FP2 June 2018 (MA)

Q1(a) $$\frac{1}{(r+3)(r+4)} = \frac{A}{r+3} + \frac{B}{r+4}$$

$$1 = A(r+4) + B(r+3)$$

$\underline{r=-4}$: $1 = -B$ $\therefore$ $B = -1$

$\underline{r=0}$: $1 = 4A + 3(-1)$ $\therefore$ $A = 1$

$$\therefore \frac{1}{(r+3)(r+4)} = \boxed{\frac{1}{r+3} - \frac{1}{r+4}}$$

b) $$\sum_{1}^{n} \frac{1}{(r+3)(r+4)} = \sum \frac{1}{r+3} - \frac{1}{r+4}$$

$\underline{n=1}$: $\frac{1}{4} - \cancel{\frac{1}{5}}$

$\underline{n=2}$: $\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}$

. . . .

$\underline{n=n-1}$: $\cancel{\frac{1}{n+2}} - \cancel{\frac{1}{n+3}}$

$\underline{n=n}$: $\cancel{\frac{1}{n+3}} - \frac{1}{n+4}$

$$= \frac{1}{4} - \frac{1}{n+4}$$

$$= \frac{(n+4) - (4)}{4(n+4)}$$

$$= \boxed{\frac{n}{4(n+4)}}$$

c) $$\sum_{r=15}^{30} \frac{1}{(r+3)(r+4)} = \sum_{r=1}^{30} \frac{1}{(r+3)(r+4)} - \sum_{r=1}^{14} \frac{1}{(r+3)(r+4)}$$

$$\therefore \sum_{15}^{30} \frac{1}{(r+3)(r+4)} = \frac{30}{4(30+4)} - \frac{14}{4(14+4)}$$

$$= \boxed{\frac{4}{153}}$$

Q2) we are given information about the z-plane (points lie on the real-plane) so make z the subject:

$$w = \frac{1-iz}{z}$$

$$zw = 1 - iz$$

$$z(w+i) = 1$$

$$z = \frac{1}{w+i}$$

$$\therefore x+iy = \frac{1}{u+iv+i} = \frac{1}{u+i(v+1)}$$

$$x+iy = \frac{u-i(v+1)}{(u+i(v+1))(u-i(v+1))} = \frac{u-i(v+1)}{u^2+(v+1)^2}$$

$$\therefore x+iy = \frac{u}{u^2+(v+1)^2} - \frac{(v+1)i}{u^2+(v+1)^2}$$

points lie on the real axis in z-plane so $y = 0$.

$$y = \frac{-(v+1)}{u^2+(v+1)^2} = 0$$

$$-v-1 = 0$$

$$\boxed{v = -1}$$

<u>alt</u>: $w = \frac{1-iz}{z}$

$$u+iv = \frac{1-i(x+iy)}{x+iy}$$

<u>$y=0$</u>: $u+iv = \frac{1-xi}{x}$

$$u+iv = \frac{1}{x} - i$$

<u>compare imaginary terms</u>: $\boxed{v = -1}$

(Q3ai) $\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$

$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$

$= \boxed{\frac{1}{4}\left(\sqrt{6} - \sqrt{2}\right)}$

ii) $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$

$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$

$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \boxed{\frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right)}$

b) $z^4 = 4\left(\cos\left(\frac{\pi}{3} + 2k\pi\right) + i\sin\left(\frac{\pi}{3} + 2k\pi\right)\right)$

$z = 4^{\frac{1}{4}}\left(\cos\left(\frac{6k\pi + \pi}{3}\right) + i\sin\left(\frac{\pi + 6k\pi}{3}\right)\right)^{\frac{1}{4}}$

$z = \sqrt{2}\left(\cos\left(\frac{6k\pi + \pi}{12}\right) + i\sin\left(\frac{6k\pi + \pi}{12}\right)\right)$

$\underline{k=0}$: $z = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

$\boxed{z = \frac{\sqrt{2}}{4}\left(\sqrt{6} + \sqrt{2}\right) + i\frac{\sqrt{2}}{4}\left(\sqrt{6} - \sqrt{2}\right)}$

$\underline{k=1}$: $z = \sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$

$\boxed{z = \frac{\sqrt{2}}{4}\left(-\sqrt{6} + \sqrt{2}\right) + i\frac{\sqrt{2}}{4}\left(\sqrt{6} + \sqrt{2}\right)}$

$\underline{k=2}$: $z = \sqrt{2}\left(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right)$

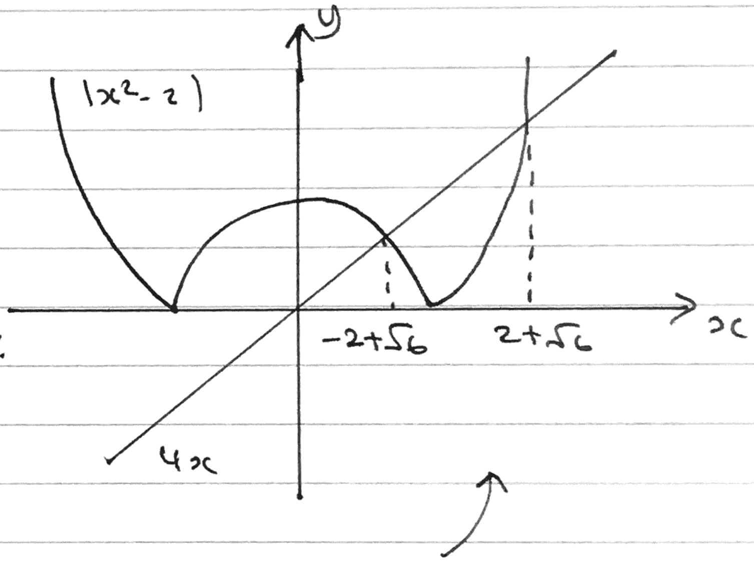
$$\boxed{z = -\frac{\sqrt{2}}{4}(\sqrt{6}+\sqrt{2}) + \frac{i\sqrt{2}}{4}(-\sqrt{6}+\sqrt{2})}$$

$\underline{k=3}$: $z = \sqrt{2}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$

$$z = \frac{\sqrt{2}}{4}(\sqrt{6}-\sqrt{2}) + \frac{i\sqrt{2}}{4}(-\sqrt{6}-\sqrt{2})$$

$$\boxed{z = \frac{\sqrt{2}}{4}(\sqrt{6}-\sqrt{2}) - \frac{i\sqrt{2}}{4}(\sqrt{6}+\sqrt{2})}$$

Q4) $4x = x^2 - 2$



$x^2 - 4x - 2 = 0$

By Quadratic formula:

$x = 2 \pm \sqrt{6}$ //

$x > 0$ for any intersection so $x = 2+\sqrt{6}$ //

$4x = -(x^2 - 2)$

$x^2 + 4x - 2 = 0$

By Quadratic formula:

$x = -2 \pm \sqrt{6}$ //

$x > 0$ for any intersection so $x = -2+\sqrt{6}$ //

region required is where the curve is above the line.

where: $\boxed{\begin{array}{l} x < -2+\sqrt{6} \\ x > 2+\sqrt{6} \end{array}}$

Q5a) $\frac{d}{dx}\left(y y'' + 3x y' - 3y^2 = 0\right)$

$\Rightarrow y'y'' + yy''' + 3y' + 3xy'' - 6yy' = 0$

$\Rightarrow y''' = \dfrac{6yy' - 3xy'' - 3y' - y'y''}{y}$ //

using original eqn:

$x=0,\ y=2,\ y'=1$: $2y'' + 3(0) - 3(2)^2 = 0$

$y'' = \frac{12}{2} = 6$ //

now finding y''': $y''' = \dfrac{6(2)(1) - 0 - 3(1) - 1(6)}{2}$

$= \boxed{\frac{3}{2}}$ //

b) $y=2,\ y'=1,\ y''=6,\ y'''=\frac{3}{2}$

$\therefore y \simeq 2 + x + \dfrac{6x^2}{2} + \dfrac{\frac{3}{2}x^3}{6}$

$\Rightarrow \boxed{y \simeq 2 + x + 3x^2 + \frac{1}{4}x^3}$

Q6a) $6\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = x - 6x^2$

AUX : $6\lambda^2 + 5\lambda - 6 = 0$

$(3\lambda - 2)(2\lambda + 3) = 0$

$\therefore \lambda = \frac{2}{3}, \quad \lambda = -\frac{3}{2}$

C.F : $y = Ae^{\frac{2}{3}x} + Be^{-\frac{3}{2}x}$

P.I : let $y = \alpha x^2 + \mu x + \varphi$

$\dot{y} = 2\alpha x + \mu$

$\ddot{y} = 2\alpha$

Substituting : $6(2\alpha) + 5(2\alpha x + \mu) - 6(\alpha x^2 + \mu x + \varphi)$

$= x - 6x^2$

$\Rightarrow 12\alpha + 10\alpha x + 5\mu - 6\alpha x^2 - 6\mu x - 6\varphi = x - 6x^2$

comparing coefficients : x^2 : $-6\alpha = -6$

$\therefore \alpha = 1$

x : $10\alpha - 6\mu = 1 \quad \therefore \mu = \frac{10-1}{6} = \frac{3}{2}$

constants : $12\alpha + 5\mu - 6\varphi = 0 \quad \therefore \varphi = 13/4$

So...

General solution : $y = Ae^{\frac{2}{3}x} + Be^{-\frac{3}{2}x} + x^2 + \frac{3}{2}x + \frac{13}{4}$

b) $y=0, x=0$: $0 = A + B + \frac{13}{4}$

$\therefore A + B = \frac{-13}{4}$ ~ (1)

$\frac{dy}{dx} = \frac{3}{2}, x=0$: $\frac{dy}{dx} = \frac{2}{3}Ae^{\frac{2}{3}x} - \frac{3}{2}Be^{-\frac{3}{2}x} + 2x + \frac{3}{2}$

$$\cancel{\frac{3}{2}} = \frac{2}{3}A - \frac{3}{2}B + \cancel{\frac{3}{2}}$$

x6 : $4A - 9B = 0$

$$A = \frac{9B}{4}$$

sub A into (1) : $\frac{9B}{4} + B = \frac{-13}{4}$

$$B\left(1 + \frac{9}{4}\right) = -\frac{13}{4}$$

$$\therefore B = \frac{-\frac{13}{4}}{1+\frac{9}{4}} = -1$$

and $A = \frac{9}{4} \times -1 = -\frac{9}{4}$

so particular solution is...

$$\boxed{y = -\frac{9}{4}e^{\frac{2}{3}x} - e^{-\frac{3}{2}x} + x^2 + \frac{3}{2}x + \frac{13}{4}}$$

Q7a) tangent parallel to initial line $\rightarrow \frac{dy}{d\theta} = 0$.

$r = 2 + \sqrt{3}\cos\theta$

$r\sin\theta = 2\sin\theta + \sqrt{3}\sin\theta\cos\theta$

$y = 2\sin\theta + \sqrt{3}\sin\theta\cos\theta = \sin\theta(2+\sqrt{3}\cos\theta)$

$\frac{dy}{d\theta} = \cos\theta(2+\sqrt{3}\cos\theta) + \sin\theta(-\sqrt{3}\sin\theta) = 0$

$\Rightarrow 2\cos\theta + \sqrt{3}\cos^2\theta - \sqrt{3}\sin^2\theta = 0$

$\sin^2\theta = 1-\cos^2\theta$

$\Rightarrow 2\cos\theta + \sqrt{3}\cos^2\theta + \sqrt{3}\cos^2\theta - \sqrt{3} = 0$

$\Rightarrow 2\sqrt{3}\cos^2\theta + 2\cos\theta - \sqrt{3} = 0$

By Quadratic formula : $\cos\theta = \frac{\sqrt{21} \pm \sqrt{3}}{6}$

$\cos\theta \le 1 \quad \therefore \quad \cos\theta = \frac{\sqrt{21} - \sqrt{3}}{6}$ //

distance $OP = r = 2 + \sqrt{3}\left(\frac{\sqrt{21} - \sqrt{3}}{6}\right)$

$= 2 + \frac{3\sqrt{7} - 3}{6}$

$= 2 + \frac{\sqrt{7}}{2} - \frac{1}{2}$

$= \frac{3}{2} + \frac{\sqrt{7}}{2} = \boxed{\frac{1}{2}(3+\sqrt{7})}$

b) $\text{Area} = \frac{1}{2}\int_0^{2\pi} r^2 \, d\theta = \frac{1}{2}\int_0^{2\pi} (2+\sqrt{3}\cos\theta)^2 \, d\theta$

$= \frac{1}{2}\int_0^{2\pi} (4 + 4\sqrt{3}\cos\theta + 3\cos^2\theta) \, d\theta$

$$\boxed{\begin{aligned} \cos 2\theta &= 2\cos^2\theta - 1 \\ \therefore \cos^2\theta &= \frac{1+\cos 2\theta}{2} \end{aligned}}$$

$=$

$= \frac{1}{2}\int_0^{2\pi} \left(4 + 4\sqrt{3}\cos\theta + \frac{3}{2} + \frac{3}{2}\cos 2\theta\right) d\theta$

$= \frac{1}{2}\int_0^{2\pi} \left(\frac{11}{2} + 4\sqrt{3}\cos\theta + \frac{3}{2}\cos 2\theta\right) d\theta$

$= \frac{1}{2}\left[\frac{11\theta}{2} + 4\sqrt{3}\sin\theta + \frac{3}{4}\sin 2\theta\right]_0^{2\pi}$

$= \frac{1}{2}\left[11\pi + 0 + 0\right] - \frac{1}{2}\left[0\right]$

$= \boxed{\frac{11\pi}{2}}$

Q8a)

$$\int (2x^5 e^{-x^2})\,dx$$

$t = x^2$

$\frac{dt}{dx} = 2x$

$dx = \frac{1}{2x}\,dt$

$\Downarrow$

$$\int \left(2x^5 e^{-t} \times \frac{1}{2x}\right) dt$$

$$= \int (x^4 e^{-t})\,dt = \int (t^2 e^{-t})\,dt$$

By parts: $\frac{dv}{dt} = e^{-t} \rightarrow v = -e^{-t}$

$u = t^2 \rightarrow u' = 2t$

$$= \left[-t^2 e^{-t}\right] + 2\int (t e^{-t})\,dt$$

By parts (again)

$\frac{dv}{dt} = e^{-t} \rightarrow v = -e^{-t}$

$u = t \rightarrow u' = 1$

$$\therefore \int (t e^{-t})\,dt = \left[-t e^{-t}\right] + \int (e^{-t})\,dt$$

so $\int (te^{-t})\,dt = -te^{-t} - e^{-t}$

$\therefore \int (t^2e^{-t})\,dt = -t^2e^{-t} - 2te^{-t} - 2e^{-t} + c$ //

$t = x^2$
$t^2 = x^4$

$\therefore \int 2x^5e^{-x^2}\,dx = -x^4e^{-x^2} - 2x^2e^{-x^2} - 2e^{-x^2} + k$

$= \boxed{e^{-x^2}(-x^4 - 2x^2 - 2) + k}$

o.e

b) $x\frac{dy}{dx} + 4y = 2x^2e^{-x^2}$

$\underline{\div x}$: $\frac{dy}{dx} + \left(\frac{4}{x}\right)y = 2xe^{-x^2}$

$I = e^{\int \frac{4}{x}\,dx} = e^{4\ln x} = x^4$ //

$\underline{\times x^4}$: $x^4\frac{dy}{dx} + 4x^3y = 2x^5e^{-x^2}$ //

$\therefore \frac{d}{dx}\left(yx^4\right) = 2x^5e^{-x^2}$

$$yx^4 = \int (2x^5 e^{-x^2})\, dx$$

from (a)... $yx^4 = e^{-x^2}(-x^4 - 2x^2 - 2) + u$

$$\therefore \boxed{y = \frac{-e^{-x^2}(x^4 + 2x^2 + 2) + u}{x^4}}$$

c) $\underline{y = 0, x = 1}$: $0 = \frac{-e^{-1}(5) + u}{1}$

$$\therefore u = 5e^{-1} = \frac{5}{e}$$

hence $\boxed{y = \frac{-e^{-x^2}(x^4 + 2x^2 + 2) + \frac{5}{e}}{x^4}}$