2007 FP2 Adapted

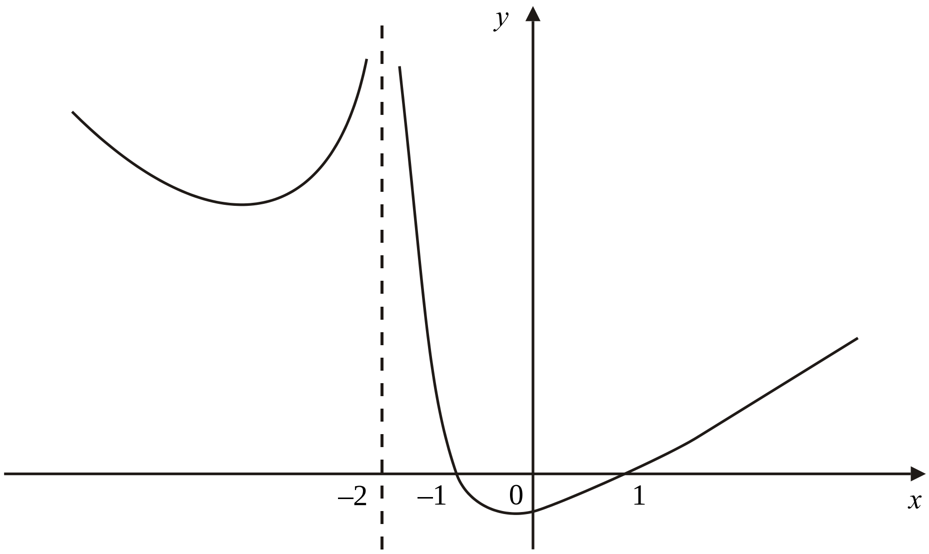
1. Obtain the general solution of the differential equation

$$x\frac{dy}{dx} + 2y = \cos x, \quad x > 0,$$

giving your answer in the form $y = f(x)$.

(Total 8 marks)

2.



The diagram above shows a sketch of the curve with equation

$$y = \frac{x^2 - 1}{|x+2|}, \quad x \neq -2.$$

The curve crosses the x-axis at $x = 1$ and $x = -1$ and the line $x = -2$ is an asymptote of the curve.

(a) Use algebra to solve the equation $\dfrac{x^2 - 1}{|x+2|} = 3(1 - x)$.

(6)

(b) Hence, or otherwise, find the set of values of x for which

$$\frac{x^2-1}{|x+2|} < 3(1-x)\ .$$

(3)
(Total 9 marks)

3. A scientist is modelling the amount of a chemical in the human bloodstream. The amount x of the chemical, measured in mg l^{-1}, at time t hours satisfies the differential equation

$$2x\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 6\left(\frac{\mathrm{dx}}{\mathrm{dt}}\right)^2 = x^2 - 3x^4, \quad x > 0.$$

(a) Show that the substitution y = $\frac{1}{x^2}$ transforms this differential equation into

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + y = 3\ . \qquad \boxed{\boldsymbol{I}}$$

(5)

(b) Find the general solution of differential equation $\boxed{\boldsymbol{I}}$.

(4)

Given that at time $t = 0$, $x = \frac{1}{2}$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$,

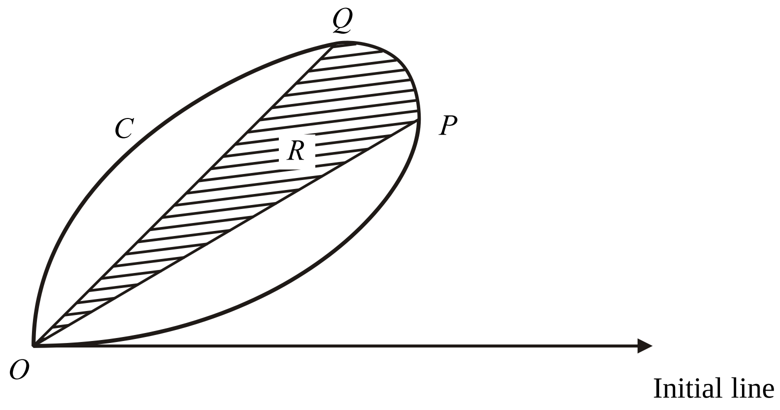
(c) find an expression for x in terms of t,

(4)

(d) write down the maximum value of x as t varies.

(1)
(Total 14 marks)

4.



The diagram above shows a sketch of the curve C with polar equation

$$r = 4\sin\theta\cos^2\theta, \qquad 0 \le \theta < \frac{\pi}{2}.$$

The tangent to C at the point P is perpendicular to the initial line.

(a) Show that P has polar coordinates $\left(\frac{3}{2}, \frac{\pi}{6}\right)$.

(6)

The point Q on C has polar coordinates $\left(\sqrt{2}, \frac{\pi}{4}\right)$.

The shaded region R is bounded by OP, OQ and C, as shown in the diagram above.

(b) Show that the area of R is given by

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\sin^2 2\theta\cos 2\theta + \frac{1}{2} - \frac{1}{2}\cos 4\theta\right) \mathrm{d}\theta$$

(3)

(c) Hence, or otherwise, find the area of R, giving your answer in the form $a + b\pi$, where a and b are rational numbers.

(5)
(Total 14 marks)

5. Find the set of values of x for which

$$\frac{x+1}{2x-3} < \frac{1}{x-3}$$

(Total 7 marks)

6. $$\frac{dy}{dx} - y \tan x = 2\sec^3 x.$$

Given that $y = 3$ at $x = 0$, find y in terms of x

(Total 7 marks)

7. For the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2x(x+3),$$

find the solution for which at $x = 0$, $\frac{dy}{dx} = 1$ and $y = 1$.

(Total 12 marks)

8. (a) Sketch the curve C with polar equation

$$r = 5 + \sqrt{3} \cos \theta, \qquad 0 \leq \theta \leq 2\pi.$$

(2)

(b) Find the polar coordinates of the points where the tangents to C are parallel to the initial line $\theta = 0$. Give your answers to 3 significant figures where appropriate.

(6)

(c) Using integration, find the area enclosed by the curve C, giving your answer in terms of π.

(6)
(Total 14 marks)

9. $$\frac{dy}{dx} = ye^{x^2}.$$

It is given that $y = 0.2$ at $x = 0$.

(a) Use the approximation $\frac{y_1 - y_0}{h} \approx \left(\frac{dy}{dx}\right)_0$, with $h = 0.1$, to obtain an estimate of the value of y at $x = 0.1$.

(2)

(b) Use your answer to part (a) and the approximation $\frac{y_2 - y_0}{2h} \approx \left(\frac{dy}{dx}\right)_1$, with $h = 0.1$, to obtain an estimate of the value of y at $x = 0.2$.

Gives your answer to 4 decimal places.

(3)
(Total 5 marks)

10. $$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = 0.$$

At $x = 0$, $y = 2$ and $\frac{dy}{dx} = -1$.

(a) Find the value of $\frac{d^3y}{dx^3}$ at $x = 0$.

(3)

(b) Express y as a series in ascending powers of x, up to and including the term in x^3.

(4)
(Total 7 marks)

11. (a) Given that $z = \cos\theta + \mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta.$$

(2)

(b) Express $32\cos^6\theta$ in the form $p\cos 6\theta + q\cos 4\theta + r\cos 2\theta + s$, where p, q, r and s are integers.

(5)

(c) Hence find the exact value of

$$\int_0^{\frac{\pi}{3}} \cos^6\theta \,\mathrm{d}\theta.$$

(4)
(Total 11 marks)

12. The transformation T from the z-plane, where $z = x + \mathrm{i}y$, to the w-plane, where

$w = u + \mathrm{i}v$, is given by

$$w = \frac{z+\mathrm{i}}{z}, \quad z \neq 0.$$

(a) The transformation T maps the points on the line with equation $y = x$ in the z-plane, other than (0, 0), to points on a line l in the w-plane. Find a cartesian equation of l.

(5)

(b) Show that the image, under T, of the line with equation $x + y + 1 = 0$ in the z-plane is a circle C in the w-plane, where C has cartesian equation

$$u^2 + v^2 - u + v = 0.$$

(7)

(c) On the same Argand diagram, sketch l and C.

(3)
(Total 15 marks)