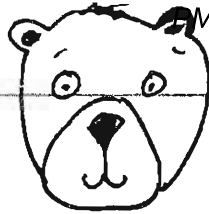


S15 FP2

**1.** (a) Use algebra to find the set of values of $x$ for which

$$x + 2 > \frac{12}{x + 3}$$

**(6)**

(b) Hence, or otherwise, find the set of values of $x$ for which

$$x + 2 > \frac{12}{|x + 3|}$$

**(1)**

a) $(x+2)(x+3)^2 > \frac{12(x+3)^{\cancel{2}}}{\cancel{(x+3)}}$

$\Rightarrow (x+2)(x+3)^2 - 12(x+3) > 0$

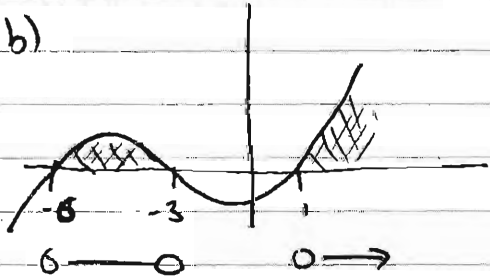
$(x+3)[(x+2)(x+3) - 12] > 0$

$(x+3)(x^2+5x+6-12) > 0$

$(x+3)(x+6)(x-1) > 0$

$-3 \quad -6 \quad 1$

b)




$>0$

$-6 < x < -3$

or

$x > 1$

**2.**

$$z = -2 + \left(2\sqrt{3}\right)\mathrm{i}$$

(a) Find the modulus and the argument of $z$.

**(3)**

Using de Moivre's theorem,

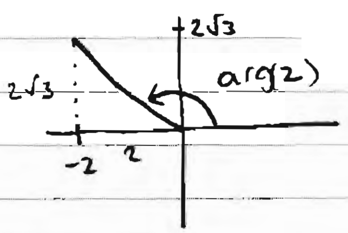
(b) find $z^6$, simplifying your answer,

**(2)**

(c) find the values of $w$ such that $w^4 = z^3$, giving your answers in the form $a + \mathrm{i}b$ where $a, b \in \mathbb{R}$.

**(4)**

a) $|z| = \sqrt{2^2+(2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$




$\arg(z) = \pi - \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$

$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

b) $z = \cos\theta + i\sin\theta \Rightarrow z = 4\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$

$z^n = \cos n\theta + i\sin n\theta \Rightarrow z^6 = 4^6\left[\cos\left(\frac{12\pi}{3}\right) + i\sin\left(\frac{12\pi}{3}\right)\right]$

$\Rightarrow z^6 = 4^6 \times 1 = 4096$

c) $z^3 = 4^3\left[\cos\left(\frac{6\pi}{3}\right) + i\sin\left(\frac{6\pi}{3}\right)\right] = 64[1] = 64$

$\therefore w^4 = 64\left[\cos(0+2\pi k) + i\sin(0+2\pi k)\right]$

$\Rightarrow w = \sqrt[4]{64}\left[\cos\left(\frac{0+2\pi k}{4}\right) + i\sin\left(\frac{0+2\pi k}{4}\right)\right]$

$w = 2\sqrt{2}\left[\cos\left(\tfrac{1}{2}\pi k\right) + i\sin\left(\tfrac{1}{2}\pi k\right)\right]$

$k=0$ $\quad w_0 = 2\sqrt{2}(\cos 0 + i\sin 0) = 2\sqrt{2}$

$k=1$ $\quad w_1 = 2\sqrt{2}\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 2\sqrt{2}(i) = 2\sqrt{2}i$

$k=2$ $\quad w_2 = 2\sqrt{2}\left(\cos(\pi) + i\sin(\pi)\right) = 2\sqrt{2}(-1) = -2\sqrt{2}$

$k=3$ $\quad w_3 = 2\sqrt{2}\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right) = 2\sqrt{2}(-i) = -2\sqrt{2}i$

3. Find, in the form $y = f(x)$, the general solution of the differential equation

$$\tan x \frac{dy}{dx} + y = 3\cos 2x \tan x, \qquad 0 < x < \frac{\pi}{2}$$

**(6)**

$(\div \tan x) \quad \frac{dy}{dx} + \frac{1}{\tan x} y = 3\cos 2x$

$IF = e^{\int \frac{1}{\tan x} dx} = e^{\int \cot x \, dx} = e^{\ln|\sin x|}$

$= \sin x$

$\Rightarrow \sin x \frac{dy}{dx} + \frac{\sin x}{\tan x} y = 3\cos 2x \sin x$

$\Rightarrow \frac{d}{dx}(y \sin x) = 3\cos 2x \sin x$

lots of methods to integrate

$\therefore y\sin x = 3\int \cos 2x \sin x \, dx$

$y \sin x = 3\int (2\cos^2 x - 1)\sin x \, dx$

$= 3\int 2\sin x \cos^2 x - \sin x \, dx$

$= 6\int \sin x \cos^2 x \, dx \quad - 3\int \sin x \, dx$

$\Rightarrow u = \cos x \qquad\qquad + 3\cos x$

$\frac{du}{dx} = -\sin x$

$dx = \frac{-du}{\sin x}$

$\therefore y \sin x = -2\cos^3 x + 3\cos x + c$

$= -6\int u^2 du$

$= -\frac{6u^3}{3}$

$\therefore y = \frac{3\cos x - 2\cos^3 x + c}{\sin x}$

$= -2\cos^3 x$

Z

**4.** (a) Show that

$$r^2(r+1)^2 - (r-1)^2 r^2 \equiv 4r^3$$

**(3)**

Given that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

(b) use the identity in (a) and the method of differences to show that

$$(1^3 + 2^3 + 3^3 + \ldots + n^3) = (1 + 2 + 3 + \ldots + n)^2$$

**(4)**

a) $r^2(r+1)^2 - (r-1)^2 r^2$

$= r^2\left[(r+1)^2 - (r-1)^2\right] = r^2\left[\cancel{r^2} + 2r + \cancel{1} - \cancel{r^2} + 2r - \cancel{1}\right] = r^2 \times 4r$

$= 4r^3$

b) $1^3 + 2^3 + \ldots + n^3 = \sum_1^n r^3$

$\sum_1^n 4r^3 = \sum_1^n r^2(r+1)^2 - (r-1)^2 r^2$

$\therefore \sum_1^n r^3 = \frac{1}{4}\sum_1^n r^2(r+1)^2 - (r-1)^2 r^2$

$= \frac{1}{4}\left[(1^2 \times \cancel{2^2} - 0) + (\cancel{2^2 \times 3^2} - \cancel{1^2 \times 2^2}) + (\cancel{3^2 \times 4^2} - \cancel{2^2 \times 3^2}) \ldots\right.$

$\left.(\cancel{(n-1)^2 n^2} - \cancel{(n-2)^2(n-1)^2}) + (n^2(n+1)^2 - \cancel{(n-1)^2 n^2})\right]$

$= \frac{1}{4}n^2(n+1)^2 \quad = \left[\frac{1}{2}n(n+1)\right]^2$

$= \left[\sum_1^n r\right]^2 = (1+2+3+\ldots+n)^2$

$\therefore 1^3 + 2^3 + \ldots + n^3 = (1+2+3+\ldots+n)^2$

**5.** A transformation $T$ from the $z$-plane to the $w$-plane is given by

$$w = \frac{z}{z + 3\mathrm{i}}, \quad z \neq -3\mathrm{i}$$

The circle with equation $|z| = 2$ is mapped by $T$ onto the curve $C$.

(a) (i) Show that $C$ is a circle.

(ii) Find the centre and radius of $C$.

**(8)**

The region $|z| \leqslant 2$ in the $z$-plane is mapped by $T$ onto the region $R$ in the $w$-plane.

(b) Shade the region $R$ on an Argand diagram.

**(2)**

a) $w(z+3i) = z \Rightarrow zw + 3wi = z$

$\Rightarrow 3wi = z - zw \Rightarrow z(1-w) = 3iw$

$\therefore z = \dfrac{3iw}{1-w}$

$|z| = 2 \Rightarrow \left|\dfrac{3iw}{1-w}\right| = 2 \Rightarrow |3iw| = 2|w-1|$

$\Rightarrow |i||3w| = 2|w-1|$

$w = u + iv \Rightarrow \quad \Rightarrow |3w| = 2|w-1|$

$\Rightarrow |3u + 3iv| = 2|(u-1) + iv|$

$\Rightarrow |3u + 3iv| = |(2u-2) + 2iv|$

$\Rightarrow 9u^2 + 9v^2 = 4u^2 + 4 - 8u + 4v^2$

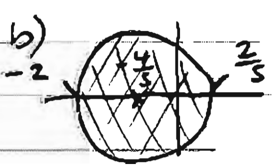
$\Rightarrow 5u^2 + 8u + 5v^2 = 4$

$\Rightarrow u^2 + \frac{8}{5}u + v^2 = \frac{4}{5}$

$\Rightarrow \left(u + \frac{4}{5}\right)^2 + v^2 = \frac{4}{5} + \frac{16}{25} = \frac{36}{25} = \left(\frac{6}{5}\right)^2$

Circle, radius $= \frac{6}{5}$ $\left(-\frac{4}{5}, 0\right)$ centre

b)



test $z = 0 + 0i$ $\quad w = \dfrac{0}{0+3i} = 0$ maps to inside.

**6.**

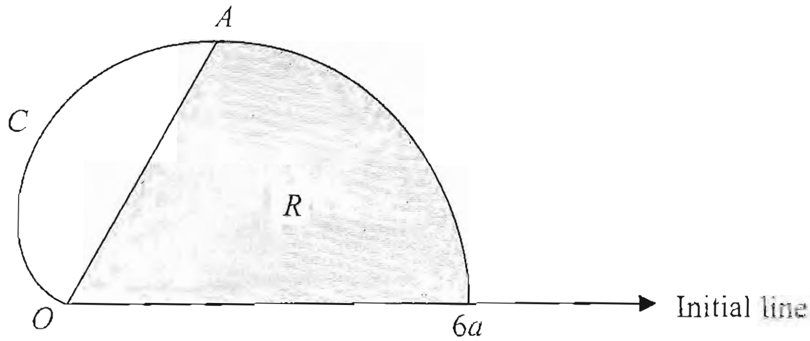



**Figure 1**

The curve $C$, shown in Figure 1, has polar equation

$$r = 3a\,(1 + \cos\theta), \qquad 0 \leqslant \theta < \pi$$

The tangent to $C$ at the point $A$ is parallel to the initial line.

(a) Find the polar coordinates of $A$.

**(6)**

The finite region $R$, shown shaded in Figure 1, is bounded by the curve $C$, the initial line and the line $OA$.

(b) Use calculus to find the area of the shaded region $R$, giving your answer in the form $a^2\left(p\pi + q\sqrt{3}\right)$, where $p$ and $q$ are rational numbers.

**(5)**

$y = r\sin\theta = 3a(1+\cos\theta)\sin\theta$

$y = 3a[(1+\cos\theta)(\sin\theta)]$ $\qquad u = 1+\cos\theta \quad v = \sin\theta$

$\qquad u' = -\sin\theta \quad v' = \cos\theta$

$\frac{dy}{d\theta} = 3a[-\sin^2\theta + \cos\theta + \cos^2\theta]$ $\qquad \frac{d}{dx}uv = vu' + uv'$

$= 3a[\cos^2\theta - 1 + \cos\theta + \cos^2\theta]$

$= 3a[2\cos^2\theta + \cos\theta - 1]$

$= 3a[(2\cos\theta - 1)(\cos\theta + 1)]$

$\frac{dy}{d\theta} = 0 \Rightarrow \cos\theta = \frac{1}{2} \quad \cos\theta = -1$

$\theta = \frac{\pi}{3} \quad \cancel{\theta = \pi} \quad 0 \le \theta < \pi$

$\therefore r = 3a(1 + \cos\frac{\pi}{3})$

$r = 3a \times \frac{3}{2} = \frac{9a}{2}$

$\therefore A\left(\frac{9}{2}a, \frac{\pi}{3}\right)$

b) $\text{Area} = \frac{1}{2}\int_0^{\pi} r^2 \, d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{3}} 9a^2(1+(\cos\theta)^2 \, d\theta$

$= \frac{9}{2}a^2 \int \cos^2\theta + 2\cos\theta + 1 \, d\theta$

$= \frac{9}{2}a^2 \int \left(\frac{1}{2}\cos 2\theta + \frac{1}{2}\right) + 2\cos\theta + 1 \, d\theta$

$= \frac{9}{4}a^2 \int \cos 2\theta + 4\cos\theta + 3 \, d\theta$

$= \frac{9}{4}a^2 \left[\frac{1}{2}\sin 2\theta + 4\sin\theta + 3\theta\right]_0^{\frac{\pi}{3}}$

$= \frac{9}{4}a^2 \left[\left(\frac{1}{2}\sin\frac{2\pi}{3} + 4\sin\frac{\pi}{3} + \pi\right) - (0+0+0)\right]$

$= \frac{9}{4}a^2 \left[\frac{9}{2}\sin\frac{\pi}{3} + \pi\right] = \frac{9}{4}a^2\left[\frac{9\sqrt{3}}{4} + \pi\right]$

$= a^2\left[\frac{81\sqrt{3}}{16} + \frac{9\pi}{4}\right]$

**7.**

$$y = \tan^2 x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

(a) Show that $\dfrac{d^2y}{dx^2} = 6\sec^4 x - 4\sec^2 x$

**(4)**

(b) Hence show that $\dfrac{d^3y}{dx^3} = 8\sec^2 x \tan x\,(A\sec^2 x + B)$, where $A$ and $B$ are constants to be found.

**(3)**

(c) Find the Taylor series expansion of $\tan^2 x$, in ascending powers of $\left(x - \frac{\pi}{3}\right)$, up to and including the term in $\left(x - \frac{\pi}{3}\right)^3$

**(4)**

$y = \tan^2 x \Rightarrow \dfrac{dy}{dx} = 2(\tan x)' \times \sec^2 x$

$\Rightarrow \dfrac{dy}{dx} = 2\tan x \sec^2 x$

$u = \tan x \qquad v = (\sec x)^2$

$u' = \sec^2 x \qquad v' = 2\sec x \times \sec x \tan x$

$= 2\sec^2 x \tan x$

$\dfrac{d^2y}{dx^2} = 4\sec^2 x \tan^2 x + 2\sec^4 x$

$= 4\sec^2 x(\sec^2 x - 1) + 2\sec^4 x$

$= 6\sec^4 x - 4\sec^2 x$ #

b) $\dfrac{d^3y}{dx^2} = 24\sec^3 x \times \sec x \tan x - 8\sec x \times \sec x \tan x$

$= 24\sec^4 x \tan x - 8\sec^2 x \tan x$

$= 8\sec^2 x \tan x\,(3\sec^2 x - 1)$ $\quad A=3$

$B=-1$

2

c) $x = \frac{\pi}{3}$ $\qquad \tan x = \tan\frac{\pi}{3} = \sqrt{3}$

$$\sec x = \frac{1}{\cos x} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

$$y = \tan^2 x = \left(\sqrt{3}\right)^2 = 3$$

$$\frac{dy}{dx} = 2\tan x \sec^2 x = 2(\sqrt{3})(2)^2 = 8\sqrt{3}$$

$$\frac{d^2y}{dx^2} = 6\sec^4 x - 4\sec^2 x = 6(2)^4 - 4(2)^2 = 80$$

$$\frac{d^3y}{dx^3} = 8\sec^2 x \tan x\left(3\sec^2 x - 1\right) = 8(2)^2(\sqrt{3})\left(3(2)^2 - 1\right) = 352\sqrt{3}$$

$$\therefore \tan^2 x \simeq 3 + 8\sqrt{3}\left(x - \frac{\pi}{3}\right) + 40\left(x - \frac{\pi}{3}\right)^2 + \frac{352}{6}\sqrt{3}\left(x - \frac{\pi}{3}\right)^3$$

**8.** (a) Show that the transformation $x = e^u$ transforms the differential equation

$$x^2 \frac{d^2y}{dx^2} - 7x\frac{dy}{dx} + 16y = 2\ln x, \qquad x > 0 \qquad \text{(I)}$$

into the differential equation

$$\frac{d^2y}{du^2} - 8\frac{dy}{du} + 16y = 2u \qquad \text{(II)}$$

**(6)**

(b) Find the general solution of the differential equation (II), expressing $y$ as a function of $u$.

**(7)**

(c) Hence obtain the general solution of the differential equation (I).

**(1)**

a) $x = e^u \Rightarrow \frac{dx}{du} = e^u = x$

$\ln x = u \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{x}\frac{dy}{du}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{1}{x}\frac{dy}{du}\right]$

$u = \frac{1}{x}$ $\quad v = \frac{dy}{du}$

$u' = \frac{-1}{x^2}$ $\quad v' = \frac{d}{dx}\left(\frac{dy}{du}\right)$

$= \frac{du}{dx} \times \frac{d}{du}\left(\frac{dy}{du}\right)$

$= \frac{1}{x} \times \frac{d^2y}{du^2}$

$= \frac{1}{x^2}\frac{d^2y}{du^2} - \frac{1}{x^2}\frac{dy}{du}.$

$x^2\frac{d^2y}{dx^2} - 7x\frac{dy}{dx} + 16y = 2\ln x$

$x^2\left(\frac{1}{x^2}\frac{d^2y}{du^2} - \frac{1}{x^2}\frac{dy}{du}\right) - 7x\left(\frac{1}{x}\frac{dy}{du}\right) + 16y = 2u$

$\Rightarrow \frac{d^2y}{du^2} - \frac{dy}{du} - 7\frac{dy}{du} + 16y = 2u \quad \therefore \frac{d^2y}{du^2} - 8\frac{dy}{du} + 16y = 2u$ #

b) PI $\quad y = au + b \qquad y'' - 8y' + 16y \equiv 2u$

$y' = a$

$y'' = 0 \qquad 0 - 8a + 16au + 16b \equiv 2u$

$16au \equiv 2u \quad \therefore a = \frac{1}{8}$

$16b - 8a \equiv 0$

$y_{PI} = \frac{1}{8}u + \frac{1}{16} \qquad 16b = 1 \quad \therefore b = \frac{1}{16}$

Cf $\quad y = Ae^{mu} \qquad y'' - 8y' + 16y = 0$

$y' = Ame^{mu}$

$y'' = Am^2e^{mu} \qquad Am^2e^{mu} - 8Ame^{mu} + 16Ae^{mu} = 0$

$\underset{\neq 0}{Ae^{mu}}(m^2 - 8m + 16) = 0$

$(m-4)^2 \quad \therefore m = 4$

$\therefore y_{cf} = (A + Bu)e^{4u}$

$\therefore y_{GS} = (A + Bu)e^{4u} + \frac{1}{8}u + \frac{1}{16}$

c) $u = \ln x \qquad y = (A + B\ln x)e^{4\ln x} + \frac{1}{8}\ln x + \frac{1}{16}$

$y = (A + B\ln x)(e^{\ln x})^4 + \frac{1}{8}\ln x + \frac{1}{16}$

$y = (A + B\ln x)x^4 + \frac{1}{8}\ln x + \frac{1}{16}$