

1. (a) Express $\frac{2}{(2r+1)(2r+3)}$ in partial fractions.

(2)

(b) Using your answer to (a), find, in terms of n ,

$$\sum_{r=1}^n \frac{3}{(2r+1)(2r+3)}$$

Give your answer as a single fraction in its simplest form.

(3)

$$\frac{2}{(2r+1)(2r+3)} = \frac{A}{2r+1} + \frac{B}{2r+3} \Rightarrow 2 = A(2r+3) + B(2r+1)$$

$$r = -\frac{1}{2} \Rightarrow A = 1 \quad r = -\frac{3}{2} \quad B = -1$$

$$\therefore = \frac{1}{2r+1} - \frac{1}{2r+3}$$

$$b) \sum_1^n \frac{3}{(2r+1)(2r+3)} = \frac{3}{2} \sum_1^n \left(\frac{2}{(2r+1)(2r+3)} \right)$$

$$= \frac{3}{2} \left[\left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{7} \right) + \left(\frac{1}{7} - \frac{1}{9} \right) + \dots + \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) + \left(\frac{1}{2n+1} - \frac{1}{2n+3} \right) \right]$$

$$= \frac{3}{2} \left[\frac{1}{3} - \frac{1}{2n+3} \right] = \frac{3}{2} \left[\frac{2n+3-3}{3(2n+3)} \right] = \frac{n}{2n+3}$$

2.

$$z = 5\sqrt{3} - 5i$$

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Find

(a) $|z|$,

(1)

(b) $\arg(z)$, in terms of π .

(2)

$$w = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Find

(c) $\left|\frac{w}{z}\right|$,

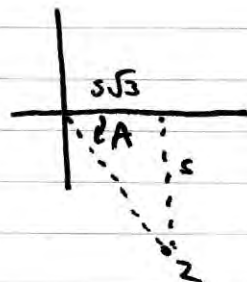
(1)

(d) $\arg\left(\frac{w}{z}\right)$, in terms of π .

(2)

$$a) |z| = \sqrt{(5\sqrt{3})^2 + (5)^2} = 10$$

$$b) A = \tan^{-1}\left(\frac{5}{5\sqrt{3}}\right) = \frac{\pi}{6} \quad \therefore \arg(z) = -\frac{\pi}{6} \quad \left(\frac{11\pi}{6}\right)$$



$$c) z = 10\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$\frac{w}{z} = \frac{2\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)}{10\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)} = \frac{1}{5}\left(\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right)$$

$$\frac{\pi}{4} - \left(-\frac{\pi}{6}\right) = \frac{5\pi}{12}$$

$$\left|\frac{w}{z}\right| = \frac{|w|}{|z|} = \frac{2}{10} = \frac{1}{5} \quad \arg\left(\frac{w}{z}\right) = \frac{5\pi}{12}$$

3.

$$\frac{d^2y}{dx^2} + 4y - \sin x = 0$$

Given that $y = \frac{1}{2}$ and $\frac{dy}{dx} = \frac{1}{8}$ at $x = 0$,

find a series expansion for y in terms of x , up to and including the term in x^3 .

(5)

$$x_0 = 0 \quad y_0 = \frac{1}{2} \quad y'_0 = \frac{1}{8}$$

$$y'' + 4y - \sin x = 0 \Rightarrow y'' + 4\left(\frac{1}{2}\right) - \sin(0) \Rightarrow y'' = -2$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) + 4\frac{d}{dx}(y) - \frac{d}{dx}(\sin x) = 0 \Rightarrow y''' + 4y' - \cos x = 0$$

$$y''' + 4\left(\frac{1}{8}\right) - \cos(0) = 0 \Rightarrow y''' = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\therefore y = \frac{1}{2} + \frac{1}{8}x - x^2 + \frac{1}{12}x^3$$

4. (a) Given that

$$z = r(\cos \theta + i \sin \theta), \quad r \in \mathbb{R}$$

prove, by induction, that $z^n = r^n(\cos n\theta + i \sin n\theta)$, $n \in \mathbb{Z}^+$

(5)

$$w = 3 \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

(b) Find the exact value of w^5 , giving your answer in the form $a + ib$, where $a, b \in \mathbb{R}$.

(2)

$$z^n = r^n (\cos n\theta + i \sin n\theta)$$

$$n=1 \quad z = r(\cos \theta + i \sin \theta) \quad z^1 = r^1 (\cos 1\theta + i \sin 1\theta)$$

$$z = r(\cos \theta + i \sin \theta) \quad \therefore \text{true for } n=1.$$

$$\text{assume true for } n=k \quad \therefore z^k = r^k (\cos k\theta + i \sin k\theta)$$

$$n=k+1 \quad z^{k+1} = z^k \times z^1 = r^k (\cos k\theta + i \sin k\theta) \times r (\cos \theta + i \sin \theta)$$

$$= r^{k+1} [(\cos k\theta \cos \theta - \sin k\theta \sin \theta) + i(\sin k\theta \cos \theta + \cos k\theta \sin \theta)]$$

$$= r^{k+1} [\cos(k\theta + \theta) + i \sin(k\theta + \theta)]$$

$$= r^{k+1} [\cos(k+1)\theta + i \sin(k+1)\theta] = z^{k+1} \quad \#$$

\therefore true for $n=1$, true for $n=k+1$ if true for $n=k$

\therefore by mathematical induction true for all $n \in \mathbb{Z}$.

$$b) \quad w^5 = 3^5 \left(\cos \frac{15\pi}{4} + i \sin \frac{15\pi}{4} \right) = 243 \left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right)$$

$$= 243 \left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right) = \frac{243\sqrt{2}}{2} + i \left(\frac{-243\sqrt{2}}{2} \right)$$

5. (a) Find the general solution of the differential equation

$$x \frac{dy}{dx} + 2y = 4x^2 \tag{5}$$

(b) Find the particular solution for which $y = 5$ at $x = 1$, giving your answer in the form $y = f(x)$. (2)

(c) (i) Find the exact values of the coordinates of the turning points of the curve with equation $y = f(x)$, making your method clear.

(ii) Sketch the curve with equation $y = f(x)$, showing the coordinates of the turning points.

a) $\frac{dy}{dx} + \frac{2}{x}y = 4x$ IF $f(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = (e^{\ln x})^2 \tag{5}$
 $= x^2$

$\therefore x^2 \frac{dy}{dx} + 2xy = 4x^3$

$\Rightarrow \frac{d}{dx}(x^2y) = 4x^3 \Rightarrow x^2y = \int 4x^3 dx \Rightarrow x^2y = x^4 + c$

$\therefore y = x^2 + cx^{-2}$

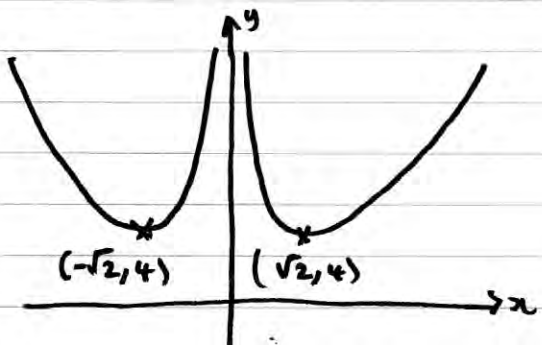
b) (1, 5) $5 = 1 + c \therefore c = 4$

$y = x^2 + \frac{4}{x^2}$

c) $y' = 2x - \frac{8}{x^3}$ TP $y' = 0$ $\frac{8}{x^3} = 2x \Rightarrow x^4 = 4 \Rightarrow x = \pm\sqrt{2}$

$y = 2 + \frac{4}{2} = 4$

$(\sqrt{2}, 4); (-\sqrt{2}, 4)$



6. (a) Use algebra to find the exact solutions of the equation

$$|2x^2 + 6x - 5| = 5 - 2x$$

(6)

- (b) On the same diagram, sketch the curve with equation $y = |2x^2 + 6x - 5|$ and the line with equation $y = 5 - 2x$, showing the x -coordinates of the points where the line crosses the curve.

(3)

- (c) Find the set of values of x for which

$$|2x^2 + 6x - 5| > 5 - 2x$$

(3)

$$2x^2 + 6x - 5 = 5 - 2x$$

$$2x^2 + 8x - 10 = 0$$

$$x^2 + 4x - 5 = 0$$

$$(x+5)(x-1) = 0$$

$$\underline{x = -5} \quad \underline{x = 1}$$

$$2x^2 + 6x - 5 = 2x - 5$$

$$2x^2 + 4x = 0$$

$$2x(x+2) = 0$$

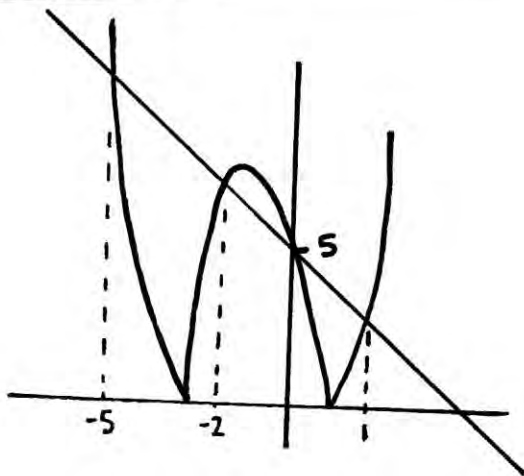
$$\underline{x = 0} \quad \underline{x = -2}$$

b) $2x^2 + 6x - 5 = 0$

$$x^2 + 3x - \frac{5}{2} = 0$$

$$\left(x + \frac{3}{2}\right)^2 = \frac{5}{2} + \frac{9}{4} = \frac{19}{4}$$

$$x = \frac{3}{2} \pm \frac{\sqrt{19}}{2} \quad (-3.7, +0.7)$$



c) $x < -5$ or $-2 < x < 0$ or $x > 1$

7. (a) Show that the transformation $y = xv$ transforms the equation

$$4x^2 \frac{d^2y}{dx^2} - 8x \frac{dy}{dx} + (8 + 4x^2)y = x^4 \tag{I}$$

into the equation

$$4 \frac{d^2v}{dx^2} + 4v = x \tag{II} \tag{6}$$

(b) Solve the differential equation (II) to find v as a function of x .

(6)

(c) Hence state the general solution of the differential equation (I).

(1)

$$y = vx \quad \frac{dy}{dx} = x \frac{dv}{dx} + v$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(x \frac{dv}{dx} \right) + \frac{d}{dx} (v) = x \frac{d^2v}{dx^2} + 2 \frac{dv}{dx}$$

$$\begin{aligned} \Rightarrow 4x^2 \left(x \frac{d^2v}{dx^2} + 2 \frac{dv}{dx} \right) - 8x \left(x \frac{dv}{dx} + v \right) + (8 + 4x^2)v &= x^4 \\ = 4x^3 \frac{d^2v}{dx^2} + 8x^2 \frac{dv}{dx} - 8x^2 \frac{dv}{dx} - 8xv + 8xv + 4x^3v &= x^4 \end{aligned}$$

$$\textcircled{\div x^3} \quad 4 \frac{d^2v}{dx^2} + 4v = x$$

$$\begin{aligned} \text{b) } v &= ax + b & 4(0) + 4ax + 4b &= x & \therefore b=0 & a=\frac{1}{4} \\ v' &= a & v_{PI} &= \frac{1}{4}x \\ v'' &= 0 \end{aligned}$$

$$\begin{aligned} v &= Ae^{mx} & \Rightarrow Ae^{mx} (4m^2 + 4) &= 0 & \Rightarrow m^2 = -1 & m = \pm i \\ v' &= Ame^{mx} & \neq 0 & = 0 & \\ v'' &= Am^2 e^{mx} & v &= Pe^{ix} + Qe^{-ix} & \Rightarrow v_{CF} &= A \cos x + B \sin x \end{aligned}$$

$$\therefore v_{GS} = A \cos x + B \sin x + \frac{1}{4}x$$

$$\text{c) } \frac{y}{x} = A \cos x + B \sin x + \frac{1}{4}x \Rightarrow y = Ax \cos x + Bx \sin x + \frac{1}{4}x^2$$

8.

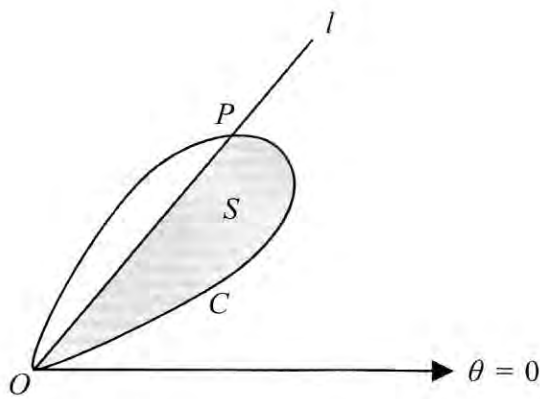


Figure 1

Figure 1 shows a curve C with polar equation $r = a \sin 2\theta$, $0 \leq \theta \leq \frac{\pi}{2}$, and a half-line l .

The half-line l meets C at the pole O and at the point P . The tangent to C at P is parallel to the initial line. The polar coordinates of P are (R, ϕ) .

(a) Show that $\cos \phi = \frac{1}{\sqrt{3}}$ (6)

(b) Find the exact value of R . (2)

The region S , shown shaded in Figure 1, is bounded by C and l .

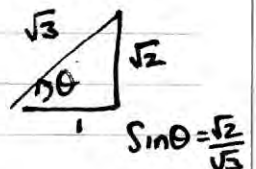
(c) Use calculus to show that the exact area of S is

$$\frac{1}{36} a^2 \left(9 \arccos \left(\frac{1}{\sqrt{3}} \right) + \sqrt{2} \right) \quad (7)$$

parallel to initial line when $\frac{dy}{d\theta} = 0$ $y = r \sin \theta = a \sin 2\theta \sin \theta$

$$\begin{aligned} \frac{dy}{d\theta} &= a 2 \cos 2\theta \sin \theta + a \sin 2\theta \cos \theta = 0 \\ 2(2 \cos^2 \theta - 1) \sin \theta &= -2 \sin \theta \cos^2 \theta \\ 4 \cos^2 \theta - 2 &= -2 \cos^2 \theta \Rightarrow 6 \cos^2 \theta = 2 \therefore \cos \theta = \frac{1}{\sqrt{3}} \end{aligned}$$

b) $r = a \sin 2\theta = a 2 \sin \theta \cos \theta$
 $= a 2 \times \frac{\sqrt{2}}{\sqrt{3}} \times \frac{1}{\sqrt{3}} = \frac{2\sqrt{2}}{3} a$



c) Area = $\frac{1}{2} a^2 \int_0^{\arccos(1/\sqrt{3})} (\sin 2\theta)^2 d\theta = \frac{1}{4} a^2 \int_0^{\arccos(1/\sqrt{3})} 1 - \cos 4\theta d\theta = \frac{1}{4} a^2 \left[\theta - \frac{1}{4} \sin 4\theta \right]_0^{\arccos(1/\sqrt{3})}$

$$\cos 4\theta = 1 - 2 \sin^2 2\theta \quad = \frac{1}{16} a^2 [4\theta - \sin 4\theta]_0^{\arccos(1/\sqrt{3})}$$

$$= \frac{1}{16} a^2 \left[4\theta - 2 \sin 2\theta \cos 2\theta \right]_0^{\arccos\left(\frac{1}{\sqrt{3}}\right)}$$

$$= \frac{1}{16} a^2 \left[4\theta - 4 \sin \theta \cos \theta (2 \cos^2 \theta - 1) \right]_0^{\arccos\left(\frac{1}{\sqrt{3}}\right)}$$

$$= \frac{1}{16} a^2 \left[4 \left(\arccos\left(\frac{1}{\sqrt{3}}\right) \right) - 4 \left(\frac{\sqrt{2}}{\sqrt{3}} \right) \left(\frac{1}{\sqrt{3}} \right) \left(\frac{2}{3} - 1 \right) \right]$$

$$= \frac{1}{16} a^2 \left[4 \arccos\left(\frac{1}{\sqrt{3}}\right) + \frac{4\sqrt{2}}{9} \right]$$

$$= \frac{1}{9} \times \frac{4}{16} a^2 \left[9 \arccos\left(\frac{1}{\sqrt{3}}\right) + \sqrt{2} \right]$$

$$= \frac{1}{36} a^2 \left[9 \arccos\left(\frac{1}{\sqrt{3}}\right) + \sqrt{2} \right] \quad \text{Q.E.D.}$$