

$$\textcircled{1} \quad \frac{x}{x+1} < \frac{2}{x+2} \Rightarrow \frac{x(x+2) - 2(x+1)}{(x+1)(x+2)} < 0 \Rightarrow \frac{x^2 - 2}{(x+1)(x+2)} < 0 \quad [6]$$

c.v's are $-2, \pm\sqrt{2}, -1$.

$$x < -2 \quad | \quad -2 < x < -\sqrt{2} \quad | \quad -\sqrt{2} < x < -1 \quad | \quad -1 < x < \sqrt{2} \quad | \quad x > \sqrt{2}$$

$>_0 \qquad <_0 \qquad >_0 \qquad <_0 \qquad >_0$

$$\text{so } \left\{ x \in \mathbb{R} \mid -2 < x < -\sqrt{2} \right\} \cup \left\{ x \in \mathbb{R} \mid -1 < x < \sqrt{2} \right\}$$

$$\textcircled{2} \textcircled{2} \quad \frac{(r-3)(r+1)(r+2) + (r+2) - (r+1)}{(r+1)(r+2)} = \frac{(r^2+3r+2)(r-3) + 1}{(r+1)(r+2)}$$

[2]

$$= \frac{r^3 - 3r^2 + 3r^2 + 2r - 9r - 6 + 1}{(r+1)(r+2)} = \frac{r^3 - 7r - 5}{(r+1)(r+2)} \equiv r-3 + \frac{1}{r+1} - \frac{1}{r+2}$$

$$\textcircled{b} \quad \sum_{r=1}^n \frac{r^3 - 7r - 5}{(r+1)(r+2)} = \sum_{r=1}^n r-3 + \frac{1}{r+1} - \frac{1}{r+2} = \frac{1}{2}n(n+1) - 3n + \sum_{r=1}^n \frac{1}{r+1} - \frac{1}{r+2}$$

where the latter sum is $\frac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{3} - \frac{1}{4}} + \dots + \frac{\frac{1}{n} - \frac{1}{n+1}}{\frac{1}{n+1} - \frac{1}{n+2}} = \frac{\frac{1}{2} - \frac{1}{n+2}}{n+2} = \frac{n}{2(n+2)}$.

[5]

$$\begin{aligned} & \frac{1}{2} - \frac{1}{3} \\ & \frac{1}{3} - \frac{1}{4} \\ & + \dots \\ & + \frac{1}{n} - \frac{1}{n+1} \\ & + \frac{1}{n+1} - \frac{1}{n+2} \end{aligned}$$

$$\text{so } \frac{\frac{1}{2}n(n+1) - 6n}{2} + \frac{n}{2(n+2)} = \frac{n(n+1)(n+2) - 6n(n+2) + n}{2(n+2)} = \frac{n(n^2+3n+2-6n-12+1)}{2(n+2)}$$

$$= \frac{n(n^2-3n-9)}{2(n+2)}$$

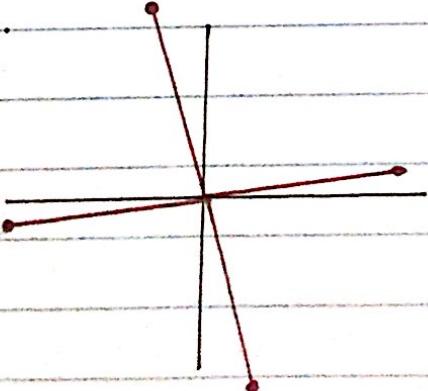
$$\textcircled{3} \textcircled{a} \quad -\frac{1}{2}\text{i} \quad \text{so} \quad \arg(8\sqrt{3} + 8i) = \frac{\pi}{6}. \quad |8\sqrt{3} + 8i| = 16. \quad [5]$$

$$\text{so } z^4 = 8\sqrt{3} + 8i \Rightarrow z^4 = 16e^{i(\frac{\pi}{6} + 2k\pi)} \Rightarrow z = 2e^{i(\frac{\pi}{24} + \frac{k\pi}{2})}$$

$$\therefore z = 2e^{i\pi/24}, 2e^{i13\pi/24}, 2e^{i25\pi/24}, 2e^{i37\pi/24}.$$

(b)

[2]



$$\textcircled{4} \quad (\text{i}) \quad p \frac{dx}{dt} + qx = c. \quad \xrightarrow{\text{(P.F.)}} \quad \frac{dx}{dt} + \frac{q}{p}x = \frac{c}{p}.$$

[4]

(ii) Find x in terms of t (given $x=0, t=0$).

Integrating Factor $e^{\int p dt} = e^{qt/p}$

$$\text{So } x e^{qt/p} = \frac{c}{p} \cdot p e^{qt/p} + c.$$

$$(0,0) : 0 = \frac{c}{q} + c \Rightarrow c = -cq.$$

$$\therefore x = \frac{c}{q} (1 - e^{-qt/p}). //$$

(b) as $t \rightarrow \infty$ we have $e^{-qt/p} \rightarrow 0$ so $x \rightarrow \frac{c}{q}.$ //

[1]

$$(\text{ii}) \quad \frac{dy}{d\theta} + 2y = \sin \theta. \quad \text{Complementary function: } \frac{dy}{d\theta} = -2y$$

[7]

$$\text{So } \ln y = -2\theta + c \Rightarrow y_{\text{CF}} = A e^{-2\theta}.$$

Guess a particular integral $y_{\text{PI}} = a \sin \theta + b \cos \theta$

$$\text{so } a \cos \theta - b \sin \theta + 2a \sin \theta + 2b \cos \theta = \sin \theta$$

$$\Rightarrow \sin \theta (2a - b) + \cos \theta (2b + a) = \sin \theta$$

$$\text{equating coefficients: } 2a - b = 1 \quad (1) \quad \text{so } 2 \times (1) + (2) : 5a = 2$$

$$2b + a = 0 \quad (2) \quad \Rightarrow a = 2/5.$$

$$\Rightarrow b = -1/5.$$

$$\text{so } y = y_{\text{CF}} + y_{\text{PI}} = A e^{-2\theta} + \frac{2}{5} \sin \theta - \frac{1}{5} \cos \theta$$

$$\text{Given } (0,0) : 0 = A - \frac{1}{5} \Rightarrow A = 1/5.$$

$$\therefore y = \frac{1}{5} (e^{-2\theta} + 2 \sin \theta - \cos \theta) //$$

$$\textcircled{5} \quad \textcircled{a} \quad \sin^5 \theta = a \sin 5\theta + b \sin 3\theta + c \sin \theta. \quad [5]$$

Recall that $\sin \theta = \text{Im } z$ if $z = \cos \theta + i \sin \theta$. Then $z - z^{-1} = 2i \sin \theta$.

$$\begin{aligned} \text{so } (2i \sin \theta)^5 &= (z - z^{-1})^5 = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5} \\ &= z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right) = 2i \sin 5\theta - 10i \sin 3\theta \\ &= 2i \sin 5\theta - 10i \sin 3\theta + 20i \sin \theta \end{aligned}$$

$$\text{so } 32i \sin^5 \theta = 2i \sin 5\theta - 10i \sin 3\theta + 20i \sin \theta$$

$$\Rightarrow \sin^5 \theta = \frac{1}{16} \sin 5\theta - \frac{5}{16} \sin 3\theta + \frac{5}{8} \sin \theta //$$

$$\begin{aligned} \textcircled{b} \quad \int_0^{\pi/3} \sin^5 \theta \, d\theta &= \frac{1}{16} \int_0^{\pi/3} \sin 5\theta - 5\sin 3\theta + 10 \sin \theta \, d\theta \quad [5] \\ &= \frac{1}{16} \left[-\frac{1}{5} \cos 5\theta + \frac{5}{3} \cos 3\theta - 10 \cos \theta \right]_0^{\pi/3} \\ &= \frac{1}{16} \left[-\frac{1}{5} - \frac{5}{3} - 5 + \frac{1}{5} - \frac{5}{3} + 10 \right] \\ &= \frac{1}{16} \left[\frac{53}{30} \right] = \frac{53}{480} // \end{aligned}$$

⑥ @ Let $f(x) = \tan x$. $f(\pi/4) = 1$. [7]

$$f'(x) = \sec^2 x \Rightarrow f'(\pi/4) = 2.$$

$$f''(x) = 2\sec x \cdot \sec x \tan x \Rightarrow f''(\pi/4) = 4$$

$$f'''(x) = \frac{d}{dx} (2\sec^2 x \tan x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$$

$$\Rightarrow f'''(\pi/4) = 16.$$

$$\text{so } f(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 //$$

⑥ Set $x = \frac{5\pi}{12}$: $\tan \frac{5\pi}{12} \approx 1 + 2\left(\frac{5\pi}{12} - \frac{\pi}{4}\right) + 2\left(\frac{5\pi}{12} - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(\frac{5\pi}{12} - \frac{\pi}{4}\right)^3$

$$= 1 + \frac{\pi}{3} + \frac{\pi^2}{18} + \frac{\pi^3}{81}. [2]$$

⑦ @ Sub $x = e^u$ into $x^2 y'' - 2xy' + 2y = -x^2$. [6]

$$\text{so } \frac{dx}{du} = e^u \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot e^{-u}.$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{du} \cdot e^{-u} \right) = \frac{d}{du} \left(\frac{dy}{du} \cdot e^{-u} \right) \cdot \frac{du}{dx} = e^{-u} \left(\frac{d^2y}{du^2} e^{-u} - e^{-u} \frac{dy}{du} \right)$$

$$= e^{-2u} \left(\frac{d^2y}{du^2} - \frac{dy}{du} \right).$$

$$\text{so } e^{2u} \cdot e^{-2u} \left(\frac{d^2y}{du^2} - \frac{dy}{du} \right) - 2e^u e^{-u} \frac{dy}{du} + 2y = -e^{-2u}$$

$$\Leftrightarrow \frac{d^2y}{du^2} - 3 \frac{dy}{du} + 2y = -e^{-2u} // \text{ as required.}$$

⑥ Aux equation: $\lambda^2 - 3\lambda + 2 = 0 \Leftrightarrow (\lambda-2)(\lambda-1) = 0 \Leftrightarrow \lambda = 2 \text{ or } \lambda = 1$

$$\text{so } y_{CF} = Ae^{2u} + Be^u. [7]$$

$$\text{guess } y_{PI} = me^{-2u} \Rightarrow y' = -2me^{-2u} \Rightarrow y'' = 4me^{-2u}.$$

$$\text{so } 4m e^{-2u} + 6m e^{-2u} + 2m e^{-2u} = -e^{-2u} \Rightarrow 12m = -1 \Rightarrow m = -\frac{1}{12}.$$

$$\therefore y = Ae^{2u} + Be^u - \frac{1}{12}e^{-2u} //$$

$$\textcircled{c} \quad y = Ax^2 + Bx - \frac{1}{12x^2}$$

[1]

$$\textcircled{8} \quad \textcircled{a} \quad \text{Intersect at } 7\cos\theta = 3 + 3\cos\theta \Leftrightarrow \cos\theta = 3/4.$$

[3]

$$\text{so } p\left(\frac{\pi}{4}, \arccos \frac{3}{4}\right), q\left(\frac{\pi}{4}, -\arccos \frac{3}{4}\right).$$

(b) By symmetry we only need to find the top half and then double. Strategy is do integrate C_2 from 0 to $\arccos 3/4$ and then C_1 from $\arccos 3/4$ to $\pi/2$.

$$\text{For convenience: } \alpha = \arccos 3/4 : \quad 9 \int_0^\alpha (1 + 2\cos\theta + \cos^2\theta) d\theta$$

$$= 9 \left[\frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta \right]_0^\alpha = 9 \left[\frac{3\alpha}{2} + 2 \times \frac{1}{4} \times \sqrt{7} + \frac{1}{4} \times \frac{3}{8} \times \sqrt{7} \right]$$

$$= 9 \left[\frac{3\alpha}{2} + \frac{19}{32} \sqrt{7} \right] = 9 \left[\frac{3\alpha}{2} + \frac{19}{32} \sqrt{7} \right]$$

$$\text{And } \int_\alpha^{\pi/2} 4q_2 (1 + \cos 2\theta) d\theta = \frac{49}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_\alpha^{\pi/2}$$

$$= \frac{49}{2} \left[\frac{\pi}{2} - \alpha - \frac{3}{16} \sqrt{7} \right].$$

$$\text{So adding: } R = \frac{49\pi}{4} - 11\alpha + \cancel{\frac{168}{32} \sqrt{7}} + \frac{3}{4} \sqrt{7}$$

$$\approx 32.5 \quad (3 \text{ s.f})$$

$$(\text{exact: } \frac{49\pi}{4} - 11\arccos 3/4 + \frac{3\sqrt{7}}{4}).$$