# FP2 June 2017 (MA)

Q1a) $$LHS = \frac{(r+1)^2}{r^2(r+1)^2} - \frac{r^2}{r^2(r+1)^2} = \frac{r^2+2r+1-r^2}{r^2(r+1)^2}$$

$$= \boxed{\frac{2r+1}{r^2(r+1)^2}}$$

b) $\sum_{r=1}^{n} \frac{1}{r^2} - \frac{1}{(r+1)^2}$ : $n=1$ : $1 - \cancel{\frac{1}{4}}$

$n=2$ : $\cancel{\frac{1}{4}} - \cancel{\frac{1}{9}}$

$\dots$

$n=n-1$ : $\cancel{\frac{1}{(n-1)^2}} - \cancel{\frac{1}{n^2}}$

$n=n$ : $\cancel{\frac{1}{n^2}} - \frac{1}{(n+1)^2}$

$$\therefore \sum_{r=1}^{n} \frac{1}{r^2} - \frac{1}{(r+1)^2} = 1 - \frac{1}{(n+1)^2}$$

$$= \frac{(n+1)^2}{(n+1)^2} - \frac{1}{(n+1)^2} = \frac{n^2+2n+1-1}{(n+1)^2}$$

$$= \frac{n^2+2n}{(n+1)^2} = \frac{n(n+2)}{(n+1)^2}$$

c) $$\sum_{r=n}^{3n} \dots = \sum_{r=1}^{3n} \dots - \sum_{r=1}^{n-1} \dots$$

$$\Rightarrow \frac{3n(3n+2)}{(3n+1)^2} - \frac{(n-1)(n-1+2)}{(n-1+1)^2} = \sum_{r=n}^{3n} \frac{2r+1}{r^2(r+1)^2} \quad //$$

$$\therefore \sum_{r=1}^{3n} \frac{2r+1}{r^2(r+1)^2} = \frac{9n^2+6n}{(3n+1)^2} - \frac{(n-1)(n+1)}{n^2}$$

$$= \frac{(9n^2+6n)(n^2)-(n^2-1)(3n+1)^2}{n^2(3n+1)^2}$$

$$= \frac{9n^4+6n^3-[n^2-1][9n^2+6n+1]}{n^2(3n+1)^2}$$

$$= \frac{9n^4+6n^3-[9n^4+6n^3+n^2-9n^2-6n-1]}{n^2(3n+1)^2}$$

$$= \frac{\cancel{9n^4}+\cancel{6n^3}-\cancel{9n^4}-\cancel{6n^3}+8n^2+6n+1}{n^2(3n+1)^2}$$

$$= \frac{8n^2+6n+1}{n^2(3n+1)^2} = \sum_{r=1}^{3n} \frac{2r+1}{r^2(r+1)^2}$$

$$\text{so } \sum_{r=1}^{3n} \frac{6r+3}{r^2(r+1)^2} = 3 \times \left[\frac{8n^2+6n+1}{n^2(3n+1)^2}\right]$$

$$= \boxed{\frac{24n^2+18n+3}{n^2(3n+1)^2}}$$

Q2) multiply both sides by $(x+2)^2$:

$$\frac{(x-2)(x+2)}{2} \leq \frac{12}{x}(x+2)$$

multiply by $2x^2$ next:

$$x^2(x-2)(x+2) \leq 12(2)x(x+2)$$

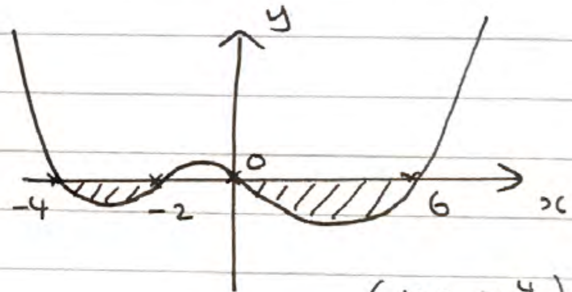
$$x^2(x+2)(x-2) \leq 24x(x+2)$$

$$x(x+2)\left[x(x-2) - 24\right] \leq 0$$

$$x(x+2)\left[x^2-2x-24\right] \leq 0$$

$$x(x+2)(x-6)(x+4) \leq 0$$

critical values: $x=0$, $x=-2$, $x=6$, $x=-4$




(+ve $x^4$).

so the region required is given by:

$$\boxed{0 < x \leq 6}\ ,\quad \boxed{-4 \leq x < -2}$$

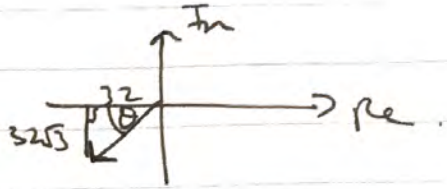
REMEMBER: in the given ~~[illegible]~~, inequality, $(x)$ and $(x+2)$ are on the denominators of the fractions. So $x \neq 0$ and $x \neq -2$.
(Those equations are not defined for these values of $x$)

3) $z^3 = -32 - 32i\sqrt{3}$

$|z^3| = \sqrt{(-32)^2 + (-32\sqrt{3})^2} = 64$ //

$\arg z^3 = -\left(\pi - \frac{\pi}{3}\right) = -\frac{2\pi}{3}$ //

$\left[\tan\theta = \sqrt{3} \;:\; \theta = \frac{\pi}{3}\right]$




$\therefore z^3 = 64\left[e^{\left(-\frac{2\pi}{3}\right)i}\right] = 64\left[e^{\left(-\frac{2\pi}{3} + 2\pi k\right)i}\right]$

$z = 64^{\frac{1}{3}}\left[e^{\left(-\frac{2\pi}{3} + \frac{6\pi k}{3}\right)i}\right]^{\frac{1}{3}}$

$z = 4e^{\left(\frac{6\pi k - 2\pi}{9}\right)i}$ //

$\underline{k=0}$: $z = 4e^{-\frac{2\pi i}{9}}$ //

$\underline{k=1}$: $z = 4e^{\frac{4\pi i}{9}}$ //

$\underline{k=-1}$: $z = 4e^{-\frac{8\pi i}{9}}$ //

4a) $y = \ln\left((1-2x)^{-1}\right)$

$$\frac{dy}{dx} = \frac{-(1-2x)^{-2}(-2)}{(1-2x)^{-1}} = 2(1-2x)^{-1} \;//$$

$$\frac{d^2y}{dx^2} = -2(1-2x)^{-2}(-2) = 4(1-2x)^{-2} \;//$$

$$\frac{d^3y}{dx^3} = -8(1-2x)^{-3}(-2) = 16(1-2x)^{-3} \;//$$

b) $f(0) = \ln(1) = 0$
$f'(0) = 2$
$f''(0) = 4(1) = 4$
$f'''(0) = 16$

$$y \simeq 0 + 2x + \frac{4x^2}{2} + \frac{16x^3}{6}$$

$$\boxed{y \simeq 2x + 2x^2 + \frac{8}{3}x^3}$$

c) $\dfrac{1}{1-2x} = \dfrac{3}{2} \longrightarrow 1-2x = \dfrac{2}{3} \quad \therefore 2x = \dfrac{1}{3}$

so $x = \frac{1}{6}$ //

<u>using $x = \frac{1}{6}$</u>: $\ln\left(\frac{1}{1-2x}\right) \longrightarrow \ln\left(\frac{3}{2}\right)$

$$\ln\left(\frac{3}{2}\right) \simeq 2\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right)^2 + \frac{8}{3}\left(\frac{1}{6}\right)^3$$

$$\simeq \boxed{0.401}$$

5a) Aux : $\lambda^2 - 2\lambda = 0$

$\lambda(\lambda - 2) = 0$

$\lambda = 0, \ \lambda = 2$

C.F : $y = A + Be^{2x}$

P.I : let $y = \lambda \sin 3x + \varphi \cos 3x$

then $\dot{y} = 3\lambda \cos 3x - 3\varphi \sin 3x$

and $\ddot{y} = -9\lambda \sin 3x - 9\varphi \cos 3x$

subbing back : $-9\lambda \sin 3x - 9\varphi \cos 3x - 6\lambda \cos 3x + 6\varphi \sin 3x = 26 \sin 3x$

comparing coefficients : $-9\lambda + 6\varphi = 26 \sim (1)$

$-9\varphi - 6\lambda = 0 \sim (2)$

from (2) : $\lambda = -\frac{3\varphi}{2}$

$\hookrightarrow$ (1) : $\frac{27}{2}\varphi + 6\varphi = 26$

$\therefore \varphi = 4/3$ //

and $\lambda = -2$ //

General solution : $\boxed{y = A + Be^{2x} - 2\sin 3x + \frac{4}{3}\cos 3x}$

b) $y=0, x=0$ : $0 = A + B + \frac{4}{3}$ $\quad \therefore A = -\frac{4}{3} - B$ //

$\frac{dy}{dx} = 2Be^{2x} - 6\cos 3x - 4\sin 3x$

$\frac{dy}{dx} = 0, x = 0$ : $0 = 2B - 6$ $\quad \therefore B = 3$ //

and so $A = -\frac{4}{3} - 3 = -\frac{13}{3}$ //

$\therefore \boxed{y = -\frac{13}{3} + 3e^{2x} - 2\sin 3x + \frac{4}{3}\cos 3x}$

6) Area $= \frac{1}{2}\int_0^{2\pi}(r^2)\,d\theta = \frac{1}{2}\int_0^{2\pi}(6+a\sin\theta)^2\,d\theta = \frac{97\pi}{2}$.

$$= \frac{1}{2}\int_0^{2\pi}\left[36+12a\sin\theta+a^2\sin^2\theta\right]d\theta$$

$$\boxed{\begin{aligned}\cos 2\theta &= 1-2\sin^2\theta \\ \sin^2\theta &= \frac{1-\cos 2\theta}{2}\end{aligned}}$$

$$= \frac{1}{2}\int_0^{2\pi}\left[36+12a\sin\theta+a^2\left[\frac{1}{2}-\frac{1}{2}\cos 2\theta\right]\right]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[36+\frac{a^2}{2}+12a\sin\theta-\frac{1}{2}a^2\cos 2\theta\right]d\theta$$

$$= \frac{1}{2}\left[36\theta+\frac{a^2}{2}\theta-12a\cos\theta-\frac{1}{4}a^2\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[72\pi+a^2\pi-12a\right]-\frac{1}{2}\left[-12a\right] = \frac{97\pi}{2}$$

$$\Rightarrow \frac{\pi(72+a^2)-12a+12a}{\cancel{2}} = \frac{97\pi}{\cancel{2}}$$

$$\Rightarrow \cancel{\pi}(72+a^2) = 97\cancel{\pi}$$

$$\Rightarrow a^2 = 97-72 = 25$$

$$\therefore a = \sqrt{25} = \boxed{5}$$

$(a>0)$.

7a) $\cos x \frac{dy}{dx} + y\sin x = 2\cos^3 x \sin x + 1$

$\div \cos x$ : $\frac{dy}{dx} + y\tan x = 2\cos^2 x \sin x + \sec x$.

$I = e^{\int \tan x \, dx} = e^{\ln|\sec x|} = \underline{\underline{\sec x}}$

$\times \sec x$ : $\sec x \frac{dy}{dx} + y\sec x \tan x = 2\cos x \sin x + \sec^2 x$

$\Rightarrow \frac{d}{dx}(y\sec x) = 2\cos x \sin x + \sec^2 x$

$\Rightarrow y\sec x = \int (\sin 2x)\, dx + \int (\sec^2 x)\, dx$

$\Rightarrow y\sec x = -\frac{1}{2}\cos 2x + \tan x + c$

$\Rightarrow y = \cos x \left(\tan x - \frac{1}{2}\cos 2x + c\right)$

b) $5\sqrt{2} = \cos\frac{\pi}{4}\left(\tan\frac{\pi}{4} - \frac{1}{2}\cos\frac{\pi}{2} + c\right)$

$5\sqrt{2} = \frac{\sqrt{2}}{2}(1 + c)$

$10 = 1 + c \quad \therefore c = 9$

$y = \cos x\left(\tan x - \frac{1}{2}\cos 2x + 9\right)$

$\underline{x = \frac{\pi}{6}}$: $y = \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}}{3} - \frac{1}{4} + 9\right) = \frac{3}{6} + \frac{\sqrt{3}}{2}\left(\frac{35}{4}\right)$

$$\boxed{y = \frac{1}{2} + \frac{35}{8}\sqrt{3}}$$

8a) <u>make z the subject</u> :

$$(1+iz)w = z+3i$$
$$w + wzi - z = 3i$$
$$z(wi-1) = 3i-w$$
$$z = \frac{3i-w}{wi-1}$$

$$|z| = \frac{|3i-w|}{|wi-1|} = 1 \,//.$$

$$\therefore |wi-1| = |3i-w|.$$

$$|(u+iv)i-1| = |3i-u-iv|$$

$$|ui-1-v| = |-u+i(3-v)|$$

$$|(-1-v)+i(u)| = |-u+i(3-v)|$$

<u>squaring</u> : $(-1-v)^2 + \cancel{u^2} = \cancel{u^2} + (3-v)^2$

$$\cancel{v^2}+2v+1 = \cancel{v^2}+9-6v$$

$$8v = 8$$

$$\boxed{v=1}$$

b) $|z-(a+bi)| = c$ in $z$-plane.

$|w| = 5$ in $w$-plane.

make $w$ the subject now:

$$w = \frac{z+3i}{1+zi}$$

$$|w| = \frac{|z+3i|}{|1+zi|} = 5$$

$$\therefore |z+3i| = 5|1+zi|$$

$$|x+iy+3i| = 5|1+(x+iy)i|$$

$$|x+i(y+3)| = 5|1+xi-y|$$

$$|x+i(y+3)| = 5|(1-y)+i(x)|$$

squaring: $x^2+(y+3)^2 = 25[(1-y)^2+x^2]$

$$x^2+y^2+6y+9 = 25[1-2y+x^2+y^2]$$

$$x^2+y^2+6y+9 = 25-50y+25x^2+25y^2$$

$$24x^2+24y^2-56y+16 = 0$$

$\div 24$: $x^2+y^2-\frac{7}{3}y+\frac{2}{3} = 0$

centre: $(\frac{7}{6}, 0)$ $\quad (x-\frac{7}{6})^2+y^2 = \frac{49}{36}-\frac{2}{3}$

$r = \frac{5}{6}$

$$(x-\frac{7}{6})^2+y^2 = \frac{25}{36}$$

$$\boxed{a = 7/6 \quad b = 0 \quad c = \frac{5}{6}}$$