

$$\textcircled{1} \quad \frac{x}{x+1} < \frac{2}{x+2} \Rightarrow \frac{x(x+2) - 2(x+1)}{(x+1)(x+2)} < 0 \Rightarrow \frac{x^2 - 2}{(x+1)(x+2)} < 0 \quad [6]$$

c.v's are $-2, \pm\sqrt{2}, -1$.

$$\begin{array}{c|c|c|c|c} x < -2 & -2 < x < -\sqrt{2} & -\sqrt{2} < x < -1 & -1 < x < \sqrt{2} & x > \sqrt{2} \\ \hline > 0 & < 0 & > 0 & < 0 & > 0 \end{array}$$

$$\text{so } \left\{ x \in \mathbb{R} \mid -2 < x < -\sqrt{2} \right\} \cup \left\{ x \in \mathbb{R} \mid -1 < x < \sqrt{2} \right\}$$

$$\textcircled{4} \text{ (i) } p \frac{dx}{dt} + qx = r. \quad (\text{I} \neq 0) \quad \Leftrightarrow \quad \frac{dx}{dt} + \frac{q}{p}x = \frac{r}{p}.$$

② Find x in terms of t (given $x=0, t=0$).

[4]

Integrating factor $e^{\int \frac{q}{p} dt} = e^{qt/p}$

$$\text{So } x e^{qt/p} = \frac{r}{p} \int e^{qt/p} dt + C.$$

$$(0,0) : 0 = \frac{r}{q} + C \Rightarrow C = -r/q.$$

$$\therefore x = \frac{r}{q} (1 - e^{-qt/p}) //$$

③ as $t \rightarrow \infty$ we have $e^{-qt/p} \rightarrow 0$ so $x \rightarrow \frac{r}{q}$.

[1]

$$\text{(ii) } \frac{dy}{d\theta} + 2y = \sin \theta. \quad \text{Complementary function: } \frac{dy}{d\theta} = -2y$$

[7]

$$\text{So } \ln y = -2\theta + C \Rightarrow y_{CF} = A e^{-2\theta}.$$

Guess a particular integral $y_{PI} = a \sin \theta + b \cos \theta$

$$\text{So } a \cos \theta - b \sin \theta + 2a \sin \theta + 2b \cos \theta = \sin \theta$$

$$\Rightarrow \sin \theta (2a - b) + \cos \theta (2b + a) = \sin \theta$$

$$\text{Equating coefficients: } 2a - b = 1 \quad \textcircled{1} \quad \text{So } 2 \times \textcircled{1} + \textcircled{2} : 5a = 2$$

$$2b + a = 0 \quad \textcircled{2} \quad \Rightarrow a = 2/5.$$

$$\Rightarrow b = -1/5.$$

$$\text{So } y = y_{CF} + y_{PI} = A e^{-2\theta} + \frac{2}{5} \sin \theta - \frac{1}{5} \cos \theta$$

$$\text{Given } (0,0) : 0 = A - \frac{1}{5} \Rightarrow A = 1/5.$$

$$\therefore y = \frac{1}{5} (e^{-2\theta} + 2 \sin \theta - \cos \theta) //$$

$$(5) \text{ a) } \sin^5 \theta = a \sin 5\theta + b \sin 3\theta + c \sin \theta. \quad [5]$$

Recall that ~~$\sin \theta$~~ if $z = \cos \theta + i \sin \theta$ then $z - z^{-1} = 2i \sin \theta$.

$$\text{so } (2i \sin \theta)^5 = (z - z^{-1})^5 = z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$$

$$= z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right) = \cancel{2i \sin 5\theta} - \cancel{10i \sin 3\theta}$$

$$= 2i \sin 5\theta - 10i \sin 3\theta + 20i \sin \theta$$

$$\text{so } 32i \sin^5 \theta = 2i \sin 5\theta - 10i \sin 3\theta + 20i \sin \theta$$

$$\Rightarrow \sin^5 \theta = \frac{1}{16} \sin 5\theta - \frac{5}{16} \sin 3\theta + \frac{5}{8} \sin \theta //$$

$$(b) \int_0^{\pi/3} \sin^5 \theta \, d\theta = \frac{1}{16} \int_0^{\pi/3} \sin 5\theta - 5 \sin 3\theta + 10 \sin \theta \, d\theta \quad [5]$$

$$= \frac{1}{16} \left[\frac{-1}{5} \cos 5\theta + \frac{5}{3} \cos 3\theta - 10 \cos \theta \right]_0^{\pi/3}$$

$$= \frac{1}{16} \left[\frac{-1}{10} - \frac{5}{3} - 5 + \frac{1}{5} - \frac{5}{3} + 10 \right]$$

$$= \frac{1}{16} \left[\frac{53}{30} \right] = \frac{53}{480} //$$

⑥ a) Let $f(x) = \tan x$. $f(\pi/4) = 1$. [7]

$$f'(x) = \sec^2 x \Rightarrow f'(\pi/4) = 2.$$

$$f''(x) = 2 \sec x \cdot \sec x \tan x \Rightarrow f''(\pi/4) = 4$$

$$f'''(x) = \frac{d}{dx} (2 \sec^2 x \tan x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x$$

$$\Rightarrow f'''(\pi/4) = 16.$$

$$\text{so } f(x) = 1 + 2 \left(x - \frac{\pi}{4}\right) + 2 \left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3} \left(x - \frac{\pi}{4}\right)^3 //$$

⑥ Set $x = \frac{5\pi}{12}$: $\tan \frac{5\pi}{12} \approx 1 + 2 \left(\frac{5\pi}{12} - \frac{\pi}{4}\right) + 2 \left(\frac{5\pi}{12} - \frac{\pi}{4}\right)^2 + \frac{8}{3} \left(\frac{5\pi}{12} - \frac{\pi}{4}\right)^3$

$$= 1 + \frac{\pi}{3} + \frac{\pi^2}{18} + \frac{\pi^3}{81}. \quad [2]$$

⑦ a) Sub $x = e^u$ into $x^2 y'' - 2xy' + 2y = -x^{-2}$. [6]

$$\text{so } \frac{dx}{du} = e^u \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot e^{-u}.$$

$$\text{and } \frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{du} \cdot e^{-u} \right) = \frac{d}{du} \left(\frac{dy}{du} \cdot e^{-u} \right) \cdot \frac{du}{dx} = e^{-u} \left(\frac{d^2 y}{du^2} e^{-u} - e^{-u} \frac{dy}{du} \right)$$

$$= e^{-2u} \left(\frac{d^2 y}{du^2} - \frac{dy}{du} \right).$$

$$\text{so } e^{2u} \cdot e^{-2u} \left(\frac{d^2 y}{du^2} - \frac{dy}{du} \right) - 2e^u e^{-u} \frac{dy}{du} + 2y = -e^{-2u}$$

$$\Rightarrow \frac{d^2 y}{du^2} - 3 \frac{dy}{du} + 2y = -e^{-2u} // \text{ as required.}$$

⑥ Aux equation: $\lambda^2 - 3\lambda + 2 = 0 \Leftrightarrow (\lambda - 2)(\lambda - 1) = 0 \Leftrightarrow \lambda = 2 \text{ or } \lambda = 1$

$$\text{so } y_{CF} = Ae^{2u} + Be^u. \quad [7]$$

$$\text{guess } y_{PI} = m e^{-2u} \Rightarrow y' = -2m e^{-2u} \Rightarrow y'' = 4m e^{-2u}.$$

$$\text{so } 4m e^{-2u} + 6m e^{-2u} + 2m e^{-2u} = -e^{-2u} \Rightarrow 12m = -1 \Rightarrow m = -\frac{1}{12}.$$

$$\therefore y = Ae^{2u} + Be^u - \frac{1}{12} e^{-2u} //$$

$$\textcircled{c} y = Ax^2 + Bx - \frac{1}{12x^2} \quad [1]$$

$$\textcircled{d} \textcircled{a} \text{ Intersect at } 7 \cos \theta = 3 + 3 \cos \theta \Leftrightarrow \cos \theta = 3/4. \quad [3]$$

$$\text{so } p \left(\frac{21}{4}, \arccos \frac{3}{4} \right), \quad \theta \left(\frac{21}{4}, -\arccos \frac{3}{4} \right).$$

\textcircled{b} By symmetry we only need to find the top half and then double. Strategy is to integrate C_2 from 0 to $\arccos 3/4$ and then C_1 from $\arccos 3/4$ to $\pi/2$. [7]

$$\text{For convenience: } \alpha = \arccos 3/4: \quad 9 \int_0^\alpha 1 + 2 \cos \theta + \cos^2 \theta \, d\theta$$

$$= 9 \left[\frac{3\theta}{2} + 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^\alpha = 9 \left[\frac{3\alpha}{2} + 2 \times \frac{4}{4} \times \sqrt{7} + \frac{1}{4} \times \frac{3}{4} \times \sqrt{7} \right]$$

$$= 9 \left[\frac{3\alpha}{2} + \frac{19}{32} \sqrt{7} \right] = 9 \left[\frac{3\alpha}{2} + \frac{19}{32} \sqrt{7} \right]$$

$$\text{And } \int_\alpha^{\pi/2} 49/2 (1 + \cos 2\theta) \, d\theta = \frac{49}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_\alpha^{\pi/2}$$

$$= \frac{49}{2} \left[\frac{\pi}{2} - \alpha - \frac{3}{16} \sqrt{7} \right].$$

$$\text{So adding: } R = \frac{49\pi}{4} - 11\alpha + \frac{3\sqrt{7}}{4}$$

$$\approx 32.5 \quad (3 \text{ s.f.})$$

$$\text{(exact: } \frac{49\pi}{4} - 11 \arccos 3/4 + \frac{3\sqrt{7}}{4} \text{)}$$