FP2 questions from old P4, P5, P6 and FP1, FP2, FP3 papers (back to June 2002)

The following pages contain questions from past papers which could conceivably appear on Edexcel's new FP2 papers from June 2009 onwards.

Mark schemes are available on a separate document, originally sent with this one.

1. Find the set of values for which

$$|x-1| > 6x-1.$$

(5)

[P4 January 2002 Qn 2]

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2. (*a*) Find the general solution of the differential equation

$$t\frac{\mathrm{d}v}{\mathrm{d}t} - v = t, \quad t > 0$$

and hence show that the solution can be written in the form $v = t(\ln t + c)$, where c is an arbitrary constant.

(6)

(*b*) This differential equation is used to model the motion of a particle which has speed v m s^{-1} at time t s. When $t = 2$ the speed of the particle is 3 m s^{-1}. Find, to 3 significant figures, the speed of the particle when $t = 4$.

(4)

[P4 January 2002 Qn 6]

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3. (*a*) Show that $y = \frac{1}{2}x^2\mathrm{e}^x$ is a solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = \mathrm{e}^x.$$

(4)

(*b*) Solve the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = \mathrm{e}^x.$$

given that at $x = 0$, $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$.

(9)

[P4 January 2002 Qn 7]

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4. The curve C has polar equation $r = 3a \cos \theta$, $-\frac{\pi}{2} \leq \frac{\pi}{2}$. The curve D has polar equation $r = a(1 + \cos \theta)$, $-\pi \leq \theta < \pi$. Given that a is a positive constant,

(a) sketch, on the same diagram, the graphs of C and D, indicating where each curve cuts the initial line.

(4)

The graphs of C intersect at the pole O and at the points P and Q.

(b) Find the polar coordinates of P and Q.

(3)

(c) Use integration to find the exact value of the area enclosed by the curve D and the lines $\theta = 0$ and $\theta = \frac{\pi}{3}$.

(7)

The region R contains all points which lie outside D and inside C.

Given that the value of the smaller area enclosed by the curve C and the line $\theta = \frac{\pi}{3}$ is

$$\frac{3a^2}{16}(2\pi - 3\sqrt{3}),$$

(d) show that the area of R is πa^2.

(4)

[P4 January 2002 Qn 8]

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5. Using algebra, find the set of values of x for which

$$2x - 5 > \frac{3}{x}.$$

(7)

[P4 June 2002 Qn 4]

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6. (a) Find the general solution of the differential equation

$$\cos x \frac{dy}{dx} + (\sin x)y = \cos^3 x.$$

(6)

(b) Show that, for $0 \leq x \leq 2\pi$, there are two points on the x-axis through which all the solution curves for this differential equation pass.

(2)

(c) Sketch the graph, for $0 \leq x \leq 2\pi$, of the particular solution for which $y = 0$ at $x = 0$.

(3)

[P4 June 2002 Qn 6]

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7. (*a*) Find the general solution of the differential equation

$$2\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 3y = 3t^2 + 11t.$$

(8)

(*b*) Find the particular solution of this differential equation for which $y = 1$ and $\frac{dy}{dt} = 1$ when $t = 0$.

(5)

(*c*) For this particular solution, calculate the value of y when $t = 1$.

(1)

[P4 June 2002 Qn 7]

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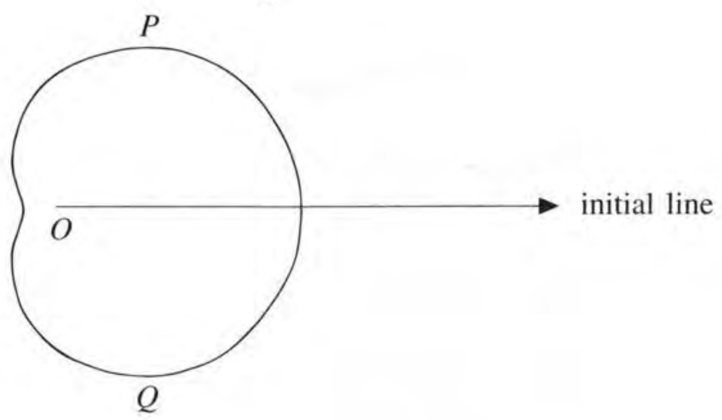
8. **Figure 1**



The curve C shown in Fig. 1 has polar equation

$$r = a(3 + \sqrt{5}\cos\theta), \quad -\pi \leq \theta < \pi.$$

(*a*) Find the polar coordinates of the points P and Q where the tangents to C are parallel to the initial line.

(6)

The curve C represents the perimeter of the surface of a swimming pool. The direct distance from P to Q is 20 m.

(*b*) Calculate the value of a.

(3)

(*c*) Find the area of the surface of the pool.

(6)

[P4 June 2002 Qn 8]

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9. (*a*) The point P represents a complex number z in an Argand diagram. Given that

$$|z - 2\text{i}| = 2|z + \text{i}|,$$

(i) find a cartesian equation for the locus of P, simplifying your answer.

(2)

(ii) sketch the locus of P.

(3)

(*b*) A transformation T from the z-plane to the w-plane is a translation $-7 + 11\text{i}$ followed by an enlargement with centre the origin and scale factor 3.

Write down the transformation T in the form

$$w = az + b, \quad a, b \in \mathbb{C}.$$

(2)

[P6 June 2002 Qn 3]

10.

$$y\frac{\text{d}^2y}{\text{d}x^2} + \left(\frac{\text{d}y}{\text{d}x}\right)^2 + y = 0.$$

(*a*) Find an expression for $\frac{\text{d}^3y}{\text{d}x^3}$.

(5)

Given that $y = 1$ and $\frac{\text{d}y}{\text{d}x} = 1$ at $x = 0$,

(*b*) find the series solution for y, in ascending powers of x, up to an including the term in x^3.

(5)

(*c*) Comment on whether it would be sensible to use your series solution to give estimates for y at $x = 0.2$ and at $x = 50$.

(2)

[P6 June 2002 Qn 4]

11.

$$z = 4\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right), \text{ and } w = 3\left(\cos\frac{2\pi}{3} + \text{i}\sin\frac{2\pi}{3}\right).$$

Express zw in the form $r(\cos\theta + \text{i}\sin\theta)$, $r > 0$, $-\pi < \theta < \pi$.

(3)

[P4 January 2003 Qn 1]

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12. (*a*) Express $\dfrac{2}{(r+1)(r+3)}$ in partial fractions.

(2)

(*b*) Hence prove that $\displaystyle\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} \equiv \frac{n(5n+13)}{6(n+2)(n+3)}$.

(5)

[P4 January 2003 Qn 3]

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13. (*a*) Sketch, on the same axes, the graphs with equation $y = |2x - 3|$, and the line with equation $y = 5x - 1$.

(2)

(*b*) Solve the inequality $|2x - 3| < 5x - 1$.

(3)

[P4 January 2003 Qn 2]

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14. (*a*) Use the substitution $y = vx$ to transform the equation

$$\frac{dy}{dx} = \frac{(4x+y)(x+y)}{x^2},\ x > 0 \qquad \text{(I)}$$

into the equation

$$x\frac{dv}{dx} = (2+v)^2. \qquad \text{(II)}$$

(4)

(*b*) Solve the differential equation II to find v as a function of x.

(5)

(*c*) Hence show that

$$y = -2x - \frac{x}{\ln x + c}, \text{ where } c \text{ is an arbitrary constant,}$$

is a general solution of the differential equation I.

(1)

[P4 January 2003 Qn 5]

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15. (*a*) Find the value of λ for which $\lambda x \cos 3x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 9y = -12 \sin 3x.$$

(4)

(*b*) Hence find the general solution of this differential equation.

(4)

The particular solution of the differential equation for which $y = 1$ and $\frac{dy}{dx} = 2$ at $x = 0$, is $y = g(x)$.

(*c*) Find $g(x)$.

(4)

(*d*) Sketch the graph of $y = g(x)$, $0 \leq x \leq \pi$.

(2)

[P4 January 2003 Qn 7]

16. **Figure 1**

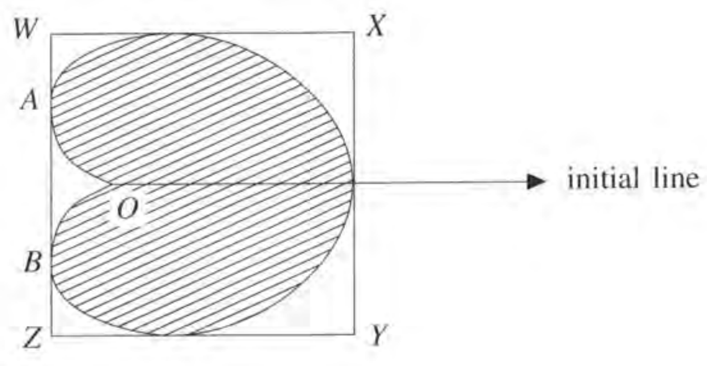

Figure 1 shows a sketch of the cardioid C with equation $r = a(1 + \cos\theta)$, $-\pi < \theta \leq \pi$. Also shown are the tangents to C that are parallel and perpendicular to the initial line. These tangents form a rectangle $WXYZ$.

(*a*) Find the area of the finite region, shaded in Fig. 1, bounded by the curve C.

(6)

(*b*) Find the polar coordinates of the points A and B where WZ touches the curve C.

(5)

(*c*) Hence find the length of WX.

(2)

Given that the length of WZ is $\dfrac{3\sqrt{3}a}{2}$,

(*d*) find the area of the rectangle $WXYZ$.

(1)

A heart-shape is modelled by the cardioid C, where $a = 10$ cm. The heart shape is cut from the rectangular card $WXYZ$, shown in Fig. 1.

(*e*) Find a numerical value for the area of card wasted in making this heart shape.

(2)

[P4 January 2003 Qn 8]

17. (*a*) Express as a simplified fraction $\frac{1}{(r-1)^2} - \frac{1}{r^2}$.

(2)

(*b*) Prove, by the method of differences, that

$$\sum_{r=2}^{n} \frac{2r-1}{r^2(r-1)^2} = 1 - \frac{1}{n^2}.$$

(3)

[P4 June 2003 Qn 1]

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18. Solve the inequality $\frac{1}{2x+1} > \frac{x}{3x-2}$.

(6)

[P4 June 2003 Qn 2]

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19. (*a*) Using the substitution $t = x^2$, or otherwise, find

$$\int x^3 e^{-x^2}\, dx.$$

(6)

(*b*) Find the general solution of the differential equation

$$x\frac{dy}{dx} + 3y = xe^{-x^2}, \quad x > 0.$$

(4)

[P4 June 2003 Qn 6]

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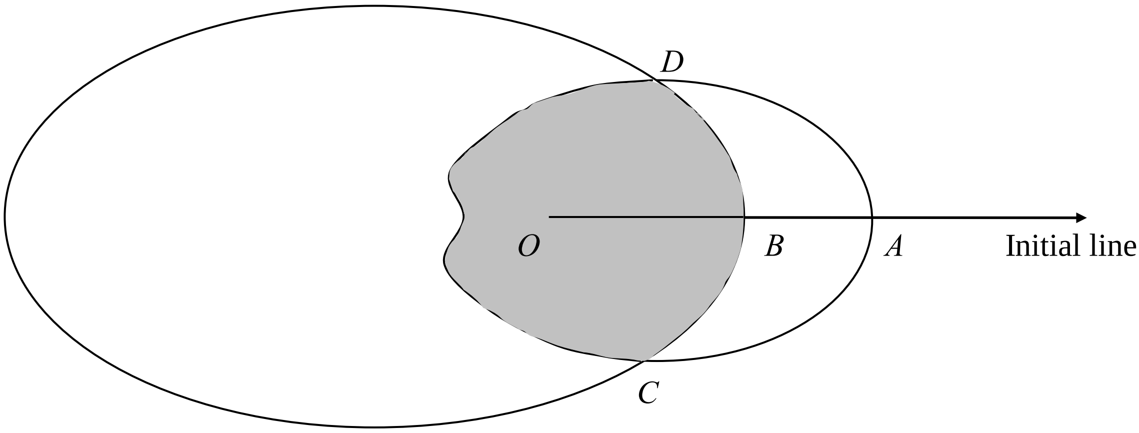
20. **Figure 1**



A logo is designed which consists of two overlapping closed curves.

The polar equations of these curves are

$$r = a(3 + 2\cos\theta) \quad \text{and}$$

$$r = a(5 - 2\cos\theta), \quad 0 \le \theta < 2\pi.$$

Figure 1 is a sketch (not to scale) of these two curves.

(*a*) Write down the polar coordinates of the points *A* and *B* where the curves meet the initial line.

(2)

(*b*) Find the polar coordinates of the points *C* and *D* where the two curves meet.

(4)

(*c*) Show that the area of the overlapping region, which is shaded in the figure, is

$$\frac{a^2}{3}(49\pi - 48\sqrt{3}).$$

(8)

[P4 June 2003 Qn 7]

21.
$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 4e^{3t}, \quad t \geq 0.$$

(*a*) Show that Kt^2e^{3t} is a particular integral of the differential equation, where K is a constant to be found.

(4)

(*b*) Find the general solution of the differential equation.

(3)

Given that a particular solution satisfies $y = 3$ and $\frac{dy}{dt} = 1$ when $t = 0$,

(*c*) find this solution.

(4)

Another particular solution which satisfies $y = 1$ and $\frac{dy}{dt} = 0$ when $t = 0$, has equation

$$y = (1 - 3t + 2t^2)e^{3t}.$$

(*d*) For this particular solution draw a sketch graph of y against t, showing where the graph crosses the t-axis. Determine also the coordinates of the minimum of the point on the sketch graph.

(5)

[P4 June 2003 Qn 8]

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22. (i) (*a*) On the same Argand diagram sketch the loci given by the following equations.

$$|z-1| = 1,$$

$$\arg(z+1) = \frac{\pi}{12},$$

$$\arg(z+1) = \frac{\pi}{2}.$$

(4)

(*b*) Shade on your diagram the region for which

$$|z-1| \leq 1 \quad \text{and} \quad \frac{\pi}{12} \leq \arg(z+1) \leq \frac{\pi}{2}.$$

(1)

(ii) (*a*) Show that the transformation

$$w = \frac{z-1}{z}, \quad z \neq 0,$$

maps $|z-1| = 1$ in the z-plane onto $|w| = |w-1|$ in the w-plane.

(3)

The region $|z-1| \leq 1$ in the z-plane is mapped onto the region T in the w-plane.

(*b*) Shade the region T on an Argand diagram.

(2)

[P6 June 2003 Qn 4]

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23. (*a*) Use de Moivre's theorem to show that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

(6)

(*b*) Hence find 3 distinct solutions of the equation $16x^5 - 20x^3 + 5x + 1 = 0$, giving your answers to 3 decimal places where appropriate.

(4)

[P6 June 2003 Qn 5]

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24. Prove by the method of differences that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1),\ n > 1.$

(6)

[P4 January 2004 Qn 1]

25.

$$\frac{dy}{dx} + y\left(1 + \frac{3}{x}\right) = \frac{1}{x^2}, \quad x > 0.$$

(*a*) Verify that x^3e^x is an integrating factor for the differential equation.

(3)

(*b*) Find the general solution of the differential equation.

(4)

(*c*) Given that $y = 1$ at $x = 1$, find y at $x = 2$.

(3)

[P4 January 2004 Qn 4]

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26. (*a*) Sketch, on the same axes, the graph of $y = |(x-2)(x-4)|$, and the line with equation $y = 6 - 2x$.

(4)

(*b*) Find the exact values of x for which $|(x-2)(x-4)| = 6 - 2x$.

(5)

(*c*) Hence solve the inequality $|(x-2)(x-4)| < 6 - 2x$.

(2)

[P4 January 2004 Qn 5]

27.

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 65 \sin 2x, \quad x > 0.$$

(*a*) Find the general solution of the differential equation.

(9)

(*b*) Show that for large values of x this general solution may be approximated by a sine function and find this sine function.

(2)

[P4 January 2004 Qn 6]

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28. (*a*) Sketch the curve with polar equation

$$r = 3\cos 2\theta, \quad -\frac{\pi}{4} \leq \theta < \frac{\pi}{4}.$$

(2)

(*b*) Find the area of the smaller finite region enclosed between the curve and the half-line $\theta = \frac{\pi}{6}$.

(6)

(*c*) Find the exact distance between the two tangents which are parallel to the initial line.

(8)

[P4 January 2004 Qn 7]

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29. Find the complete set of values of x for which

$$|x^2 - 2| > 2x.$$

(7)

[P4 June 2004 Qn 4]

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30. (*a*) Find the general solution of the differential equation

$$\frac{dy}{dx} + 2y = x.$$

(5)

Given that $y = 1$ at $x = 0$,

(*b*) find the exact values of the coordinates of the minimum point of the particular solution curve,

(4)

(*c*) draw a sketch of this particular solution curve.

(2)

[P4 June 2004 Qn 6]

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31. (*a*) Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 2e^{-t}.$$

(6)

(*b*) Find the particular solution that satisfies $y = 1$ and $\frac{dy}{dt} = 1$ at $t = 0$.

(6)

[P4 June 2004 Qn 7]

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32. **Figure 1**

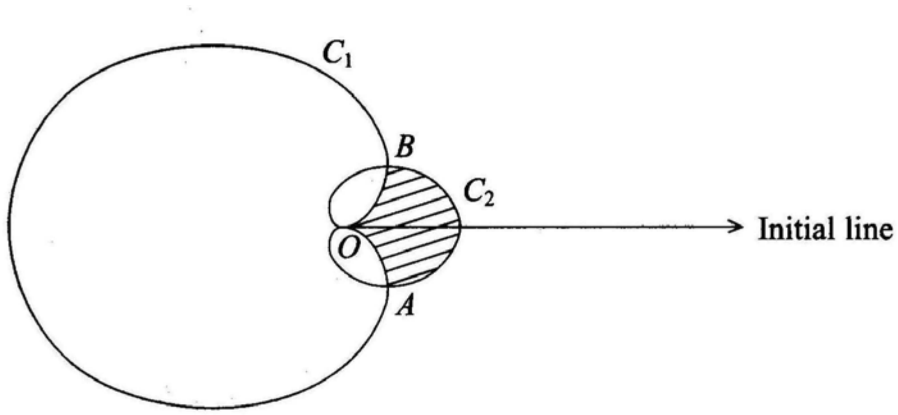

Figure 1 is a sketch of the two curves C_1 and C_2 with polar equations

$$C_1 : r = 3a(1 - \cos\theta), \qquad -\pi \leq \theta < \pi$$

and $$C_2 : r = a(1 + \cos\theta), \qquad -\pi \leq \theta < \pi.$$

The curves meet at the pole O, and at the points A and B.

(a) Find, in terms of a, the polar coordinates of the points A and B.

(4)

(b) Show that the length of the line AB is $\frac{3\sqrt{3}}{2}a$.

(2)

The region inside C_2 and outside C_1 is shown shaded in Fig. 1.

(c) Find, in terms of a, the area of this region.

(7)

A badge is designed which has the shape of the shaded region.

Given that the length of the line AB is 4.5 cm,

(d) calculate the area of this badge, giving your answer to three significant figures.

(3)

[P4 June 2004 Qn 8]

33. Given that $y = \tan x$,

(a) find $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ and $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3}$.

(3)

(b) Find the Taylor series expansion of $\tan x$ in ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to and including the term in $\left(x - \frac{\pi}{4}\right)^3$.

(3)

(c) Hence show that $\tan \frac{3\pi}{10} \approx 1 + \frac{\pi}{10} + \frac{\pi^2}{200} + \frac{\pi^3}{3000}$.

(2)

[P6 June 2004 Qn 2]

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34. (a) Prove by induction that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^x \cos x) = 2^{\frac{1}{2}n} \mathrm{e}^x \cos (x + \tfrac{1}{4} n\pi), \qquad n \geq 1.$$

(8)

(b) Find the Maclaurin series expansion of $\mathrm{e}^x \cos x$, in ascending powers of x, up to and including the term in x^4.

(3)

[P6 June 2004 Qn 4]

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35. The transformation T from the complex z-plane to the complex w-plane is given by

$$w = \frac{z+1}{z+\mathrm{i}}, \qquad z \neq -\mathrm{i}.$$

(*a*) Show that T maps points on the half-line $\arg(z) = \frac{\pi}{4}$ in the z-plane into points on the circle
$|w| = 1$ in the w-plane.
(4)

(*b*) Find the image under T in the w-plane of the circle $|z| = 1$ in the z-plane.
(6)

(*c*) Sketch on separate diagrams the circle $|z| = 1$ in the z-plane and its image under T in the w-plane.
(2)

(*d*) Mark on your sketches the point P, where $z = \mathrm{i}$, and its image Q under T in the w-plane.
(2)

[P6 June 2004 Qn 7]

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36. (*a*) Sketch the graph of $y = |x - 2a|$, given that $a > 0$.
(2)

(*b*) Solve $|x - 2a| > 2x + a$, where $a > 0$.
(3)

[FP1/P4 January 2005 Qn 1]

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37. Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\cot 2x = \sin x, \qquad 0 < x < \frac{\pi}{2},$$

giving your answer in the form $y = \mathrm{f}(x)$.
(7)

[FP1/P4 January 2005 Qn 3]

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38. (*a*) Express $\frac{1}{r(r+2)}$ in partial fractions.

(2)

(*b*) Hence prove, by the method of differences, that

$$\sum_{r=1}^{n} \frac{4}{r(r+2)} = \frac{n(3n+5)}{(n+1)(n+2)}.$$

(5)

(*c*) Find the value of $\sum_{r=50}^{100} \frac{4}{r(r+2)}$, to 4 decimal places.

(3)

[FP1/P4 January 2005 Qn 5]

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39. (*a*) Show that the transformation $y = xv$ transforms the equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (2 + 9x^2)y = x^5, \qquad \text{I}$$

into the equation

$$\frac{d^2v}{dx^2} + 9v = x^2. \qquad \text{II}$$

(5)

(*b*) Solve the differential equation II to find v as a function of x.

(6)

(*c*) Hence state the general solution of the differential equation I.

(1)

[FP1/P4 January 2005 Qn 6]

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40. The curve C has polar equation $r = 6\cos\theta, \quad -\frac{\pi}{2} \le \theta < \frac{\pi}{2},$

and the line D has polar equation $r = 3\sec\left(\frac{\pi}{3} - \theta\right), \quad -\frac{\pi}{6} \le \theta < \frac{5\pi}{6}.$

(*a*) Find a cartesian equation of C and a cartesian equation of D.

(5)

(*b*) Sketch on the same diagram the graphs of C and D, indicating where each cuts the initial line.

(3)

The graphs of C and D intersect at the points P and Q.

(*c*) Find the polar coordinates of P and Q.

(5)

[FP1/P4 January 2005 Qn 7]

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41. (*a*) By expressing $\frac{2}{4r^2 - 1}$ in partial fractions, or otherwise, prove that

$$\sum_{r=1}^{n} \frac{2}{4r^2 - 1} = 1 - \frac{1}{2n+1}.$$

(3)

(*b*) Hence find the exact value of $\sum_{r=11}^{20} \frac{2}{4r^2 - 1}$.

(2)

[FP1/P4 June 2005 Qn 1]

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42. Find the general solution of the differential equation

$$(x+1)\frac{dy}{dx} + 2y = \frac{1}{x}, \qquad x > 0.$$

giving your answer in the form $y = f(x)$.

(7)

[FP1/P4 June 2005 Qn 3]

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43. (*a*) On the same diagram, sketch the graphs of $y = |x^2 - 4|$ and $y = |2x - 1|$, showing the coordinates of the points where the graphs meet the axes.

(4)

(*b*) Solve $|x^2 - 4| = |2x - 1|$, giving your answers in surd form where appropriate.

(5)

(*c*) Hence, or otherwise, find the set of values of x for which of $|x^2 - 4| > |2x - 1|$.

(3)

[FP1/P4 June 2005 Qn 6]

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44. (*a*) Find the general solution of the differential equation

$$2\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 2x = 2t + 9.$$

(6)

(*b*) Find the particular solution of this differential equation for which $x = 3$ and $\frac{dx}{dt} = -1$ when $t = 0$.

(4)

The particular solution in part (*b*) is used to model the motion of a particle P on the x-axis. At time t seconds ($t \geq 0$), P is x metres from the origin O.

(*c*) Show that the minimum distance between O and P is $\frac{1}{2}(5 + \ln 2)$ m and justify that the distance is a minimum.

(4)

[FP1/P4 June 2005 Qn 7]

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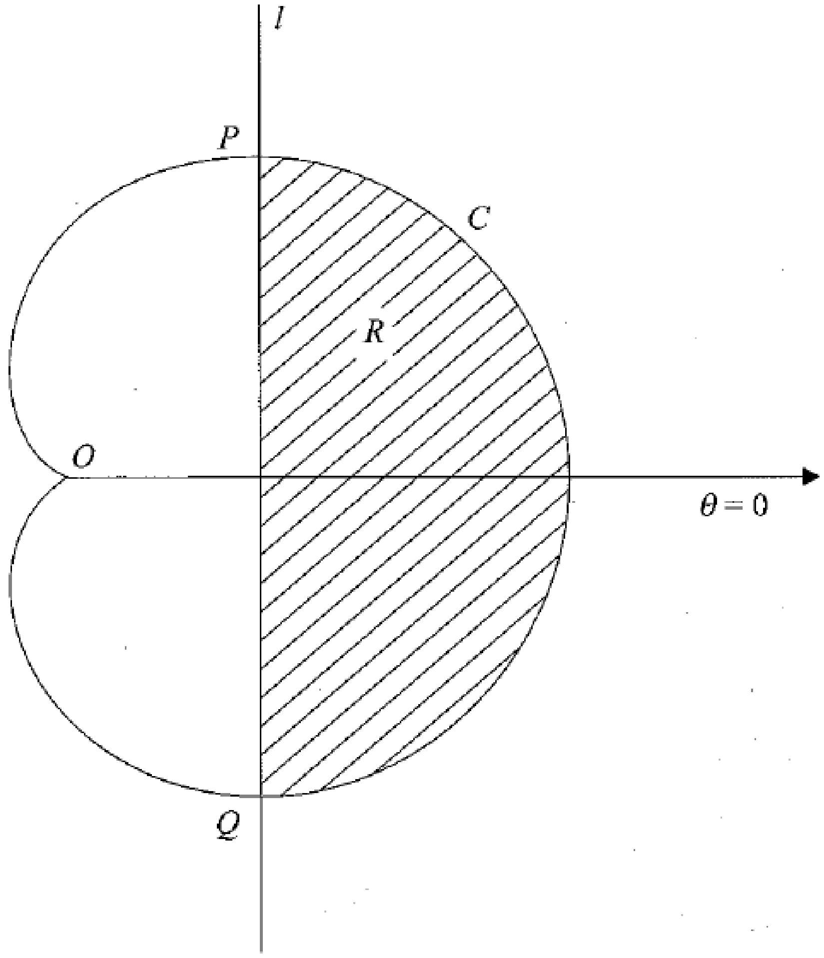
45. **Figure 1**



The curve C which passes through O has polar equation

$$r = 4a(1 + \cos\theta), \quad -\pi < \theta \leq \pi.$$

The line l has polar equation

$$r = 3a \sec\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

The line l cuts C at the points P and Q, as shown in Figure 1.

(a) Prove that $PQ = 6\sqrt{3}a$.

(6)

The region R, shown shaded in Figure 1, is bounded by l and C.

(b) Use calculus to find the exact area of R.

(7)

[FP1/P4 June 2005 Qn8]

46. A complex number z is represented by the point P in the Argand diagram. Given that

$$|z - 3\text{i}| = 3,$$

(*a*) sketch the locus of P.

(2)

(*b*) Find the complex number z which satisfies both $|z - 3\text{i}| = 3$ and $\arg(z - 3\text{i}) = \frac{3}{4}\pi$.

(4)

The transformation T from the z-plane to the w-plane is given by

$$w = \frac{2\text{i}}{w}.$$

(*c*) Show that T maps $|z - 3\text{i}| = 3$ to a line in the w-plane, and give the cartesian equation of this line.

(5)

[FP3/P6 June 2005 Qn 4]

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47. (*a*) Given that $z = \text{e}^{\text{i}\theta}$, show that

$$z^n - \frac{1}{z^n} = 2\text{i} \sin n\theta,$$

where n is a positive integer.

(2)

(*b*) Show that

$$\sin^5 \theta = \frac{1}{16}(\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta).$$

(5)

(*c*) Hence solve, in the interval $0 \le \theta < 2\pi$,

$$\sin 5\theta - 5 \sin 3\theta + 6 \sin \theta = 0.$$

(5)

[FP3/P6 June 2005 Qn 5]

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48. Find the set of values of x for which

$$\frac{x^2}{x-2} > 2x.$$

(6)

[FP1/P4 January 2006 Qn 2]

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49. (*a*) Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0.$$

(4)

(*b*) Given that $x = 1$ and $\frac{dx}{dt} = 1$ at $t = 0$, find the particular solution of the differential equation, giving your answer in the form $x = f(t)$.

(5)

(*c*) Sketch the curve with equation $x = f(t)$, $0 \leq t \leq \pi$, showing the coordinates, as multiples of π, of the points where the curve cuts the t-axis.

(4)

[FP1/P4 January 2006 Qn 4]

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50. (*a*) Show that the substitution $y = vx$ transforms the differential equation

$$\frac{dy}{dx} = \frac{3x-4y}{4x+3y} \qquad \text{(I)}$$

into the differential equation

$$x\frac{dv}{dx} = -\frac{3v^2+8v-3}{3v+4} \qquad \text{(II).}$$

(4)

(*b*) By solving differential equation (II), find a general solution of differential equation (I).

(5)

(*c*) Given that $y = 7$ at $x = 1$, show that the particular solution of differential equation (I) can be written as

$$(3y - x)(y + 3x) = 200.$$

(5)

[FP1/P4 January 2006 Qn 6]

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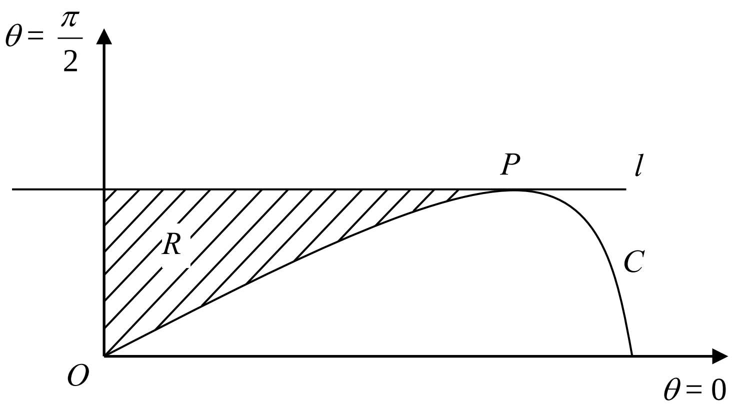
51. **Figure 1**



A curve C has polar equation $r^2 = a^2 \cos 2\theta$, $0 \leq \theta \leq \frac{\pi}{4}$. The line l is parallel to the initial line, and l is the tangent to C at the point P, as shown in Figure 1.

(*a*) (i) Show that, for any point on C, $r^2 \sin^2 \theta$ can be expressed in terms of $\sin \theta$ and a only.

(1)

(ii) Hence, using differentiation, show that the polar coordinates of P are $\left(\frac{a}{\sqrt{2}}, \frac{\pi}{6}\right)$.

(6)

The shaded region R, shown in Figure 1, is bounded by C, the line l and the half-line with equation $\theta = \frac{\pi}{2}$.

(*b*) Show that the area of R is $\frac{a^2}{16}(3\sqrt{3} - 4)$.

(8)

[FP1/P4 January 2006 Qn 7]

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52. Solve the equation

$$z^5 = \text{i},$$

giving your answers in the form $\cos \theta + \text{i} \sin \theta$.

(5)

[FP3/P6 January 2006 Qn 1]

53.
$$(1 + 2x)\frac{dy}{dx} = x + 4y^2.$$

(*a*) Show that

$$(1 + 2x)\frac{d^2 y}{dx^2} = 1 + 2(4y - 1)\frac{dy}{dx}. \qquad (1)$$

(2)

(*b*) Differentiate equation (1) with respect to x to obtain an equation involving

$\frac{d^3 y}{dx^3}$, $\frac{d^2 y}{dx^2}$, $\frac{dy}{dx}$, x and y.

(3)

Given that $y = \frac{1}{2}$ at $x = 0$,

(*c*) find a series solution for y, in ascending powers of x, up to and including the term in x^3.

(6)

[FP3/P6 January 2006 Qn 6]

---

54. In the Argand diagram the point P represents the complex number z.

Given that $\arg\left(\frac{z - 2\text{i}}{z + 2}\right) = \frac{\pi}{2}$,

(*a*) sketch the locus of P,

(4)

(*b*) deduce the value of $|z + 1 - \text{i}|$.

(2)

The transformation T from the z-plane to the w-plane is defined by

$$w = \frac{2(1 + \text{i})}{z + 2}, \quad z \neq -2.$$

(*c*) Show that the locus of P in the z-plane is mapped to part of a straight line in the w-plane, and show this in an Argand diagram.

(6)

[FP3/P6 January 2006 Qn 8]

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55. **Figure 1**

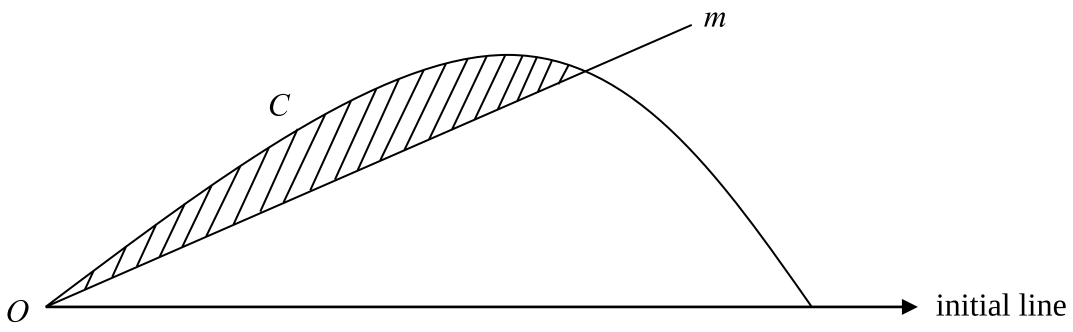

Figure 1 shows a curve C with polar equation $r = 4a \cos 2\theta$, $0 \leq \theta \leq \frac{\pi}{4}$, and a line m with polar equation $\theta = \frac{\pi}{8}$. The shaded region, shown in Figure 1, is bounded by C and m. Use calculus to show that the area of the shaded region is $\frac{1}{2}a^2(\pi - 2)$.

(7)

[FP1 June 2006 Qn 2]

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56. Given that $3x \sin 2x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 4y = k \cos 2x,$$

where k is a constant,

(*a*) calculate the value of k,

(4)

(*b*) find the particular solution of the differential equation for which at $x = 0$, $y = 2$, and for which at $x = \frac{\pi}{4}$, $y = \frac{\pi}{2}$.

(4)

[FP1 June 2006 Qn 3]

---

57. Given that for all real values of r,

$$(2r + 1)^3 - (2r - 1)^3 = Ar^2 + B,$$

where A and B are constants,

(a) find the value of A and the value of B.

(2)

(b) Hence, or otherwise, prove that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n + 1)(2n + 1)$.

(5)

(c) Calculate $\sum_{r=1}^{40} (3r - 1)^2$.

(3)

[FP1 June 2006 Qn 5]

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58. (a) Use algebra to find the exact solutions of the equation

$$|2x^2 + x - 6| = 6 - 3x.$$

(6)

(b) On the same diagram, sketch the curve with equation $y = |2x^2 + x - 6|$ and the line with equation $y = 6 - 3x$.

(3)

(c) Find the set of values of x for which

$$|2x^2 + x - 6| > 6 - 3x.$$

(3)

[FP1 June 2006 Qn 7]

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59. During an industrial process, the mass of salt, S kg, dissolved in a liquid t minutes after the process begins is modelled by the differential equation

$$\frac{dS}{dt} + \frac{2S}{120 - t} = \frac{1}{4}, \quad 0 \leq t < 120.$$

Given that $S = 6$ when $t = 0$,

(*a*) find S in terms of t,

(8)

(*b*) calculate the maximum mass of salt that the model predicts will be dissolved in the liquid at any one time during the process.

(4)

[FP1 June 2006 Qn 8]

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60. (*a*) Find the Taylor expansion of cos $2x$ in ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to and including the term in $\left(x - \frac{\pi}{4}\right)^5$.

(5)

(*b*) Use your answer to (*a*) to obtain an estimate of cos 2, giving your answer to 6 decimal places.

(3)

[FP3 June 2006 Qn 2]

---

61. (*a*) Use de Moivre's theorem to show that

$$\sin 5\theta = \sin \theta (16 \cos^4 \theta - 12 \cos^2 \theta + 1).$$

(5)

(*b*) Hence, or otherwise, solve, for $0 \leq \theta < \pi$,

$$\sin 5\theta + \cos \theta \sin 2\theta = 0.$$

(6)

[FP3 June 2006 Qn 3]

---

62. The point P represents a complex number z on an Argand diagram, where

$$|z - 6 + 3\text{i}| = 3|z + 2 - \text{i}|.$$

(*a*) Show that the locus of P is a circle, giving the coordinates of the centre and the radius of this circle.

(7)

The point Q represents a complex number z on an Argand diagram, where

$$\tan [\arg (z + 6)] = \tfrac{1}{2}.$$

(*b*) On the same Argand diagram, sketch the locus of P and the locus of Q.

(5)

(*c*) On your diagram, shade the region which satisfies both

$$|z - 6 + 3\text{i}| > 3|z + 2 - \text{i}| \text{ and } \tan [\arg (z + 6)] > \tfrac{1}{2}.$$

(2)

[FP3 June 2006 Qn 6]

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63. Obtain the general solution of the differential equation

$$x\frac{\text{d}y}{\text{d}x} + 2y = \cos x, \quad x > 0.$$

giving your answer in the form $y = \text{f}(x)$.

(8)

[FP1 January 2007 Qn 2]

---

64. (*a*) Show that

$$\frac{r^3 - r + 1}{r(r+1)} \equiv r - 1 + \frac{1}{r} - \frac{1}{r+1}, \quad \text{for } r \neq 0, -1.$$

(3)

(*b*) Find $\sum_{r=1}^{n} \frac{r^3 - r + 1}{r(r+1)}$, expressing your answer as a single fraction in its simplest form.

(6)

[FP1 January 2007 Qn 4]

65.

Figure 1

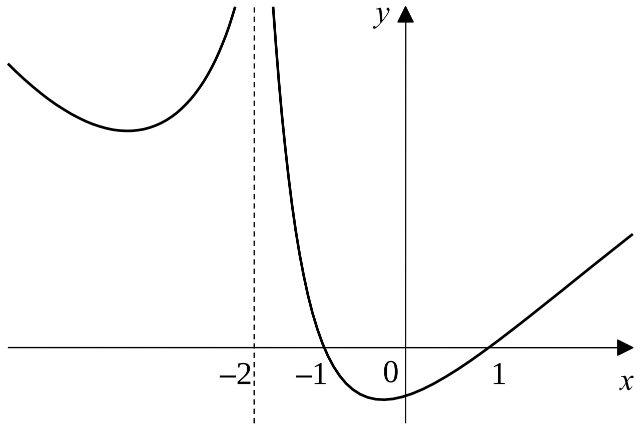

Figure 1 shows a sketch of the curve with equation

$$y = \frac{x^2 - 1}{|x+2|}, \quad x \neq -2.$$

The curve crosses the x-axis at $x = 1$ and $x = -1$ and the line $x = -2$ is an asymptote of the curve.

(*a*) Use algebra to solve the equation $\frac{x^2 - 1}{|x+2|} = 3(1 - x)$.

(6)

(*b*) Hence, or otherwise, find the set of values of x for which

$$\frac{x^2 - 1}{|x+2|} < 3(1 - x).$$

(3)

[FP1 January 2007 Qn 5]

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66. A scientist is modelling the amount of a chemical in the human bloodstream. The amount x of the chemical, measured in mg l^{-1}, at time t hours satisfies the differential equation

$$2x\frac{d^2x}{dt^2} - 6\left(\frac{dx}{dt}\right)^2 = x^2 - 3x^4, \quad x > 0.$$

(*a*) Show that the substitution $y = \dfrac{1}{x^2}$ transforms this differential equation into

$$\frac{d^2y}{dt^2} + y = 3. \qquad [\text{ I }]$$

(5)

(*b*) Find the general solution of differential equation [I].

(4)

Given that at time $t = 0$, $x = \dfrac{1}{2}$ and $\dfrac{dx}{dt} = 0$,

(*c*) find an expression for x in term of t,

(4)

(*d*) write down the maximum value of x as t varies.

(1)

[FP1 January 2007 Qn 7]

67.

Figure 2

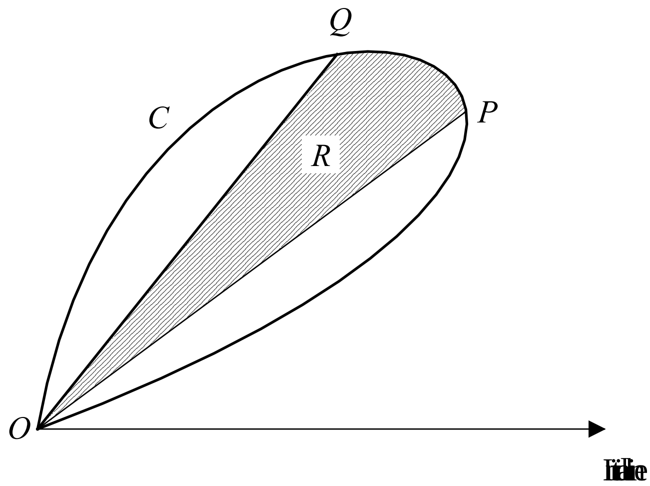

Figure 2 shows a sketch of the curve C with polar equation

$$r = 4\sin\theta\cos^2\theta, \quad 0 \le \theta < \frac{\pi}{2}.$$

The tangent to C at the point P is perpendicular to the initial line.

(a) Show that P has polar coordinates $\left(\frac{3}{2}, \frac{\pi}{6}\right)$.

(6)

The point Q on C has polar coordinates $\left(\sqrt{2}, \frac{\pi}{4}\right)$.

The shaded region R is bounded by OP, OQ and C, as shown in Figure 2.

(b) Show that the area of R is given by

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\sin^2 2\theta\cos 2\theta + \frac{1}{2} - \frac{1}{2}\cos 4\theta\right) \,\mathrm{d}\theta.$$

(3)

(c) Hence, or otherwise, find the area of R, giving your answer in the form $a + b\pi$, where a and b are rational numbers.

(4)

[FP1 January 2007 Qn 8]

68. Find the set of values of x for which

$$\frac{x+1}{2x-3} < \frac{1}{x-3}.$$

(7)

[FP1 June 2007 Qn 1]

---

69. $$\frac{dy}{dx} - y \tan x = 2 \sec^3 x.$$

Given that $y = 3$ at $x = 0$, find y in terms of x.

(7)

[FP1 June 2007 Qn 2]

---

70. (*a*) Show that $(r + 1)^3 - (r - 1)^3 \equiv 6r^2 + 2$.

(2)

(*b*) Hence show that $\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n + 1)(2n + 1)$.

(5)

(*c*) Show that $\sum_{r=n}^{2n} r^2 = \frac{1}{6} n(n + 1)(an + b)$, where a and b are constants to be found.

(4)

[FP1 June 2007 Qn 3]

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71. For the differential equation

$$\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 2y = 2x(x + 3),$$

find the solution for which at $x = 0$, $\frac{dy}{dx} = 1$ and $y = 1$.

(12)

[FP1 June 2007 Qn 5]

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72. (*a*) Sketch the curve C with polar equation

$$r = 5 + \sqrt{3}\cos\theta, \quad 0 \leq \theta < 2\pi.$$

(2)

(*b*) Find the polar coordinates of the points where the tangents to C are parallel to the initial line $\theta = 0$. Give your answers to 3 significant figures where appropriate.

(6)

(*c*) Using integration, find the area enclosed by the curve C, giving your answer in terms of π.

(6)

[FP1 June 2007 Qn 7]

73.

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = 0.$$

At $x = 0$, $y = 2$ and $\frac{dy}{dx} = -1$.

(*a*) Find the value of $\frac{d^3y}{dx^3}$ at $x = 0$.

(3)

(*b*) Express y as a series in ascending powers of x, up to and including the term in x^3.

(4)

[FP3 June 2007 Qn 2]

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74. (*a*) Given that $z = \cos\theta + \text{i}\sin\theta$, use de Moivre's theorem to show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta.$$

(2)

(*b*) Express $32\cos^6\theta$ in the form $p\cos 6\theta + q\cos 4\theta + r\cos 2\theta + s$, where p, q, r and s are integers.

(5)

(*c*) Hence find the exact value of $\int_0^{\frac{\pi}{3}} \cos^6\theta \, d\theta$.

(4)

[FP3 June 2007 Qn 4]

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75. The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane, where $w = u + \mathrm{i}v$, is given by

$$w = \frac{z + \mathrm{i}}{z}, \quad z \neq 0.$$

(*a*) The transformation T maps the points on the line with equation $y = x$ in the z-plane, other than (0, 0), to points on a line l in the w-plane. Find a cartesian equation of l.

(5)

(*b*) Show that the image, under T, of the line with equation $x + y + 1 = 0$ in the z-plane is a circle C in the w-plane, where C has cartesian equation

$$u^2 + v^2 - u + v = 0.$$

(7)

(*c*) On the same Argand diagram, sketch l and C.

(3)

[FP3 June 2007 Qn 8]

---

76. Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = x$$

to obtain y as a function of x.

(5)

[FP1 January 2008 Qn 1]

---

77. (*a*) Simplify the expression $\dfrac{(x+3)(x+9)}{x-1} - (3x - 5)$, giving your answer in the form $\dfrac{a(x+b)(x+c)}{x-1}$, where a, b and c are integers.

(4)

(*b*) Hence, or otherwise, solve the inequality

$$\frac{(x+3)(x+9)}{x-1} > 3x - 5.$$

(4)

[FP1 January 2008 Qn 3]

---

78. (*a*) Express $\frac{5r+4}{r(r+1)(r+2)}$ in partial fractions.

(4)

(*b*) Hence, or otherwise, show that

$$\sum_{r=1}^{n} \frac{5r+4}{r(r+1)(r+2)} = \frac{7n^2+11n}{2(n+1)(n+2)}.$$

(5)

[FP1 January 2008 Qn 5]

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79. (*a*) Find the general solution of the differential equation

$$3\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = x^2.$$

(8)

(*b*) Find the particular solution for which, at $x = 0$, $y = 2$ and $\frac{dy}{dx} = 3$.

(6)

[FP1 January 2008 Qn 7]

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80. **Figure 1**

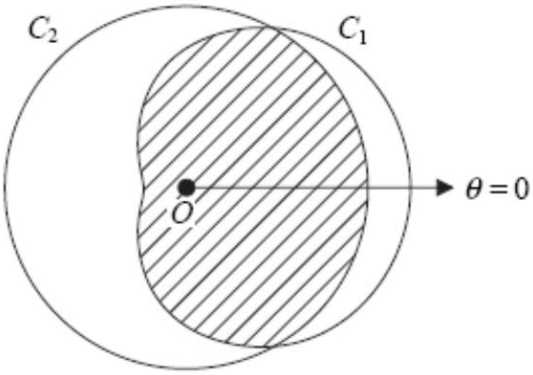

Figure 1 shows the curve C_1 which has polar equation $r = a(3 + 2\cos\theta)$, $0 \leq \theta < 2\pi$, and the circle C_2 with equation $r = 4a$, $0 \leq \theta < 2\pi$, where a is a positive constant.

(*a*) Find, in terms of a, the polar coordinates of the points where the curve C_1 meets the circle C_2.

(4)

The regions enclosed by the curves C_1 and C_2 overlap and this common region R is shaded in the figure.

(*b*) Find, in terms of a, an exact expression for the area of the shaded region R.

(8)

(*c*) In a single diagram, copy the two curves in Figure 1 and also sketch the curve C_3 with polar equation $r = 2a\cos\theta$, $0 \leq \theta < 2\pi$. Show clearly the coordinates of the points of intersection of C_1, C_2 and C_3 with the initial line, $\theta = 0$.

(3)

[FP1 January 2008 Qn 8]

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81. (*a*) Find, in terms of k, the general solution of the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = kt + 5, \quad \text{where } k \text{ is a constant and } t > 0.$$

(7)

For large values of t, this general solution may be approximated by a linear function.

(*b*) Given that $k = 6$, find the equation of this linear function.

(2)

[FP1 June 2008 Qn 4]

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82. (*a*) Find, in the simplest surd form where appropriate, the exact values of x for which

$$\frac{x}{2} + 3 = \left|\frac{4}{x}\right|.$$

(5)

(*b*) Sketch, on the same axes, the line with equation $y = \frac{x}{2} + 3$ and the graph of $y = \left|\frac{4}{x}\right|, x \neq 0$.

(3)

(*c*) Find the set of values of x for which $\frac{x}{2} + 3 > \left|\frac{4}{x}\right|$.

(2)

[FP1 June 2008 Qn 5]

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83. (*a*) Express $\frac{2}{(r+1)(r+3)}$ in partial fractions.

(2)

(*b*) Hence prove, by the method of differences, that

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \frac{n(an+b)}{6(n+2)(n+3)},$$

where a and b are constants to be found.

(6)

(*c*) Find the value of $\sum_{r=21}^{30} \frac{2}{(r+1)(r+3)}$, to 5 decimal places.

(3)

[FP1 June 2008 Qn 6]

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84. (*a*) Show that the substitution $y = vx$ transforms the differential equation

$$\frac{dy}{dx} = \frac{x}{y} + \frac{3y}{x}, \quad x > 0, \quad y > 0 \qquad \text{(I)}$$

into the differential equation

$$x\frac{dv}{dx} = 2v + \frac{1}{v}. \qquad \text{(II)}$$

(3)

(*b*) By solving differential equation (II), find a general solution of differential equation (I) in the form $y = f(x)$.

(7)

Given that $y = 3$ at $x = 1$,

(*c*) find the particular solution of differential equation (I).

(2)

[FP1 June 2008 Qn 7]

85.

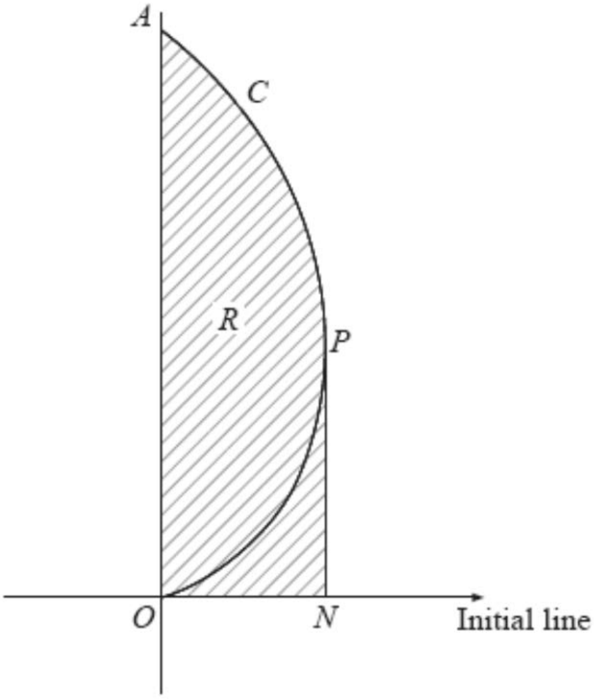

Figure 1

The curve C shown in Figure 1 has polar equation

$$r = 4(1 - \cos\theta), \qquad 0 \le \theta \le \frac{\pi}{2}.$$

At the point P on C, the tangent to C is parallel to the line $\theta = \frac{\pi}{2}$.

(*a*) Show that P has polar coordinates $\left(2, \frac{\pi}{3}\right)$.

(5)

The curve C meets the line $\theta = \frac{\pi}{2}$ at the point A. The tangent to C at P meets the initial line at the point N. The finite region R, shown shaded in Figure 1, is bounded by the initial line, the line $\theta = \frac{\pi}{2}$, the arc AP of C and the line PN.

(*b*) Calculate the exact area of R.

(8)

[FP1 June 2008 Qn 8]

86.

$$(x^2 + 1)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2y^2 + (1 - 2x)\frac{\mathrm{d}y}{\mathrm{d}x}. \qquad \text{(I)}$$

(*a*) By differentiating equation (I) with respect to x, show that

$$(x^2 + 1)\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = (1 - 4x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (4y - 2)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

(3)

Given that $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ at $x = 0$,

(*b*) find the series solution for y, in ascending powers of x, up to and including the term in x^3.

(4)

(*c*) Use your series to estimate the value of y at $x = -0.5$, giving your answer to two decimal places.

(1)

[FP3 June 2008 Qn 3]

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87. The point P represents a complex number z on an Argand diagram such that

$$|z - 3| = 2|z|.$$

(*a*) Show that, as z varies, the locus of P is a circle, and give the coordinates of the centre and the radius of the circle.

(5)

The point Q represents a complex number z on an Argand diagram such that

$$|z + 3| = |z - \mathrm{i}\sqrt{3}|.$$

(*b*) Sketch, on the same Argand diagram, the locus of P and the locus of Q as z varies.

(5)

(*c*) On your diagram shade the region which satisfies

$$|z - 3| \geq 2|z| \text{ and } |z + 3| \geq |z - \mathrm{i}\sqrt{3}|.$$

(2)

[FP3 June 2008 Qn 4]

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88. De Moivre's theorem states that

$$(\cos \theta + \mathrm{i} \sin \theta)^n = \cos n\theta + \mathrm{i} \sin n\theta \text{ for } n \in \mathbb{R}.$$

(*a*) Use induction to prove de Moivre's theorem for $n \in \mathbb{Z}^+$.

(5)

(*b*) Show that

$$\cos 5\theta = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta.$$

(5)

(*c*) Hence show that $2 \cos \dfrac{\pi}{10}$ is a root of the equation $x^4 - 5x^2 + 5 = 0$.

(3)

[FP3 June 2008 Qn 6]
