$	A1(ag) **[3]**	Completion www in real part	
2	(a)	(ii)	$\theta = 18^\circ \Rightarrow \cos 5\theta = 0$ *			
			$\Rightarrow 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$			
			$\cos\theta \neq 0 \Rightarrow 16\cos^4\theta - 20\cos^2\theta + 5 = 0$	B1	This equation s.o.i.	
			$\Rightarrow \cos^2\theta = \dfrac{20 \pm \sqrt{20^2 - 4\times16\times5}}{2\times16}$	M1 A1	Solving a 3-term quadratic Unsimplified values of $\cos^2\theta$	Allow one error
			$\Rightarrow \cos\theta = \pm\left(\dfrac{5+\sqrt{5}}{8}\right)^{\frac{1}{2}}$ or $\pm\left(\dfrac{5-\sqrt{5}}{8}\right)^{\frac{1}{2}}$			SC Answers unsupported www B1
			$\cos 18^\circ$ is closest to 1 $\Rightarrow \cos 18^\circ = \left(\dfrac{5+\sqrt{5}}{8}\right)^{\frac{1}{2}}$	A1(ag)	Justifying selection of this root	To include *
			$\cos^2 18^\circ + \sin^2 18^\circ = 1$	M1	Using $\cos^2\theta + \sin^2\theta = 1$	
			$\Rightarrow \dfrac{5+\sqrt{5}}{8} + \sin^2 18^\circ = 1$			
			$\Rightarrow \sin^2 18^\circ = \dfrac{3-\sqrt{5}}{8}$ and $\sin 18^\circ > 0$			
			$\Rightarrow \sin 18^\circ = \left(\dfrac{3-\sqrt{5}}{8}\right)^{\frac{1}{2}}$	A1	Must have this form	
				[6]		

Question			Answer	Marks	Guidance	
2	(b)	(i)	$4\sqrt{3}+4\text{j}=8e^{\text{j}\frac{\pi}{6}}$	B1B1	8, $\frac{\pi}{6}$	Condone decimal equivalents for arguments throughout (to 2 s.f.).
			Cube roots are $re^{\text{j}\theta}$			
			$r^3=8\Rightarrow r=2$	B1ft	$\sqrt[3]{\text{their }8}$	
			$3\theta=\frac{\pi}{6}\Rightarrow\theta=\frac{\pi}{18}$	B1ft	$\frac{1}{3}$ of their $\frac{\pi}{6}$	Radians only
			$\pm\frac{2\pi}{3}$	M1		
			$\Rightarrow\quad\theta=\frac{\pi}{18},\frac{13\pi}{18},\frac{25\pi}{18}$	A1	Accept $-\frac{11\pi}{18}$	Radians only
				B1 [7]	Approx. order 3 rotational symmetry. 1^{st} root in $0 < \arg z < \pi/4$ 2^{nd} root in 2^{nd} quadrant 3^{rd} root in $5\pi/4 < \arg z < 3\pi/2$	Ignore numbers etc. on diagram
2	(b)	(ii)	$\arg w=\frac{1}{2}\left(\frac{\pi}{18}+\frac{13\pi}{18}\right)=\frac{7\pi}{18}$	B1		
			$n=18$	B1 [2]		

Question			Answer	Marks	Guidance	
3	(i)		$\det(\mathbf{A}) = k(4+9)+7(-4-3)+4(-6+2)$	M1A1	Obtaining $\det(\mathbf{A})$ in terms of k	Allow one error. isw unsimplified $65 - 13k$ M1A0 and allows B1 below
			$=13k-65$			
			$\Rightarrow$ no inverse if $k = 5$	B1(ag)	May be verified separately	
			$\mathbf{A}^{-1} = \frac{1}{13k-65}\begin{pmatrix} 13 & -26 & -13 \\ 7 & -2k-4 & -3k+8 \\ -4 & 3k-7 & -2k+14 \end{pmatrix}$	M1	At least 4 cofactors correct (including one involving k)	M0 if more than 1 is multiplied by the corresponding element
				A1	Six signed cofactors correct	
				M1	Transposing and ÷ by $\det(\mathbf{A})$. Dependent on previous M1M1	
				A1		Mark final answer
				[7]		
3	(ii)		When $k = 4$,	M1	Substituting $k = 4$	
			$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-13}\begin{pmatrix} 13 & -26 & -13 \\ 7 & -12 & -4 \\ -4 & 5 & 6 \end{pmatrix}\begin{pmatrix} p \\ 1 \\ 2 \end{pmatrix}$	M2	Correct use of inverse	One correct element. Condone missing determinant. M0 if wrong order
			OR e.g. $\begin{matrix} 6x-13y=p+4 \\ 4x-13y=3p-4 \end{matrix} \Rightarrow x=-p+4$ M2		Eliminating one unknown in two different ways and reaching one unknown in terms of p	
			M1		Finding the other two unknowns	
			$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-13}\begin{pmatrix} 13p-52 \\ 7p-20 \\ -4p+17 \end{pmatrix}$	A2	Dependent on all M marks. Terms must be collected. Give A1 for one correct	$x=-p+4$, $y=-\frac{7}{13}p+\frac{20}{13}$, $z=\frac{4}{13}p-\frac{17}{13}$ $\lambda \times$ correct vector ($\lambda \neq 0$) A1
				[5]		

Question			Answer	Marks	Guidance	
3	(iii)		e.g. $7x-13y=p+4, 7x-13y=3p-4$ (or $4x+13z=7-2p, 4x+13z=-1$) (or $8y+14z=p-10, 4y+7z=-3$) For solutions, $p+4=3p-4$ $\Rightarrow \quad p=4$	M2 A1	Eliminating one unknown in two different ways & obtaining a value of p	Or $7x-13y=8$ Or $8x+26z=3p-14$ Or $4y+7z=5-2p$
			OR M2 $p=4$ A1		A method leading to an equation from which p could be found	E.g. setting z = 0, augmented matrix, adjoint matrix, etc.
			$x=\lambda,\ y=\frac{7}{13}\lambda-\frac{8}{13},\ z=-\frac{4}{13}\lambda-\frac{1}{13}$ Straight line	M1 A1 B1 **[6]**	Obtaining general soln. by e.g. setting one unknown = λ and finding equations involving the other two and λ Any correct form Accept "sheaf", "pages of a book", etc.	Accept unknown instead of λ $x=\frac{13}{7}\lambda+\frac{8}{7},\ y=\lambda,\ z=-\frac{4}{7}\lambda-\frac{3}{7}$ $x=-\frac{13}{4}\lambda-\frac{1}{4},\ y=-\frac{7}{4}\lambda-\frac{3}{4},\ z=\lambda$ Independent of all previous marks. Ignore other comments

Question			Answer	Marks	Guidance	
4	(i)		$\cosh u = \frac{e^u + e^{-u}}{2} \Rightarrow \cosh^2 u = \frac{e^{2u} + 2 + e^{-2u}}{4}$			
			$\sinh u = \frac{e^u - e^{-u}}{2} \Rightarrow \sinh^2 u = \frac{e^{2u} - 2 + e^{-2u}}{4}$	B1	Numerators of both expressions	Accept other variables
			$\Rightarrow \cosh^2 u - \sinh^2 u = 1$	B1(ag)	Completion www	
			OR $\cosh u + \sinh u = e^u$ $\cosh u - \sinh u = e^{-u}$ $\Rightarrow \cosh^2 u - \sinh^2 u = e^u \times e^{-u}$ B1 $\Rightarrow \cosh^2 u - \sinh^2 u = 1$ B1(ag)		Both expressions s.o.i. and multiplication Completion www	
				[2]		
4	(ii)		$y = \text{arsinh}\, x \Rightarrow x = \sinh y$			
			$\Rightarrow \frac{dx}{dy} = \cosh y$	M1	$\sinh y = \ldots$ and differentiating w.r.t. y or x	Or $\cosh y \frac{dy}{dx} = 1$ or differentiating (*)
			$\Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$	A1	o.e.	
			$\Rightarrow \frac{dy}{dx} = (\pm)\frac{1}{\sqrt{1+\sinh^2 y}} = (\pm)\frac{1}{\sqrt{1+x^2}}$	A1(ag)	Completion www with valid intermediate step	$\frac{dy}{dx} = \pm\frac{1}{\sqrt{1+x^2}}$ as final answer or $\pm$ not considered scores max. 3/4
			y is an increasing function so take + sign	B1	Validly rejecting negative value	Or $\cosh y \geq 1$, or $\cosh y > 0$
			$x = \sinh y \Rightarrow x = \frac{e^y - e^{-y}}{2}$	B1	x in exponential form	
			$\Rightarrow e^y - e^{-y} = 2x$ $\Rightarrow e^{2y} - 2xe^y - 1 = 0$ $\Rightarrow (e^y - x)^2 = 1 + x^2$	M1	Obtaining quadratic in e^y	
			$\Rightarrow e^y = x \pm \sqrt{1+x^2}$	M1	Solving to reach e^y . Dep. on M1 above	Allow one slip
			$\Rightarrow y = \ln\left(x(\pm)\sqrt{1+x^2}\right)$ (*)	A1(ag)	Completion www	
			$x - \sqrt{1+x^2} < 0$ so take + sign	B1	Validly rejecting negative root	e.g. $e^y > 0$
				[9]		

Question			Answer	Marks	Guidance	
4	(iii)		$\int_0^2 \frac{1}{\sqrt{4+9x^2}}\,dx = \frac{1}{3}\int_0^2 \frac{1}{\sqrt{\frac{4}{9}+x^2}}\,dx$			
			$= \frac{1}{3}\left[\text{arsinh}\,\frac{3x}{2}\right]_0^2$	M1 A1A1	Integral involving arsinh $\frac{1}{3}$, $\frac{3x}{2}$ o.e.	
			$= \frac{1}{3}\text{arsinh}3$			
			OR $= \frac{1}{3}\left[\ln\left(x+\sqrt{x^2+\frac{4}{9}}\right)\right]_0^2$ M1 A1A1		Integral in form $\ln\left(kx+\sqrt{k^2x^2+...}\right)$ $\frac{1}{3}$, $x+\sqrt{x^2+\frac{4}{9}}$ or $3x+\sqrt{9x^2+4}$	Or $\frac{3x}{2}+\sqrt{\frac{9x^2}{4}+1}$
			OR $x = \frac{2}{3}\sinh u \Rightarrow \frac{dx}{du} = \frac{2}{3}\cosh u$ M1 A1 $\int_0^2 \frac{1}{\sqrt{4+9x^2}}\,dx = \int_0^{\ln(3+\sqrt{10})} \frac{1}{3}\,du$ A1		Using a sinh substitution Correct substitution $\int \frac{1}{3}\,du$	
			$= \frac{1}{3}\ln\left(3+\sqrt{10}\right)$	A1(ag) **[4]**	Completion with valid intermediate step(s)	Condone omitted brackets

Question			Answer	Marks	Guidance	
4	(iv)		$\int_0^1 \frac{1}{\sqrt{1+x^2}} \operatorname{arsinh} x \, dx$			
			$= \left[(\operatorname{arsinh} x)^2 \right]_0^1 - \int_0^1 \frac{1}{\sqrt{1+x^2}} \operatorname{arsinh} x \, dx$	M1	Parts with $u = \operatorname{arsinh} x$, $v' = \frac{1}{\sqrt{1+x^2}}$	Allow one error Allow equivalent form
			OR $\int \frac{1}{\sqrt{1+x^2}} \operatorname{arsinh} x \, dx = \int u \, du$ M1		Substitution with u = arsinh x or x = sinh u	Must reach $\int u \, du$
			OR inspection M1		Recognising integrand as $k(\operatorname{arsinh} x)^2$	$k \neq 0$
			$\Rightarrow \frac{1}{2}(\operatorname{arsinh} x)^2$	A1	A correct indefinite integrand	
			$\Rightarrow I = \frac{1}{2}\left(\ln\left(1+\sqrt{2}\right)\right)^2$	A1	This answer only	Mark final answer
				[3]		