1. (a) Express $\frac{2}{4r^2-1}$ in partial fractions.

(2)

(b) Hence use the method of differences to show that

$$\sum_{r=1}^{n} \frac{1}{4r^2-1} = \frac{n}{2n+1}$$

(3)

a) $\frac{2}{(2r-1)(2r+1)} = \frac{A}{2r-1} + \frac{B}{2r+1} \Rightarrow 2 = A(2r+1) + B(2r-1)$

$r = \frac{1}{2} \Rightarrow \underline{A=1} \quad r = -\frac{1}{2} \quad \underline{B=-1}$

$\therefore = \frac{1}{2r-1} - \frac{1}{2r+1}$

b) $\sum_{r=1}^{n} \frac{2}{4r^2-1} = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \ldots + \left(\cancel{\frac{1}{2n-3}} - \cancel{\frac{1}{2n-1}}\right) + \left(\cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right)$

$= 1 - \frac{1}{2n+1} = \frac{2n + \cancel{1} - \cancel{1}}{2n+1} = \frac{2n}{2n+1}$

$\therefore 2\sum_{r=1}^{n} \frac{1}{4r^2-1} = \frac{2n}{2n+1} \quad \therefore \sum_{1}^{n} \frac{1}{4r^2-1} = \frac{n}{2n+1}$ #

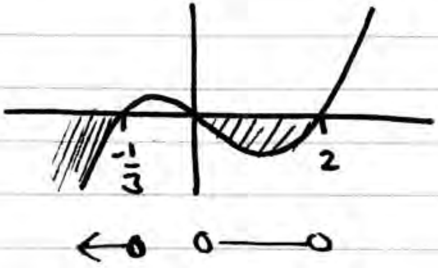
2. Using algebra, find the set of values of x for which

$$3x - 5 < \frac{2}{x}$$

(5)

$$3x - 5 - \frac{2}{x} < 0 \Rightarrow \frac{3x^2 - 5x - 2}{x} < 0$$

$$\Rightarrow \frac{(3x+1)(x-2)}{x} < 0$$



$$\therefore x < -\frac{1}{3} \text{ or } 0 < x < 2$$

3. (a) Find the general solution of the differential equation

$$\frac{dy}{dx} + 2y\tan x = e^{4x}\cos^2 x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

giving your answer in the form $y = f(x)$.

(6)

(b) Find the particular solution for which $y = 1$ at $x = 0$

(2)

$$IF = e^{2\int \tan x \, dx} = \left(e^{\ln|\sec x|}\right)^2 = \sec^2 x$$

$$\Rightarrow \sec^2 x \frac{dy}{dx} + 2y\tan x \sec^2 x = e^{4x}\cancel{\cos^2 x}\,\cancel{\sec^2 x}$$

$$\therefore \frac{d}{dx}(y\sec^2 x) = e^{4x} \Rightarrow y\sec^2 x = \int e^{4x}\,dx = \frac{1}{4}e^{4x} + c$$

$$\therefore y = \frac{1}{4}e^{4x}\cos^2 x + c\cos^2 x$$

$x=0, y=1 \quad 1 = \frac{1}{4}(1)(1)^2 + c(1)^2$

$$1 = \frac{1}{4} + c \quad \therefore c = \frac{3}{4}$$

$$\therefore y = \frac{1}{4}\cos^2 x\,(e^{4x} + 3)$$

4.

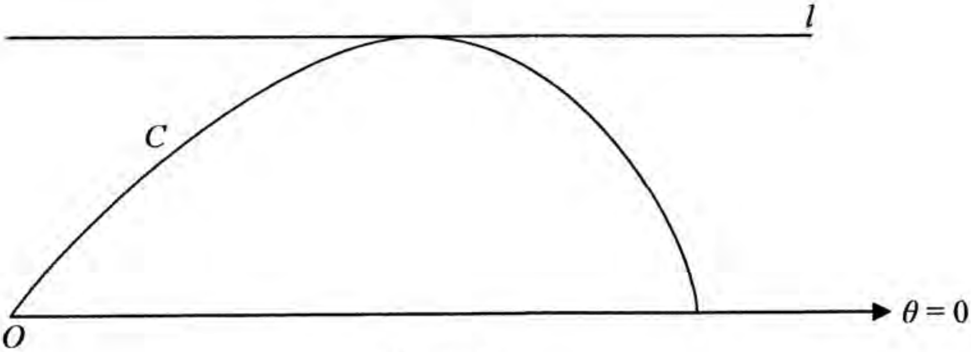


Figure 1

Figure 1 shows the curve C with polar equation

$$r = 2\cos 2\theta, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{4}$$

The line l is parallel to the initial line and is a tangent to C.

Find an equation of l, giving your answer in the form $r = f(\theta)$.

(9)

$\frac{dy}{d\theta} = 0 \quad y = r\sin\theta = 2\cos 2\theta \sin\theta = 2(1 - 2\sin^2\theta)\sin\theta$

$\therefore y = 2\sin\theta - 2\sin^3\theta$

$\frac{dy}{d\theta} = 2\cos\theta - 6\sin^2\theta\cos\theta = 0 \Rightarrow 2\cancel{\cos\theta} = 6\sin^2\theta\cancel{\cos\theta}$

$\therefore \sin^2\theta = \frac{1}{3} \quad \therefore \sin\theta = \pm\frac{1}{\sqrt{3}}$

$\theta = 0.6154\ldots$

$r = 2(1 - 2\sin^2\theta) = 2(1 - 2(\frac{1}{3}))$

$= 2(\frac{1}{3}) = \frac{2}{3}$

$y = r\sin\theta = \frac{2}{3}(\frac{1}{\sqrt{3}}) = \frac{2\sqrt{3}}{9} \quad \therefore L \Rightarrow y = \frac{2\sqrt{3}}{9}$

$r = \frac{y}{\sin\theta} \Rightarrow r = y\operatorname{cosec}\theta$

$\therefore r = \frac{2\sqrt{3}}{9}\operatorname{cosec}\theta$

5.

$$y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 2y = 0$$

(a) Find an expression for $\frac{d^3y}{dx^3}$ in terms of $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and y.

(4)

Given that $y = 2$ and $\frac{dy}{dx} = 0.5$ at $x = 0$,

(b) find a series solution for y in ascending powers of x, up to and including the term in x^3.

(5)

$$\frac{d}{dx}\left(y\frac{d^2y}{dx^2}\right) + 2\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 + 2\frac{d}{dx}(y) = 0$$

$$y\frac{d^3y}{dx^3} + \frac{dy}{dx}\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$$

$$y\frac{d^3y}{dx^3} + 5\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + 2\frac{dy}{dx} = 0$$

$x_0 = 0 \quad y_0 = 2 \quad y_0' = \frac{1}{2}$

$2y_0'' + 2\left(\frac{1}{2}\right)^2 + 2(2) = 0 \Rightarrow 2y_0'' = -\frac{9}{2} \quad \therefore y_0'' = \frac{-9}{4}$

$2y_0''' + 5\left(\frac{1}{2}\right)\left(\frac{-9}{4}\right) + 2\left(\frac{1}{2}\right) = 0 \Rightarrow 2y_0''' = -1 + \frac{45}{8} = \frac{37}{8}$

$y_0''' = \frac{37}{16}$

$\therefore y = 2 + \frac{1}{2}x + -\frac{9}{8}x^2 + \frac{37}{96}x^3$

6. The transformation T maps points from the z-plane, where $z = x + iy$, to the w-plane, where $w = u + iv$.

The transformation T is given by

$$w = \frac{z}{iz + 1}, \quad z \neq i$$

The transformation T maps the line l in the z-plane onto the line with equation $v = -1$ in the w-plane.

(a) Find a cartesian equation of l in terms of x and y.

(5)

The transformation T maps the line with equation $y = \frac{1}{2}$ in the z-plane onto the curve C in the w-plane.

(b) (i) Show that C is a circle with centre the origin.

(ii) Write down a cartesian equation of C in terms of u and v.

(6)

$$u + iv \equiv \frac{x+iy}{i(x+iy)+1} = \frac{x+iy}{(1-y)+ix} = \left[\frac{x+iy}{(1-y)+ix}\right]\left[\frac{(1-y)-ix}{(1-y)-ix}\right]$$

$$= \frac{(x(1-y)+xy)+i(y(1-y)-x^2)}{(1-y)^2+x^2}$$

$$v=-1 \Rightarrow \frac{y-y^2-x^2}{(1-y)^2+x^2} = -1 \Rightarrow y - \cancel{y^2} - \cancel{x^2} = -1 + 2y - \cancel{y^2} - \cancel{x^2}$$

$$\therefore y = 1$$

b) $wiz + w = z \Rightarrow w = z - wiz \Rightarrow w = z(1 - wi)$

$$z = \frac{w}{1-wi} \Rightarrow x+iy \equiv \frac{u+iv}{1-(u+iv)i} = \left[\frac{u+iv}{(1+v)-ui}\right]\left[\frac{(1+v)+ui}{(1+v)+ui}\right]$$

$$x+iy \equiv \frac{(u(1+v)-uv)+i(v(1+v)+u^2)}{(1+v)^2+u^2}$$

$$y = \frac{1}{2} \quad \therefore \frac{v+v^2+u^2}{(1+v)^2+u^2} = \frac{1}{2} \Rightarrow \cancel{2v} + 2v^2 + 2u^2 = 1 + \cancel{2v} + v^2 + u^2$$

$$\Rightarrow u^2 + v^2 = 1$$

Circle centre O, r = 1

ii) $u^2 + v^2 = 1$

7. (a) Use de Moivre's theorem to show that

$$\sin 5\theta \equiv 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$$

(5)

(b) Hence find the five distinct solutions of the equation

$$16x^5 - 20x^3 + 5x + \frac{1}{2} = 0$$

giving your answers to 3 decimal places where necessary.

(5)

(c) Use the identity given in (a) to find

$$\int_0^{\frac{\pi}{4}} (4\sin^5 \theta - 5\sin^3\theta)\mathrm{d}\theta$$

expressing your answer in the form $a\sqrt{2} + b$, where a and b are rational numbers.

(4)

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

Comparing Imaginary parts

$\Rightarrow \sin 5\theta \equiv 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$= 5(1 - 2\sin^2\theta + \sin^4\theta)\cancel{\sin\theta} - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$

$= (5 - 10\sin^2\theta + 5\sin^4\theta)\sin\theta + (10\sin^2\theta - 10)\sin^3\theta + \sin^5\theta$

$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta + 10\sin^5\theta - 10\sin^3\theta + \sin^5\theta$

$\therefore \sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

b) if $x = \sin\theta \Rightarrow \quad 16\sin^5\theta - 20\sin^5\theta + 5\sin\theta = -\frac{1}{2}$

$\Rightarrow \sin 5\theta = -\frac{1}{2} \Rightarrow 5\theta = \frac{7}{6}\pi, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}, \frac{31\pi}{6}$

$\therefore \theta = \frac{7}{30}\pi, \frac{11\pi}{30}, \frac{19\pi}{30}, \frac{23\pi}{30}, \frac{31\pi}{30}, \ldots$

$x = \sin\theta \quad \therefore \quad x = 0.823,$

$x = 0.669; 0.914; -0.105, -0.5, -0.978$

2

b) $\frac{1}{4}\sin 5\theta = 4\sin^5\theta - 5\sin^3\theta + \frac{5}{4}\sin\theta$

$$\therefore 4\sin^5\theta - 5\sin^3\theta = \frac{1}{4}(\sin 5\theta - 5\sin\theta)$$

$$\frac{1}{4}\int_0^{\frac{\pi}{4}} \sin 5\theta - 5\sin\theta \, d\theta = \frac{1}{4}\left[-\frac{1}{5}\cos 5\theta + 5\cos\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left[\left(\frac{\sqrt{2}}{10} + \frac{5\sqrt{2}}{2}\right) - \left(-\frac{1}{5} + 5\right)\right]$$

$$= \frac{13}{20}\sqrt{20} - \frac{6}{5}$$

8. (a) Show that the substitution $x = e^z$ transforms the differential equation

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 3\ln x, \quad x > 0 \qquad \text{(I)}$$

into the equation

$$\frac{d^2y}{dz^2} + \frac{dy}{dz} - 2y = 3z \qquad \text{(II)}$$

(7)

(b) Find the general solution of the differential equation (II).

(6)

(c) Hence obtain the general solution of the differential equation (I) giving your answer in the form $y = f(x)$.

(1)

$$\frac{dx}{dz} = e^z \Rightarrow \frac{dz}{dx} = e^{-z} \qquad x = e^z \quad x^2 = e^{2z}$$

$$\ln x = \ln e^z = z$$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = e^{-z}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = e^{-z}\left[\frac{d}{dx}\left(\frac{dy}{dz}\right)\right] + \frac{dy}{dz}\left[\frac{d}{dx}e^{-z}\right]$$

$$= e^{-z}\frac{d^2y}{dz^2}\frac{dz}{dx} + \frac{dy}{dz} - e^{-z}\frac{dz}{dx}$$

$$= e^{-2z}\frac{d^2y}{dz^2} - e^{-2z}\frac{dy}{dz} = e^{-2z}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$\therefore x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 3\ln x \Rightarrow$$

$$\cancel{e^{2z}}\left(\cancel{e^{-2z}}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\right) + 2\cancel{e^{z}}\left(\cancel{e^{-z}}\frac{dy}{dz}\right) - 2y = 3z$$

$$= \frac{d^2y}{dz^2} + \frac{dy}{dz} - 2y = 3z \quad \ast$$

(f) $y = Ae^{mz}$ $\quad y'' + y' - 2y = 0$

$y' = Ame^{mz}$ $\quad Ae^{mz}(m^2 + m - 2) = 0$ $\quad y_{GS} = Ae^{z} + Be^{-2z}$

$y'' = Am^2e^{mz}$ $\quad \neq 0 \quad (m+2)(m-1) = 0$

$\therefore m = -2 \quad m = 1$

(PI) $y = a + bz + cz^2$ $\quad -2y \quad -2a - 2bz - 2cz^2$

$y' = b + 2cz$ $\quad +y' = b + 2cz$

$y'' = 2c$ $\quad +y'' \quad 2c$

$3z = (-2a + b + 2c) + (-2b + 2c)z - 2cz^2$

$\therefore c = 0 \quad -2b + 2c = 3 \quad \therefore b = -\frac{3}{2}$

$-2a - \frac{3}{2} + 0 = 0 \quad \therefore 2a = -\frac{3}{2} \quad \therefore a = -\frac{3}{4}$

$\therefore y_{PI} = -\frac{3}{4} - \frac{3}{2}z$

$\therefore y_{GS} = Ae^{z} + Be^{-2z} - \frac{3}{2}z - \frac{3}{4}$ $\quad x = e^{z} \therefore \ln x = z$

$\therefore y = Ae^{\ln x} + Be^{-2\ln x} - \frac{3}{2}\ln x - \frac{3}{4}$

$y = Ax + \frac{B}{x^2} - \frac{3}{2}\ln x - \frac{3}{4}$