

1. (a) Express $\frac{2}{(r+2)(r+4)}$ in partial fractions.

(1)

(b) Hence show that

$$\sum_{r=1}^n \frac{2}{(r+2)(r+4)} = \frac{n(7n+25)}{12(n+3)(n+4)}$$

(5)

$$\frac{A}{r+2} + \frac{B}{r+4} \Rightarrow 2 = A(r+4) + B(r+2)$$

$$r = -2 \Rightarrow \underline{A=1} \quad r = -4 \Rightarrow \underline{B=-1}$$

$$\therefore = \frac{1}{r+2} - \frac{1}{r+4}$$

$$\text{b) } \sum_{r=1}^n \frac{2}{(r+2)(r+4)} = \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \dots + \left(\frac{1}{n+1} - \frac{1}{n+3}\right) + \left(\frac{1}{n+2} - \frac{1}{n+4}\right)$$

$$= \frac{1}{3} + \frac{1}{4} + \frac{-1}{n+3} - \frac{1}{n+4} = \frac{7}{12} - \frac{1}{n+3} - \frac{1}{n+4}$$

$$= \frac{7(n+3)(n+4) - 12(n+4) - 12(n+3)}{12(n+3)(n+4)}$$

$$= \frac{7n^2 + 49n + 84 - 12n - 48 - 12n - 36}{12(n+3)(n+4)}$$

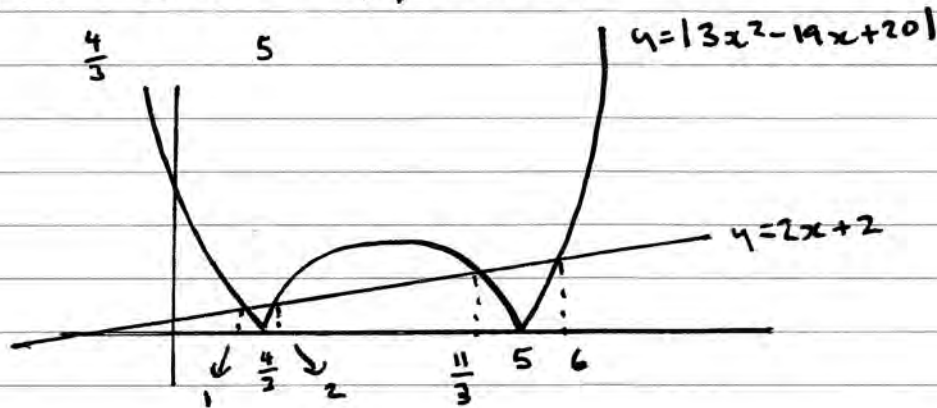
$$= \frac{7n^2 + 25n}{12(n+3)(n+4)} = \frac{n(7n+25)}{12(n+3)(n+4)}$$

2. Use algebra to find the set of values of x for which

$$|3x^2 - 19x + 20| < 2x + 2$$

(6)

$$|(3x - 4)(x - 5)| < 2x + 2$$



$$3x^2 - 19x + 20 = 2x + 2$$

$$3x^2 - 21x + 18 = 0$$

$$x^2 - 7x + 6 = 0$$

$$(x - 6)(x - 1) = 0$$

$$x = \frac{6}{2} \quad x = \frac{1}{2}$$

$$1 < x < 2$$

$$3x^2 - 19x + 20 = -2x - 2$$

$$3x^2 - 17x + 22 = 0$$

$$(3x - 11)(x - 2) = 0$$

$$x = \frac{11}{3} \quad x = \frac{2}{2}$$

$$\frac{11}{3} < x < 6$$

3.

$$y = \sqrt{8 + e^x}, \quad x \in \mathbb{R}$$

Find the series expansion for y in ascending powers of x , up to and including the term in x^2 , giving each coefficient in its simplest form.

(8)

$$f(x) = (8 + e^x)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(8 + e^x)^{-\frac{1}{2}} x e^x$$

$$f''(x) = -\frac{1}{4}(8 + e^x)^{-\frac{3}{2}} x e^{2x} + \frac{1}{2} e^x (8 + e^x)^{-\frac{1}{2}}$$

$$f(0) = (8 + 1)^{\frac{1}{2}} = 3$$

$$f'(0) = \frac{1}{2}(8 + 1)^{-\frac{1}{2}} \times 1 = \frac{1}{6}$$

$$f''(0) = -\frac{1}{4}(8 + 1)^{-\frac{3}{2}} + \frac{1}{2}(8 + 1)^{-\frac{1}{2}} = \frac{17}{108}$$

$$\therefore f(x) = 3 + \frac{1}{6}x + \frac{17}{216}x^2 \dots$$

4. (a) Use de Moivre's theorem to show that

$$\cos 6\theta = 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1 \quad (5)$$

(b) Hence solve for $0 \leq \theta \leq \frac{\pi}{2}$

$$64 \cos^6 \theta - 96 \cos^4 \theta + 36 \cos^2 \theta - 3 = 0$$

giving your answers as exact multiples of π .

(5)

$\cos 6\theta$:-

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$$(\cos \theta + i \sin \theta)^6 = \cos 6\theta + i \sin 6\theta$$

$$\begin{aligned} (\cos \theta + i \sin \theta)^6 &= \cos^6 \theta + 6i \cos^5 \theta \sin \theta - 15 \cos^4 \theta \sin^2 \theta - 20i \cos^3 \theta \sin^3 \theta + 15 \cos^2 \theta \sin^4 \theta \\ &\quad + 6i \cos \theta \sin^5 \theta - \sin^6 \theta \end{aligned}$$

equating real parts

\Rightarrow

$$\cos 6\theta = \cos^6 \theta - 15 \cos^4 \theta (1 - \cos^2 \theta) + 15 \cos^2 \theta (1 - 2 \cos^2 \theta + \cos^4 \theta) - (1 - 3 \cos^2 \theta + 3 \cos^4 \theta - \cos^6 \theta)$$

$$\cos 6\theta = \cos^6 \theta - 15 \cos^4 \theta + 15 \cos^6 \theta + 15 \cos^2 \theta - 30 \cos^4 \theta + 15 \cos^6 \theta - 1 + 3 \cos^2 \theta - 3 \cos^4 \theta + \cos^6 \theta$$

$$\cos 6\theta = \underline{32 \cos^6 \theta} - 48 \cos^4 \theta + 18 \cos^2 \theta - 1$$

$$b) 2 \cos 6\theta - 1 = (64 \cos^6 \theta - 96 \cos^4 \theta + 36 \cos^2 \theta - 2) - 1$$

$$\therefore \cos 6\theta = \frac{1}{2} \Rightarrow 6\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$$

$$\therefore \theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}$$

5. (a) Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 27e^{-x} \tag{6}$$

(b) Find the particular solution that satisfies $y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$ (6)

(CF) $y = Ae^{mx}$ $y'' + 2y' + 10y = 0$
 $y' = Am e^{mx}$ $Ae^{mx}(m^2 + 2m + 10) = 0$
 $y'' = Am^2 e^{mx}$ $\neq 0 = 0 \quad (m+1)^2 - 1 = -10$
 $\Rightarrow m = -1 \pm 3i$

$$\Rightarrow y_{cf} = e^{-x} [A \cos 3x + B \sin 3x]$$

(PI) $y = \lambda e^{-x}$ $10y = 10\lambda e^{-x}$
 $y' = -\lambda e^{-x}$ $+ 2y' = -2\lambda e^{-x}$
 $y'' = \lambda e^{-x}$ $+ y'' = \lambda e^{-x}$
 $\frac{27e^{-x}}{27e^{-x}} = \frac{9\lambda e^{-x}}{9\lambda e^{-x}} \therefore \lambda = 3 \quad y_{PI} = 3e^{-x}$

$$\therefore y_{GS} = e^{-x} [A \cos 3x + B \sin 3x + 3]$$

$x=0, y=0 \quad 0 = A + 3 \therefore A = -3 \quad y_{GS} = e^{-x} [B \sin 3x - 3(\cos 3x + 3)]$

$$y' = e^{-x} (3B \cos 3x + 3 \sin 3x) + e^{-x} (B \sin 3x - 3(\cos 3x + 3))$$

$x=0, y'=0 \quad 0 = 3B - 3 + 3 \therefore B = 0$

$$\therefore y = 3e^{-x} (1 - \cos 3x)$$

6. The transformation T from the z -plane, where $z = x + iy$, to the w -plane, where $w = u + iv$, is given by

$$w = \frac{4(1-i)z - 8i}{2(-1+i)z - i}, \quad z \neq \frac{1}{4} - \frac{1}{4}i$$

The transformation T maps the points on the line l with equation $y = x$ in the z -plane to a circle C in the w -plane.

- (a) Show that

$$w = \frac{ax^2 + bxi + c}{16x^2 + 1}$$

where a , b and c are real constants to be found.

(6)

- (b) Hence show that the circle C has equation

$$(u - 3)^2 + v^2 = k^2$$

where k is a constant to be found.

(4)

$$a) z = x + iy \quad x = y \Rightarrow z = x + ix = x(1+i)$$

$$w = \frac{4(1-i)(1+i)x - 8i}{2(-1+i)(1+i)x - i} = \frac{(8x - 8i)}{(-4x - i)} \times \frac{(-4x + i)}{(-4x + i)}$$

$$\therefore w = \frac{-32x^2 + 40xi + 8}{16x^2 + 1}$$

$$b) \ddot{w} = u + iv \equiv \frac{8 - 32x^2}{16x^2 + 1} + \frac{40xi}{16x^2 + 1}$$

$$\therefore u = \frac{8 - 32x^2}{16x^2 + 1} \Rightarrow 16x^2u + u = 8 - 32x^2$$

$$16x^2u + 32x^2 = 8 - u$$

$$16x^2(u + 2) = 8 - u$$

$$16x^2 = \frac{8 - u}{u + 2}$$

$$\therefore 1600x^2 = 100 \left(\frac{8 - u}{u + 2} \right)$$

$$16x^2 + 1 = \frac{8 - u + u + 2}{u + 2}$$

$$16x^2 + 1 = \frac{10}{u + 2}$$

$$v = \frac{40x}{16x^2 + 1} \Rightarrow v^2 = \frac{1600x^2}{(16x^2 + 1)^2} = 100 \left(\frac{8 - u}{u + 2} \right) \left(\frac{u + 2}{10} \right)^2$$

$$\Rightarrow v^2 = \frac{100(8 - u)(u + 2)^2}{100(u + 2)}$$

$$\therefore v^2 = (8 - u)(u + 2) \Rightarrow v^2 = -u^2 + 6u + 16$$

$$\Rightarrow u^2 - 6u + v^2 = 16$$

$$\Rightarrow (u - 3)^2 + v^2 = 25$$

$$\therefore u = 3$$

7. (a) Show that the substitution $v = y^{-3}$ transforms the differential equation

$$x \frac{dy}{dx} + y = 2x^4 y^4 \quad (I)$$

into the differential equation

$$\frac{dv}{dx} - \frac{3v}{x} = -6x^3 \quad (II)$$

(5)

(b) By solving differential equation (II), find a general solution of differential equation (I) in the form $y^3 = f(x)$.

(6)

$$v = y^{-3} \Rightarrow \frac{dv}{dx} = -3y^{-4} \frac{dy}{dx} \quad \frac{dy}{dx} = -\frac{1}{3}y^4 \frac{dv}{dx}$$

$$\Rightarrow x \frac{dy}{dx} + y = 2x^4 y^4$$

$$= -\frac{1}{3}xy^4 \frac{dv}{dx} + y = 2x^4 y^4$$

$$\div xy^4 \left(\frac{1}{3} \frac{dv}{dx} + \frac{1}{xy^3} \right) = 2x^3 \quad (x-3) \quad \frac{dv}{dx} - \frac{3}{xy^3} = -6x^3$$

$$\frac{1}{y^3} = v \quad \therefore \frac{dv}{dx} - \frac{3}{x}v = -6x^3$$

$$IF = e^{-\int \frac{3}{x} dx} = e^{-3 \ln x} = (e^{\ln x})^{-3} = x^{-3} = \frac{1}{x^3}$$

$$\Rightarrow \frac{1}{x^3} \frac{dv}{dx} - \frac{3}{x^4}v = -6 \quad \Rightarrow \frac{d}{dx} \left(\frac{v}{x^3} \right) = -6$$

$$\Rightarrow \frac{v}{x^3} = \int -6 dx = -6x + C \quad \therefore v = Cx^3 - 6x^4$$

$$v = \frac{1}{y^3} \quad \therefore \frac{1}{y^3} = Cx^3 - 6x^4 \quad \therefore y^3 = \frac{1}{Cx^3 - 6x^4}$$

8.

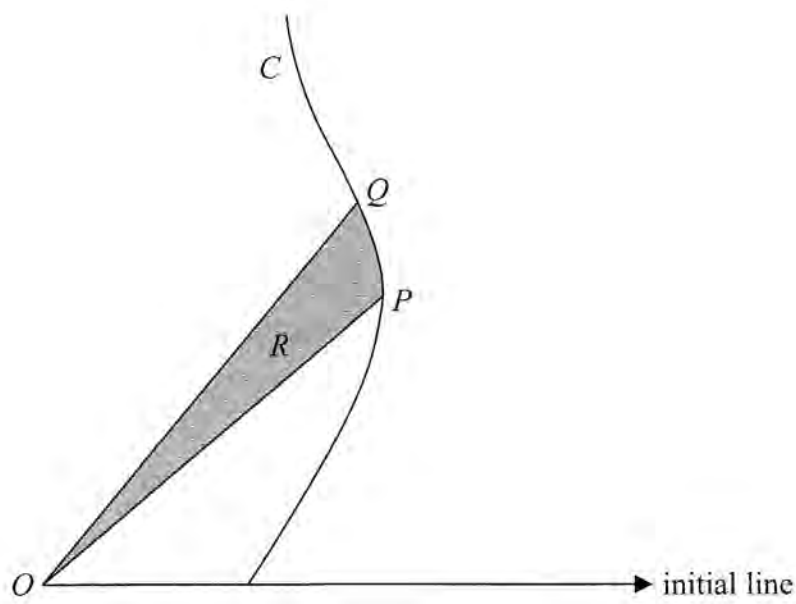


Figure 1

Figure 1 shows a sketch of part of the curve C with polar equation

$$r = 1 + \tan \theta, \quad 0 \leq \theta < \frac{\pi}{2}$$

The tangent to the curve C at the point P is perpendicular to the initial line.

(a) Find the polar coordinates of the point P .

(5)

The point Q lies on the curve C , where $\theta = \frac{\pi}{3}$

The shaded region R is bounded by OP , OQ and the curve C , as shown in Figure 1

(b) Find the exact area of R , giving your answer in the form

$$\frac{1}{2} (\ln p + \sqrt{q} + r)$$

where p , q and r are integers to be found.

(7)

$$\frac{dy}{dx} = 0 \quad x = r \cos \theta = (1 + \tan \theta) \cos \theta = \cos \theta + \frac{\sin \theta \cos \theta}{\cancel{\cos \theta}}$$

$$\therefore x = \cos \theta + \sin \theta$$

$$\frac{dx}{d\theta} = -\sin \theta + \cos \theta = 0 \Rightarrow \sin \theta = \cos \theta$$

$$\Rightarrow \tan \theta = 1 \therefore \theta = \frac{\pi}{4}$$

$$r = 1 + \tan \frac{\pi}{4} \therefore r = 2 \quad P(2, \frac{\pi}{4})$$

$$b) \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1 + \tan \theta)^2 d\theta = \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1 + 2 \tan \theta + \tan^2 \theta d\theta = \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 2 \tan \theta + \sec^2 \theta d\theta$$

$$= \frac{1}{2} [2 \ln |\sec \theta| + \tan \theta]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{1}{2} [(2 \ln 2 + \sqrt{3}) - (2 \ln \sqrt{2} + 1)]$$

$$= \frac{1}{2} [2 \ln \frac{2}{\sqrt{2}} + \sqrt{3} - 1] = \frac{1}{2} [\ln (\frac{2}{\sqrt{2}})^2 + \sqrt{3} - 1]$$

$$= \frac{1}{2} [\ln 2 + \sqrt{3} - 1]$$