

1. (a) Sketch, on the same axes, the graph with equation $y = |3x - 1|$, and the line with equation $y = 4x + 3$.

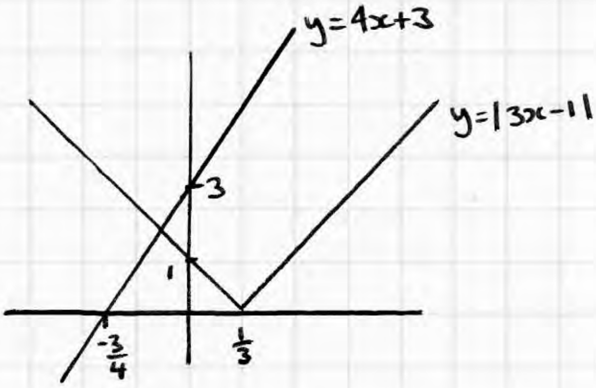
Show the coordinates of the points at which the graphs meet the x-axis.

(3)

(b) Solve the inequality $|3x-1| < 4x+3$.

(3)

(Total 6 marks)



$|3x-1| = 4x+3$

$3x-1 = 4x+3$

$\cancel{x=-4}$

$3x-1=-4x-3$

$7x=-2$

$x=-\frac{2}{7}$

$\therefore \; x > -\frac{2}{7}$

2. (a) Express $\frac{2}{(2r+1)(2r+3)}$ in partial fractions.

(2)

(b) Hence prove that $\sum_{r=1}^{n} \frac{2}{(2r+1)(2r+3)} = \frac{2n}{3(2n+3)}$.

(3)

(Total 5 marks)

a) $\frac{2}{(2r+1)(2r+3)} = \frac{A}{2r+1} + \frac{B}{2r+3} \Rightarrow 2 = A(2r+3) + B(2r+1)$ $\quad r=-\frac{1}{2} \Rightarrow A=1$

$r=-\frac{3}{2} \Rightarrow B=-1$

$= \frac{1}{2r+1} - \frac{1}{2r+3}$

b) $\sum_{1}^{n} \frac{2}{(2r+1)(2r+3)} = \underbrace{\left(\frac{1}{3}-\cancel{\frac{1}{5}}\right)}_{r=1} + \underbrace{\left(\cancel{\frac{1}{5}}-\cancel{\frac{1}{7}}\right)}_{r=2} + \underbrace{\left(\cancel{\frac{1}{7}}-\cancel{\frac{1}{9}}\right)}_{r=3} + \ldots + \underbrace{\left(\cancel{\frac{1}{2n-1}}-\cancel{\frac{1}{2n+1}}\right)}_{r=n-1} + \underbrace{\left(\cancel{\frac{1}{2n+1}}-\frac{1}{2n+3}\right)}_{r=n}$

$= \frac{1}{3} - \frac{1}{2n+3} = \frac{2n+\cancel{3}-\cancel{3}}{3(2n+3)} = \frac{2n}{3(2n+3)}$ #

3. (a) Given that $y = \ln(1+5x)$, $|x| < 0.2$, find $\frac{d^3y}{dx^3}$.

(4)

(b) Hence obtain the Maclaurin series for $\ln(1+5x)$, $|x| < 0.2$, up to and including the term in x^3.

(3)

(Total 7 marks)

$y = \ln(1+5x)$ $\qquad y(0) = \ln 1 = 0$

$y' = \frac{5}{1+5x} = 5(1+5x)^{-1}$ $\qquad y'(0) = 5$ $\qquad \therefore \; y = 5x - \frac{25}{2}x^2 + \frac{250}{6}x^3$

$y'' = -25(1+5x)^{-2}$ $\qquad y''(0) = -25$

$y''' = 250(1+5x)^{-3}$ $\qquad y'''(0) = 250$

4. Use the Taylor Series method to find the series solution, ascending up to and including the term in x^3, of the differential equation

$$\frac{d^2y}{dx^2}+y\frac{dy}{dx}+y^2=3x+4,$$

given that $\frac{dy}{dx}=y=1$ at $x=0$.

(Total 8 marks)

$x_0=0 \quad y_0=1 \quad y_0'=1$

$y''+(1)(1)+(1)^2=4$

$y''=2$

$$\frac{d^3y}{dx^3}+\frac{d}{dx}\left(y\frac{dy}{dx}\right)+\frac{d}{dx}(y^2)=\frac{d}{dx}(3x+4)$$

$y'''+(1)(2)+(1)^2+2(1)(1)=3$

$$\frac{d^3y}{dx^3}+y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+2y\frac{dy}{dx}=3$$

$y'''=-2$

$$\therefore \quad y=1+x+x^2-\frac{2}{6}x^3$$

5.

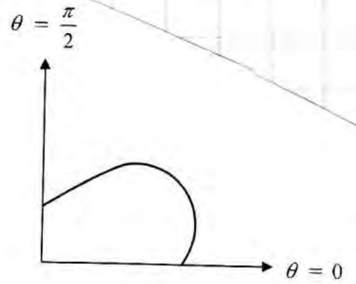


Figure 1

The curve C, shown in Figure 1, has polar equation, $r=2+\sin 3\theta$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$

Use integration to calculate the exact value of the area enclosed by C, the line $\theta=0$ and the line $\theta=\frac{\pi}{2}$.

(Total 7 marks)

$$\text{Area}=\frac{1}{2}\int_0^{\frac{\pi}{2}}(2+\sin 3\theta)^2\,d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}4+4\sin 3\theta+\sin^2 3\theta\,d\theta$$

$\cos 6\theta=1-2\sin^2 3\theta$

$\Rightarrow \sin^2 3\theta=-\frac{1}{2}\cos 6\theta+\frac{1}{2}$

$$\text{Area}=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{9}{2}+4\sin 3\theta+-\frac{1}{2}\cos 6\theta\,d\theta$$

$$\therefore \text{Area}=\frac{1}{4}\int_0^{\frac{\pi}{2}}9+8\sin 3\theta-\cos 6\theta\,d\theta=\frac{1}{4}\left[9\theta-\frac{8}{3}\cos 3\theta-\frac{1}{6}\sin 6\theta\right]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{4}\left[\left(\frac{9\pi}{2}\right)-\left(-\frac{8}{3}\right)\right]=\frac{9\pi}{8}+\frac{2}{3}$$

6. (a) Use de Moivre's Theorem to show that

$$\sin 5\theta=16\sin^5\theta-20\sin^3\theta+5\sin\theta.$$

(5)

(b) Hence or otherwise, prove that the only real solutions of the equation

$$\sin 5\theta=5\sin\theta,$$

are given by $\theta=n\pi$, where n is an integer.

(4)

(Total 9 marks)

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$(\cos\theta+i\sin\theta)^5=\cos 5\theta+i\sin 5\theta$

$(\cos\theta+i\sin\theta)^5=\cos^5\theta+5i\cos^4\theta\sin\theta-10\cos^3\theta\sin^2\theta-10i\cos^2\theta\sin^3\theta+5\cos\theta\sin^4\theta+i\sin^5\theta$

equating Imaginary Parts $\Rightarrow \sin 5\theta \equiv 5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$

$=5(1-2\sin^2\theta+\sin^4\theta)\sin\theta-10(1-\sin^2\theta)\sin^3\theta+\sin^5\theta$

$=5\sin\theta-10\sin^3\theta+5\sin^5\theta-10\sin^3\theta+10\sin^5\theta+\sin^5\theta$

$=5\sin\theta-20\sin^3\theta+16\sin^5\theta$

b) $\sin 5\theta = 5\cancel{\sin\theta} = 16\sin^5\theta - 20\sin^3\theta + \cancel{5\sin\theta}$

$\Rightarrow 4\sin^3\theta\,(4\sin^2\theta - 5) = 0$

$\Rightarrow \sin^3\theta = 0$ $\quad\quad \sin^2\theta = \frac{5}{4}$ $\quad\quad \sin\theta > 1$ or $\sin\theta < -1$ not possible.

$\sin\theta = 0$ $\quad\quad \sin\theta = \pm\sqrt{\frac{5}{4}} \Rightarrow$ no solutions

$\theta = n\pi \quad n \in \mathbb{Z}$

7. A population P is growing at a rate which is modelled by the differential equation

$$\frac{dP}{dt} - 0.1P = 0.05t\,,$$

where t years is the time that has elapsed from the start of observations.

It is given that the population is 10 000 at the start of the observations.

(a) Solve the differential equation to obtain an expression for P in terms of t.

(7)

(b) Show that the population doubles between the sixth and seventh year after the observations began.

(2)

(Total 9 marks)

IF $f(x) = e^{-\int 0.1\,dt} = e^{-0.1t}$

$e^{-0.1t}\frac{dp}{dt} - 0.1e^{-0.1t}p = 0.05te^{-0.1t}$

$\frac{d}{dt}(pe^{-0.1t}) = 0.05te^{-0.1t}$

$\Rightarrow pe^{-0.1t} = \int 0.05te^{-0.1t}\,dt$

$u = 0.05t \quad v = -10e^{-0.1t}$

$u' = 0.05 \quad v' = e^{-0.1t}$

$\Rightarrow pe^{-0.1t} = -\frac{1}{2}e^{-0.1t} \times t + \int \frac{1}{2}e^{-0.1t}\,dt$

$\Rightarrow pe^{-0.1t} = -\frac{1}{2}te^{-0.1t} - 5e^{-0.1t} + c$

$\therefore p = ce^{0.1t} - \frac{1}{2}t - 5$

$P = 10000 \quad t = 0 \quad \Rightarrow \quad 10000 = c - 5 \quad \therefore \quad c = 10005$

$P = 10005e^{0.1t} - \frac{1}{2}t - 5$

b) $t = 6 \quad P = 18222.3$

$t = 7 \quad P = 20139.1$

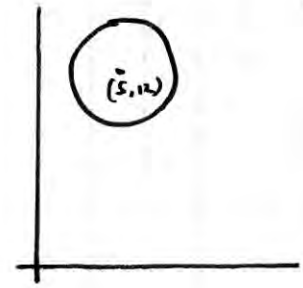
$\therefore P = 20000$ between year 6 and year 7.

8. A complex number z satisfies the equation

$$|z-5-12\text{i}|=3.$$

(a) Describe in geometrical terms with the aid of a sketch, the locus of the point which represents z in the Argand diagram. **(3)**

Circle, centre $(5,12)$
radius $=3$



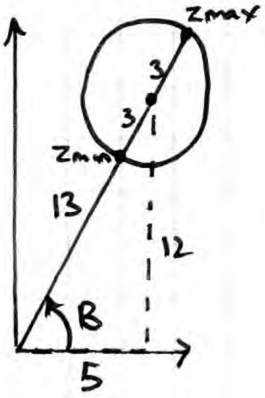
For points on this locus, find

(b) the maximum and minimum values for $|z|$, **(4)**

(c) the maximum and minimum values for arg z, giving your answers in radians to 2 decimal places. **(4)**

(Total 11 marks)

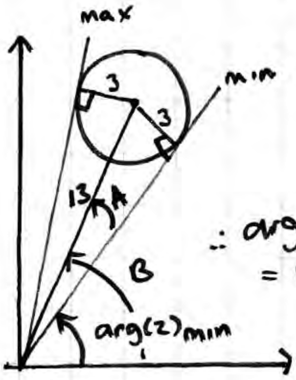
b)



$\therefore\ |z|_{min} = 13-3=10$

$|z|_{max} = 13+3=16$

c)



$\therefore\ \arg z_{min} = B-A$

$B = \tan^{-1}\left(\frac{12}{5}\right) = 1.176^c$

$A = \sin^{-1}\left(\frac{3}{13}\right) = 0.233^c$

$\therefore\ \arg(z)_{min} = 0.94^c$

$\arg(z)_{max} = 1.41^c$

9. Resonance in an electrical circuit is modelled by the differential equation

$$\frac{d^2V}{dt^2}+64V=\cos 8t,$$

where V represents the voltage in the circuit and t represents time.

(a) Find the value of λ for which $\lambda t\sin 8t$ is a particular integral of the differential equation. **(5)**

(b) Find the general solution of the differential equation. **(4)**

Given that $V=0$ and $\frac{dV}{dt}=0$ when $t=0$,

(c) find the particular solution of the equation. **(3)**

(d) Describe the behaviour of V as t becomes large, according to this model. **(1)**

(Total 13 marks)

$v = \lambda t\sin 8t$

$v' = 8\lambda t\cos 8t + \lambda\sin 8t$

$v'' = -64\lambda t\sin 8t + 8\lambda\cos 8t + 8\lambda\cos 8t$

$v'' = -64\lambda t\sin 8t + 16\lambda\cos 8t$

$+\ 64v = 16\lambda\cos 8t$

$\cos 8t = 16\lambda\cos 8t$

$\therefore\ \lambda = \frac{1}{16}$

$V_{PI} = \frac{1}{16}t\sin 8t$

$v = Ae^{mt}$ $\quad V''+64V=0$

$v' = Ame^{mt}$ $\quad Ae^{mt}(m^2+64)=0$

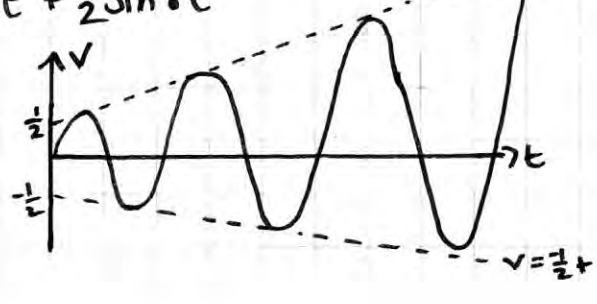
$v'' = Am^2e^{mt}$ $\quad \neq 0 \quad =0 \Rightarrow m=\pm 8i \Rightarrow V_{CF} = A\cos 8t + B\sin 8t$

$\Rightarrow V = A\cos 8t + (B+\frac{1}{2}t)\sin 8t$

$t=0, V=0 \Rightarrow A=0 \quad \Rightarrow V=(B+\frac{1}{2}t)\sin 8t$

$V' = (B+\frac{1}{2}t)8\cos 8t + \frac{1}{2}\sin 8t$

$t=0, V'=0 \Rightarrow B=0 \quad \therefore V=\frac{1}{2}t\sin 8t$



d) as $t\to\infty$ the amplitude of the curve $\to\infty$ $\Rightarrow$ as $t\to\infty$ $V\to\pm\infty$