FP2 2003 Adapted

1. (i) (*a*) On the same Argand diagram sketch the loci given by the following equations.

$$|z-1| = 1, \quad ,, \arg(z+1) = \frac{\pi}{12}, \qquad , \arg(z+1) = \frac{\pi}{2}.$$ **(4)**

(*b*) Shade on your diagram the region for which

$$|z-1| \le 1 \quad \text{and} \quad \frac{\pi}{12} \le \arg(z+1) \le \frac{\pi}{2}.$$ **(1)**

(ii) (*a*) Show that the transformation $w = \frac{z-1}{z}, \quad z \ne 0,$

maps $|z-1| = 1$ in the *z*-plane onto $|w| = |w-1|$ in the *w*-plane. **(3)**

The region $|z-1| \le 1$ in the *z*-plane is mapped onto the region *T* in the *w*-plane.

(*b*) Shade the region *T* on an Argand diagram. **(2)**

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2. (*a*) Use de Moivre's theorem to show that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

(6)

(*b*) Hence find 3 distinct solutions of the equation $16x^5 - 20x^3 + 5x + 1 = 0$, giving your answers to 3 decimal places where appropriate.

(4)

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3. $$\frac{dy}{dx} = x^2 - y^2, \quad y = 1 \text{ at } x = 0. \quad \text{(I)}$$

(*b*) By differentiating (I) twice with respect to *x*, show that

$$\frac{d^3y}{dx^3} + 2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 - 2 = 0.$$ **(4)**

(*c*) Hence, for (I), find the series solution for *y* in ascending powers of *x* up to and including the term in x^3. (4)

4. **(*a*)** **Express as a simplified single fraction** $\dfrac{1}{(r-1)^2} - \dfrac{1}{r^2}$. **(2)**

(*b*) **Hence prove, by the method of differences, that** $\displaystyle\sum_{r=2}^{n} \frac{2r-1}{r^2(r-1)^2} = 1 - \frac{1}{n^2}$. **(3)**

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5. **Solve the inequality** $\dfrac{1}{2x+1} > \dfrac{x}{3x-2}$. **(6)**

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6. **(*a*)** **Using the substitution $t = x^2$, or otherwise, find**

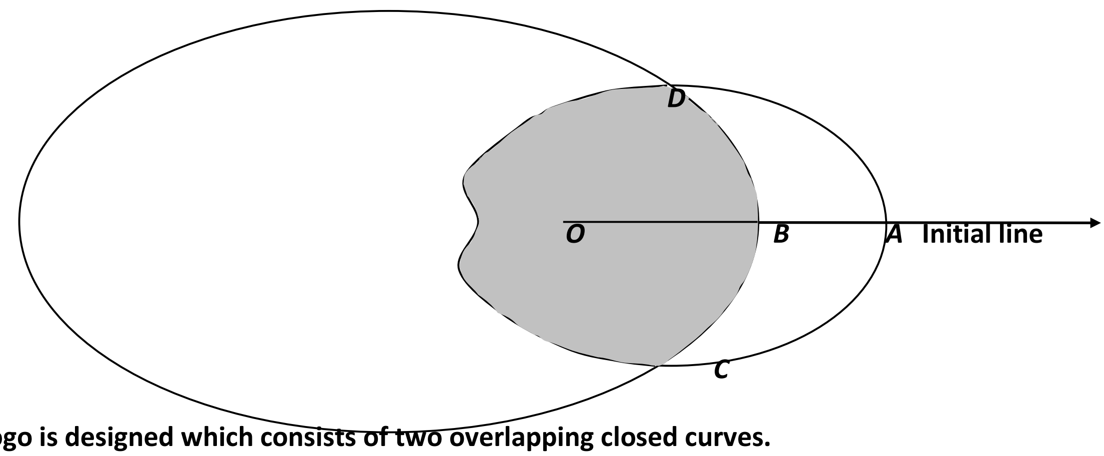
$$\int x^3 e^{-x^2} \, dx.$$ **(6)**

(*b*) **Find the general solution of the differential equation**

$$x\frac{dy}{dx} + 3y = xe^{-x^2}, \quad x > 0.$$ **(4)**

7.

Figure 1



A logo is designed which consists of two overlapping closed curves.

The polar equations of these curves are $r = a(3 + 2\cos\theta)$ **and**

$$r = a(5 - 2\cos\theta), \quad 0 \le \theta < 2\pi.$$

Figure 1 is a sketch (not to scale) of these two curves.

(*a*) **Write down the polar corrdinates of the points *A* and *B* where the curves meet the initial line.(2)**

(*b*) **Find the polar coordinates of the points *C* and *D* where the two curves meet. (4)**

(*c*) Show that the area of the overlapping region, which is shaded in the figure, is

$$\frac{a^2}{3}(49\pi - 48\sqrt{3}) \qquad \textbf{(8)}$$

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8. $$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 6\frac{\mathrm{d}y}{\mathrm{d}t} + 9y = 4e^{3t}, \quad t \geq 0.$$

(*a*) Show that Kt^2e^{3t} is a particular integral of the differential equation, where K is a constant to be found. (4)

(*b*) Find the general solution of the differential equation. (3)

Given that a particular solution satisfies $y = 3$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 1$ when $t = 0$,

(*c*) find this solution.(4)

Another particular solution which satisfies $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$ when $t = 0$, has equation

$$y = (1 - 3t + 2t^2)e^{3t}.$$

(*d*) For this particular solution draw a sketch graph of y against t, showing where the graph crosses the t-axis. Determine also the coordinates of the minimum of the point on the sketch graph.

(5)

9. $$z = 4\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right), \text{ and } w = 3\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right).$$

Express zw in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, $r > 0$, $-\pi < \theta < \pi$.

(3)

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10. (*a*) Sketch, on the same axes, the graphs with equation $y = |2x - 3|$, and the line with equation $y = 5x - 1$.

(2)

(*b*) Solve the inequality $|2x - 3| < 5x - 1$. (3)

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11. **(*a*)** **Express** $\dfrac{2}{(r+1)(r+3)}$ **in partial fractions.** **(2)**

(*b*) **Hence prove that** $\displaystyle\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} \equiv \frac{n(5n+13)}{6(n+2)(n+3)}$. **(5)**

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12. **(*a*)** **Use the substitution *y* = *vx* to transform the equation**

$$\frac{dy}{dx} = \frac{(4x+y)(x+y)}{x^2}, \; x > 0 \qquad \text{(I)}$$

into the equation $$x\frac{dv}{dx} = (2+v)^2. \qquad \text{(II)}$$ **(4)**

(*b*) **Solve the differential equation II to find *v* as a function of *x*** **(5)**

(*c*) **Hence show that** $y = -2x - \dfrac{x}{\ln x + c}$, **where *c* is an arbitrary constant,is a general solution of the differential equation I.** **(1)**

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13. **Given that *z* = 3 – 3i express, in the form *a* + i*b*, where *a* and *b* are real numbers,**

(*a*) z^2, **(2)** **(*b*)** $\dfrac{1}{z}$. **(2)**

(*c*) **Find the exact value of each of** $|z|$, $|z^2|$ **and** $\left|\dfrac{1}{z}\right|$. **(2)**

The complex numbers *z*, z^2 and $\dfrac{1}{z}$ are represented by the points *A*, *B* and *C* respectively on an Argand diagram.

The real number 1 is represented by the point *D*, and *O* is the origin.

(*d*) **Show the points *A*, *B*, *C* and *D* on an Argand diagram.** **(2)**

(*e*) **Prove that** $\triangle OAB$ **is similar to** $\triangle OCD$. **(3)**

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14. **(*a*) Find the value of λ for which $\lambda x \cos 3x$ is a particular integral of the differential equation**

$$\frac{d^2y}{dx^2} + 9y = -12 \sin 3x. \qquad (4)$$

(*b*) **Hence find the general solution of this differential equation.(4)**

The particular solution of the differential equation for which *y* = 1 and $\dfrac{dy}{dx} = 2$ at *x* = 0, is *y* = g(*x*).

(*c*) **Find g(*x*).** **(4)**

(*d*) Sketch the graph of $y = g(x)$, $0 \le x \le \pi$. **(2)**

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15. **Figure 1**

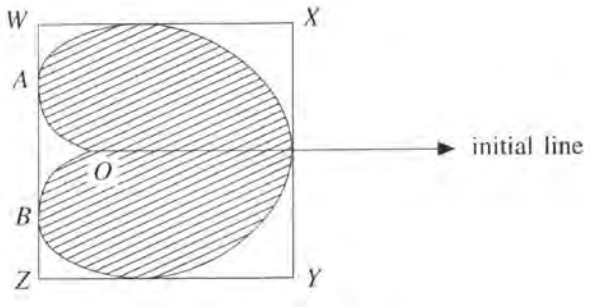

Figure 1 shows a sketch of the cardioid C with equation $r = a(1 + \cos\theta)$, $-\pi < \theta \le \pi$. Also shown are the tangents to C that are parallel and perpendicular to the initial line. These tangents form a rectangle $WXYZ$.

(*a*) Find the area of the finite region, shaded in Fig. 1, bounded by the curve C. **(6)**

(*b*) Find the polar coordinates of the points A and B where WZ touches the curve C. **(5)**

(*c*) Hence find the length of WX. **(2)**

Given that the length of WZ is $\frac{3\sqrt{3}a}{2}$,

(*d*) find the area of the rectangle $WXYZ$. **(1)**

A heart-shape is modelled by the cardioid C, where $a = 10$ cm. The heart shape is cut from the rectangular card $WXYZ$, shown in Fig. 1.

(*e*) Find a numerical value for the area of card wasted in making this heart shape. **(2)**

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8. A transformation T from the z-plane to the w-plane is defined by

$$w = \frac{z+1}{iz-1}, \quad z \ne -i,$$

where $z = x + iy$, $w = u + iv$ and x, y, u and v are real.

T transforms the circle $|z| = 1$ in the z-plane onto a straight line L in the w-plane.

(a) Find an equation of L giving your answer in terms of u and v. **(5 marks)**

(b) Show that T transforms the line $\text{Im } z = 0$ in the z-plane onto a circle C in the w-plane, giving the centre and radius of this circle. **(6 marks)**

(c) On a single Argand diagram sketch L and C. **(3 marks)**

Question: Solve $$x^5 = -(9\sqrt{3})i$$