FP2 Specimen

1. $\frac{x}{x-3} > \frac{1}{x-2} \Rightarrow \frac{x(x-2)^2(x-3)^{\cancel{2}}}{\cancel{x-3}} > \frac{(x-2)^{\cancel{2}}(x-3)^2}{\cancel{x-2}}$

$\Rightarrow x(x-3)(x-2)^2 - (x-2)(x-3)^2 > 0$

$\Rightarrow (x-2)(x-3)[x(x-2)-(x-3)] > 0$

$\Rightarrow (x-2)(x-3)[x^2-3x+3] > 0$

$\to (x-\frac{3}{2})^2+\frac{3}{4} \Rightarrow$ always positive. no roots

both +ve when $\underline{x>3}$

both -ve when $\underline{x<2}$ $\quad \therefore (x>3)\cup(x<2)$

2) $\frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{(r+1)^2-r^2}{r^2(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}$

b) $\sum \frac{2r+1}{r^2(r+1)^2}$

$r=1 \quad \left(\frac{1}{1} - \cancel{\frac{1}{4}}\right)$

$+$

$r=2 \quad \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{9}}\right)$

$+$

$r=3 \quad \left(\cancel{\frac{1}{9}} - \cancel{\frac{1}{16}}\right)$

$+$

$\dots$

$+$

$r=n-1 \quad \left(\cancel{\frac{1}{(n-1)^2}} - \cancel{\frac{1}{n^2}}\right)$

$+$

$r=n \quad \left(\cancel{\frac{1}{n^2}} - \frac{1}{(n+1)^2}\right)$

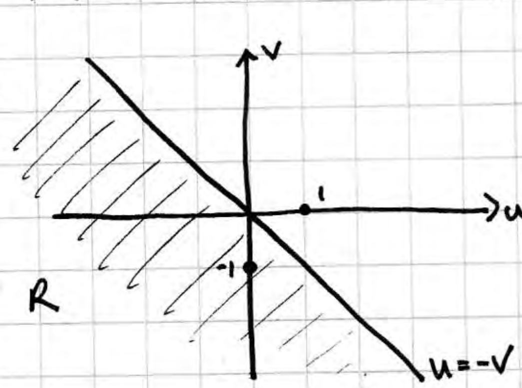
$\therefore \sum \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}$

3) $w = \frac{z-i}{z+1} \Rightarrow wz + w = z - i \Rightarrow w + i = z - wz$

$\Rightarrow w + i = z(1-w) \Rightarrow |w+i| = |z||1-w|$

$|z| = 1 \Rightarrow |w+i| = |w-1| \quad \therefore |z|=1$ maps to $u=-v$ in w-plane.

b)



4) $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x$ $\quad x=1$, $y=0$, $y'=2$

$y'' + yy' = x$

$y'' + 0 = 1 \Rightarrow \underline{y''=1}$

$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) + \frac{d}{dx}\left(y\frac{dy}{dx}\right) = \frac{d}{dx}(x)$

$\frac{d^3y}{dx^3} + y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1$

$y''' + yy'' + (y')^2 = 1$

$y''' + 0 + (2)^2 = 1 \Rightarrow \underline{y'''=-3}$

$\therefore y = 2(x-1) + \frac{1}{2}(x-1)^2 - \frac{3}{6}(x-1)^3 \; ... \; \therefore y = 2(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{2}(x-1)^3$

5) $\frac{dS}{dt} - 0.1S = t$ $\quad$ IF $f(x) = e^{\int -0.1\,dt} = e^{-0.1t}$

$\Rightarrow e^{-0.1t}\frac{dS}{dt} - (0.1e^{-0.1t})S = te^{-0.1t} \Rightarrow \frac{d}{dt}(Se^{-0.1t}) = te^{-0.1t}$

$\Rightarrow Se^{-0.1t} = \int te^{-0.1t}\,dt$ $\quad u=t \quad v=-10e^{-0.1t}$

$u'=1 \quad v'=e^{-0.1t}$

$\Rightarrow Se^{-0.1t} = -10te^{-0.1t} + \int 10e^{-0.1t}\,dt$

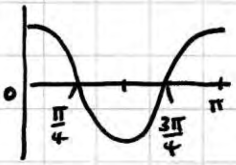
$\Rightarrow Se^{-0.1t} = -10te^{-0.1t} - 100e^{-0.1t} + c$

$\therefore S = -10t - 100 + ce^{0.1t}$

$t=0 \quad S = -100 + c = 200 \therefore c = 300 \Rightarrow S = 300e^{0.1t} - 10t - 100$

$t=10 \quad S = 300e - 200 \Rightarrow S = \underline{£615 \text{ million}}$

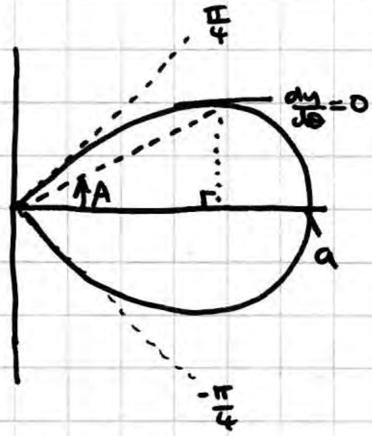
6) $r^2 = a^2 \cos 2\theta$



$r_{max} = a$ at $\theta = 0$

$r_{min} = 0$ at $\theta = \frac{\pi}{4}, -\frac{\pi}{4}$

undefined $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$



b) parallel to initial line when $\frac{dy}{d\theta} = 0$

$y = r\sin\theta = a(\cos 2\theta)^{\frac{1}{2}} \times \sin\theta$

$\frac{dy}{d\theta} = \cancel{\tfrac{1}{2}} a (\cos 2\theta)^{-\frac{1}{2}} \underset{\times}{} - \cancel{2}\sin 2\theta \times \sin\theta + a(\cos 2\theta)^{\frac{1}{2}} \times \cos\theta$

$\Rightarrow \cancel{a}(\cos 2\theta)^{-\frac{1}{2}} \sin 2\theta \sin\theta = \cancel{a}(\cos 2\theta)^{\frac{1}{2}} \times \cos\theta$

$\Rightarrow 2\sin^2\theta \cancel{\cos\theta} = \cos 2\theta \cancel{\cos\theta}$

$\Rightarrow 2\sin^2\theta = 1 - 2\sin^2\theta \Rightarrow \sin^2\theta = \frac{1}{4} \quad \sin\theta = \pm\frac{1}{2} \quad \theta = \frac{\pi}{6}, -\frac{\pi}{6}$

$r^2 = a^2 \cos\frac{\pi}{3} \Rightarrow r^2 = \frac{1}{2}a^2 \Rightarrow r = \frac{1}{\sqrt{2}}a \Rightarrow r = \frac{\sqrt{2}}{2}a$

$\left(\frac{\sqrt{2}}{2}a, \frac{\pi}{6}\right) ; \left(\frac{\sqrt{2}}{2}a, \frac{-\pi}{6}\right)$

c) Area $= \frac{1}{2}a^2 \int_{-\frac{\pi}{4}}^{\pi/4} \cos 2\theta \, d\theta = \frac{1}{2}a^2 \left[\frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$= \frac{1}{2}a^2 \left[\left(\frac{1}{2}\right) - \left(-\frac{1}{2}\right)\right] = \frac{1}{2}a^2$

7) $x = e^t \quad \Rightarrow \frac{dx}{dt} = e^t \quad \frac{dt}{dx} = e^{-t}$

$\frac{dy}{dx} = \frac{dt}{dx} \times \frac{dy}{dt} = e^{-t}\frac{dy}{dt}$

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(e^{-t}\frac{dy}{dt}\right) = \left[\frac{d}{dx}(e^{-t})\right]\frac{dy}{dt} + e^{-t}\left[\frac{d}{dx}\left(\frac{dy}{dt}\right)\right]$

$= \left(-e^{-t}\frac{dt}{dx}\right)\frac{dy}{dt} + e^{-t}\left(\frac{d^2y}{dt^2}\right)\frac{dt}{dx}$

$= -e^{-t} \times e^{-t}\frac{dy}{dt} + e^{-t}\left(\frac{d^2y}{dt^2}\right)e^{-t} = e^{-2t}\left[\frac{d^2y}{dt^2} - \frac{dy}{dt}\right]$

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7b) $$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3$$

$$\cancel{e^{2t}}\left[\cancel{e^{-2t}}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right] - 2\cancel{e^{t}}\left[\cancel{e^{-t}}\frac{dy}{dt}\right] + 2y = e^{3t}$$

$$\Rightarrow \frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}$$

7c) $y = Ae^{3t}$ $\qquad y'' - 3y' + 2y = 0$

$y' = Ame^{3t}$ $\qquad Ae^{mt}(m^2 - 3m + 2) = 0$

$y'' = Am^2e^{3t}$ $\qquad \neq 0 \qquad = 0 \Rightarrow (m-2)(m-1) = 0 \Rightarrow m = 2 \quad m = 1$

$$\therefore y_{cf} = Ae^{t} + Be^{2t}$$

$y = \lambda e^{3t}$ $\qquad y'' - 3y' + 2y' = e^{3t}$

$y' = 3\lambda e^{3t}$ $\qquad \cancel{9\lambda e^{3t}} - \cancel{9\lambda e^{3t}} + 2\lambda e^{3t} = e^{3t} \qquad \therefore \lambda = \frac{1}{2}$

$y'' = 9\lambda e^{3t}$

$$\therefore y_{PI} = \frac{1}{2}e^{3t} \qquad \therefore y = Ae^{t} + Be^{2t} + \frac{1}{2}e^{3t}$$

$$\Rightarrow y = Ax + Bx^2 + \frac{1}{2}x^3$$

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8) $z = e^{i\theta} \Rightarrow z^p = (e^{i\theta})^p = e^{ip\theta}$ $\quad z^{-p} = (e^{i\theta})^{-p} = e^{-ip\theta}$

$$\Rightarrow z^p + \frac{1}{z^p} = e^{ip\theta} + e^{-ip\theta}$$

$$= \begin{array}{l} \cos p\theta + i\sin p\theta \\ + \cos(-p\theta) + i\sin(-p\theta) \end{array} \Rightarrow \begin{array}{l} \cos p\theta + i\sin p\theta \\ + \cos p\theta - i\sin p\theta \\ \hline \underline{2\cos p\theta} \end{array}$$

b) $\cos^4\theta = A\cos 4\theta + B\cos 2\theta + C$

1 1
1 2 1
1 3 3 1
1 4 6 4 1

$$\left(z+\frac{1}{z}\right)^4 = (2\cos\theta)^4 = 16\cos^4\theta$$

$$\left(z+\frac{1}{z}\right)^4 = z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z\left(\frac{1}{z^3}\right) + \left(\frac{1}{z^4}\right)$$

$$= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

$$\Rightarrow 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$$

$$\therefore \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

c) $\text{Volume} = \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y^2\,dx = \pi\int_{-\frac{\pi}{2}=\theta}^{\frac{\pi}{2}=\theta} \cos^4\theta\,d\theta = \frac{1}{8}\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 4\theta + 4\cos 2\theta + 3\,d\theta$

$$= \frac{1}{8}\pi\left[\frac{1}{4}\sin 4\theta + 2\sin 2\theta + 3\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{8}\pi\left[\left(\frac{3\pi}{2}\right) - \left(-\frac{3\pi}{2}\right)\right] = \underline{\underline{\frac{3}{8}\pi^2}}$$