1. $$\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 2\cos x$$

(a) Find $\frac{d^3y}{dx^3}$ in terms of x, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(3)

At $x = 0$, $y = 1$ and $\frac{dy}{dx} = 3$

(b) Find the value of $\frac{d^3y}{dx^3}$ at $x = 0$

(1)

(c) Express y as a series in ascending powers of x, up to and including the term in x^3.

(3)

a) $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) + \frac{d}{dx}\left(x\frac{dy}{dx}\right) = \frac{d}{dx}2\cos x \Rightarrow \frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -2\sin x$

$\therefore \frac{d^3y}{dx^3} = -x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2\sin x \qquad y''' = -xy'' - y' - 2\sin x$

b) $x=0 \quad y_0 = 1 \quad y_0' = 3$

$y'' = -xy' + 2\cos x \Rightarrow y_0'' = 0 + 2\cos(0) = 2 \quad \therefore y_0'' = 2$

$y''' = -xy'' - y' - 2\sin x \Rightarrow y_0''' = 0 - 3 - 2\sin(0) \quad \therefore y_0''' = -3$

c) $y = y_0 + y_0'x + y_0''\frac{x^2}{2} + y_0'''\frac{x^3}{6}$

$y = 1 + 3x + x^2 - \frac{1}{2}x^3$

2. (a) Sketch, on the same axes,

(i) $y = |2x - 3|$

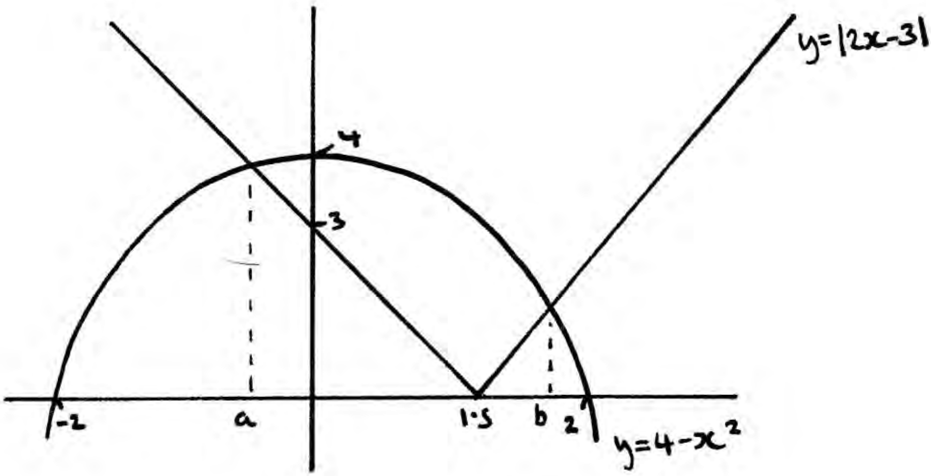
(ii) $y = 4 - x^2$

(3)

(b) Find the set of values of x for which

$$4 - x^2 > |2x - 3|$$

(6)



$|2x-3| = 4-x^2$

$2x-3 = 4-x^2$

$x^2+2x-7=0$

$(x+1)^2 = 8$

$x = -1 \pm 2\sqrt{2}$

$b = \underline{1.83}, -3.\cancel{8}3$

$2x-3 = x^2-4$

$x^2-2x-1=0$

$(x-1)^2 = 2$

$x-1 = \pm\sqrt{2}$

$x = 1 \pm \sqrt{2}$

$2.\cancel{4}1, -\underline{0.41} = a$

$a = 1-\sqrt{2}$

$b = -1+2\sqrt{2}$

$4-x^2 > |2x-3|$

smaller

when $1-\sqrt{2} < x < -1+2\sqrt{2}$

3.
$$f(x) = \ln(1 + \sin kx)$$

where k is a constant, $x \in \mathbb{R}$ and $-\frac{\pi}{2} < kx < \frac{3\pi}{2}$

(a) Find $f'(x)$

(2)

(b) Show that $f''(x) = \frac{-k^2}{1 + \sin kx}$

(3)

(c) Find the Maclaurin series of $f(x)$, in ascending powers of x, up to and including the term in x^3.

(4)

$f(x) = \ln(1 + \sin kx)$

$f'(x) = \frac{k\cos kx}{1 + \sin kx}$

$u = k\cos kx \quad v = 1 + \sin kx$

$u' = -k^2 \sin kx \quad v' = k\cos kx$

$$f''(x) = \frac{(1+\sin kx)(-k^2 \sin kx) - (k\cos kx)(k\cos kx)}{(1+\sin kx)^2}$$

$$= \frac{-k^2 \sin kx - k^2 \sin^2 kx - k^2 \cos^2 kx}{(1+\sin kx)^2}$$

$$= \frac{-k^2(\sin kx + \sin^2 kx + \cos^2 kx)}{(1+\sin kx)^2} = \frac{-k^2 \cancel{(\sin kx + 1)}}{(1+\sin kx)^{\cancel{2}}}$$

$\therefore f''(x) = \frac{-k^2}{1 + \sin kx}$

c) $f''(x) = -k^2(1+\sin kx)^{-1} \Rightarrow f'''(x) = k^2(1+\sin kx)^{-2} \times k\cos kx$

$$= \frac{k^3 \cos kx}{(1+\sin kx)^2}$$

$f(0) = \ln 1 = 0$

$f'(0) = k$

$f''(0) = -k^2$

$f'''(0) = k^3$

$\therefore f(x) = kx - \frac{k^2x^2}{2} + \frac{k^3x^3}{6} \ldots$

4. Find the general solution of the differential equation

$$x\frac{dy}{dx} + (1 + x\cot x)y = \sin x, \qquad 0 < x < \pi$$

giving your answer in the form $y = f(x)$.

(9)

$$\frac{dy}{dx} + \left(\frac{1+x\cot x}{x}\right)y = \frac{\sin x}{x}$$

$$\text{IF} \Rightarrow f(x) = e^{\int \frac{1+x\cot x}{x}\,dx} = e^{\int \frac{1}{x} + \cot x \, dt}$$

$$= e^{\ln x + \ln|\sin x|} = e^{\ln x} \times e^{\ln|\sin x|} = x\sin x$$

$$\Rightarrow x\sin x\frac{dy}{dx} + x\sin x\left(\frac{1+x\cot x}{x}\right)y = \frac{\cancel{x}\sin x \times \sin x}{\cancel{x}}$$

$$\Rightarrow \frac{d}{dx}(x\sin x\, y) = \sin^2 x \qquad \Rightarrow x\sin x\, y = \int \tfrac{1}{2} - \tfrac{1}{2}\cos 2x \, dx$$

$$\cos 2x = 1 - 2\sin^2 x \qquad \Rightarrow x\sin x\, y = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + c$$

$$\sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x$$

$$\therefore y = \frac{1}{2\sin x} - \frac{\sin 2x}{4x\sin x} + \frac{c}{x\sin x}$$

$$\Rightarrow y = \tfrac{1}{2}\operatorname{cosec} x - \frac{\cos x}{2x} + \frac{c}{x\sin x}$$

Z

5. (a) Express $\dfrac{2}{r(r+1)(r+2)}$ in partial fractions.

(3)

(b) Using your answer to part (a) and the method of differences, show that

$$\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)} = \frac{n(n+3)}{2(n+1)(n+2)}$$

(4)

a) $\dfrac{2}{r(r+1)(r+2)} = \dfrac{A}{r} + \dfrac{B}{r+1} + \dfrac{C}{r+2} \Rightarrow 2 = A(r+1)(r+2) + B(r)(r+2) + C(r)(r+1)$

$r=0 \Rightarrow 2 = 2A \therefore \underline{A=1}$ $\quad r=-1 \quad 2=-B \therefore \underline{B=-2}$ $\quad r=-2 \Rightarrow 2 = +2C$ $\therefore \underline{C=+1}$

$\therefore \quad = \dfrac{1}{r} - \dfrac{2}{r+1} + \dfrac{1}{r+2}$

b)

$r=1 \quad \cancel{\dfrac{1}{1}} - \cancel{\dfrac{2}{2}} + \cancel{\dfrac{1}{3}}$

$r=2 \quad \dfrac{1}{2} - \cancel{\dfrac{2}{3}} + \cancel{\dfrac{1}{4}}$

$r=3 \quad \cancel{\dfrac{1}{3}} - \cancel{\dfrac{2}{4}} + \cancel{\dfrac{1}{5}}$

$r=4 \quad \cancel{\dfrac{1}{4}} - \cancel{\dfrac{2}{5}} + \dfrac{1}{6}$

$\vdots$

$r=n-2 \quad \cancel{\dfrac{1}{n-2}} - \cancel{\dfrac{2}{n-1}} + \cancel{\dfrac{1}{n}}$

$r=n-1 \quad \cancel{\dfrac{1}{n-1}} - \cancel{\dfrac{2}{n}} + \dfrac{1}{n+1}$

$r=n \quad \cancel{\dfrac{1}{n}} - \dfrac{2}{n+1} + \dfrac{1}{n+2}$

$$\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)}$$

$$= \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}$$

$$= \frac{(n+1)(n+2) - (2)(n+2) + 1(2)(n+1)}{2(n+1)(n+2)}$$

$$= \frac{n^2+3n+2-2n-4+2n+2}{2(n+1)(n+2)}$$

$$= \frac{n^2+3n}{2(n+1)(n+2)}$$

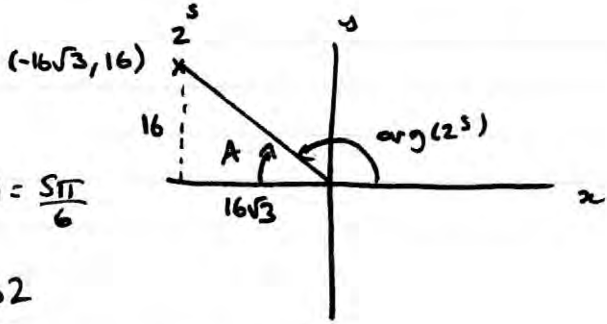
$$= \frac{n(n+3)}{2(n+1)(n+2)} \quad \#$$

6. Solve the equation

$$z^5 = -16\sqrt{3} + 16\mathrm{i}$$

giving your answers in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta < \pi$.

(8)



$\tan A = \frac{16}{16\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\therefore A = \frac{\pi}{6} \quad \therefore \arg(z^5) = \frac{5\pi}{6}$

$r = \sqrt{16^2 + (16\sqrt{3})^2} = 32$

$\therefore z^5 = 32\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$\Rightarrow z = 32^{\frac{1}{5}}\left(\cos\left(\frac{5\pi}{6} + 2k\pi\right) + i\sin\left(\frac{5\pi}{6} + 2k\pi\right)\right)^{\frac{1}{5}}$

$\Rightarrow z = 2\left[\cos\left(\frac{12k+5}{6}\right)\pi + i\sin\left(\frac{12k+5}{6}\right)\pi\right)^{\frac{1}{5}}$

$\Rightarrow z = 2\left[\cos\left(\frac{12k+5}{30}\right)\pi + i\sin\left(\frac{12k+5}{30}\right)\pi\right]$

$k = -2 \quad z = 2\left(\cos\frac{-19\pi}{30} + i\sin\frac{-19\pi}{30}\right)$

$k = -1 \quad z = 2\left(\cos\frac{-7\pi}{30} + i\sin\frac{-7\pi}{30}\right)$

$k = 0 \quad z = 2\left(\cos\frac{5\pi}{30} + i\sin\frac{5\pi}{30}\right)$

$k = 1 \quad z = 2\left(\cos\frac{17\pi}{30} + i\sin\frac{17\pi}{30}\right)$

$k = 2 \quad z = 2\left(\cos\frac{29\pi}{30} + i\sin\frac{29\pi}{30}\right)$

7. (a) Find the value of the constant λ for which $y = \lambda x e^{2x}$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 4y = 6e^{2x}$$

(4)

(b) Hence, or otherwise, find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 4y = 6e^{2x}$$

(3)

a) $y = \lambda x e^{2x}$

$y' = 2\lambda x e^{2x} + \lambda e^{2x} = (2\lambda x + \lambda)e^{2x}$

$y'' = 2(2\lambda x + \lambda)e^{2x} + 2\lambda e^{2x} = (4\lambda x + 4\lambda)e^{2x}$

$y'' - 4y = 6e^{2x} \Rightarrow (4\cancel{\lambda x} + 4\lambda - 4\cancel{\lambda x})e^{2x} = 6e^{2x}$

$\therefore 4\lambda = 6 \quad \lambda = \frac{3}{2}$

$y_{PI} = \frac{3}{2}xe^{2x}$

b) $y = Ae^{mt}$ $\Rightarrow$ $Am^2e^{mt} - 4Ae^{mt} = 0$

$y' = Ame^{mt}$ $\quad Ae^{mt}(m^2 - 4) = 0$

$y'' = Am^2e^{mt}$ $\quad \neq 0 \quad \Rightarrow m = \pm 2$

$\therefore y_{CF} = Ae^{2x} + Be^{-2x}$

$\therefore y_{GS} = Ae^{2x} + Be^{-2x} + \frac{3}{2}xe^{2x}$

$= (A + \frac{3}{2}x)e^{2x} + Be^{-2x}$

8. A complex number z is represented by the point P on an Argand diagram.

(a) Given that $|z| = 1$, sketch the locus of P.

(1)

The transformation T from the z-plane to the w-plane is given by

$$w = \frac{z + 7\mathrm{i}}{z - 2\mathrm{i}}$$

(b) Show that T maps $|z| = 1$ onto a circle in the w-plane.

(5)

(c) Show that this circle has its centre at $w = -5$ and find its radius.

(2)

b) $wz - 2wi = z + 7i$

$wz - z = 7i + 2wi$

$z(w-1) = 7i + 2wi$

$|z||w-1| = |7i + 2wi|$

$1\,|w-1| = |\cancel{i}||7+2w|$

$|(u-1)+iv| = |(2u+7)+2iv|$

$(u-1)^2 + v^2 = (2u+7)^2 + 4v^2$

$u^2 - 2u + 1 + v^2 = 4u^2 + 28u + 49 + 4v^2$

$3u^2 + 30u + 3v^2 = -48$

$\div 3$ $\quad u^2 + 10u + v^2 = -16$

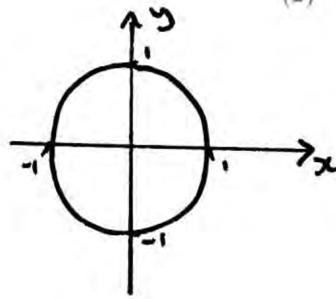
$(u+5)^2 + v^2 = -16 + 25 = 9$

$\therefore (u+5)^2 + v^2 = 3^2$

T maps $|z| = 1$ to a circle

$C(-5, 0)$ $r = 3$ in the w-plane

a)



9.

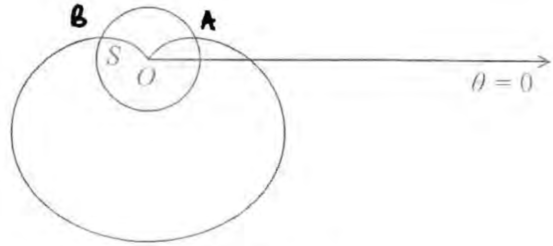


Figure 1

Figure 1 shows a sketch of the curves given by the polar equations

$$r = 1 \quad \text{and} \quad r = 2 - 2\sin\theta$$

(a) Find the coordinates of the points where the curves intersect.

(3)

The region S, between the curves, for which $r < 1$ and for which $r < 2 - 2\sin\theta$, is shown shaded in Figure 1.

(b) Find, by integration, the area of the shaded region S, giving your answer in the form $a\pi + b\sqrt{3}$, where a and b are rational numbers.

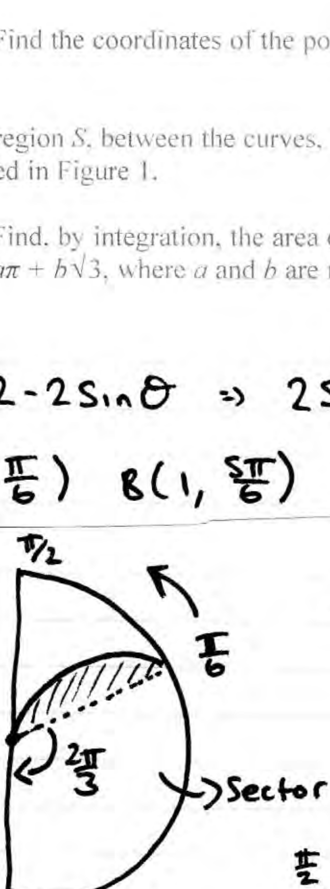
(8)

a) $1 = 2 - 2\sin\theta \Rightarrow 2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$A(1, \frac{\pi}{6}) \quad B(1, \frac{5\pi}{6})$

b)

2x



$$\text{Area} = 2 \times \left[\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2 - 2\sin\theta)^2 \, d\theta + \frac{1}{2} r^2 \left(\frac{2\pi}{3}\right) \right]$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4 - 8\sin\theta + 4\sin^2\theta \, d\theta + \frac{2\pi}{3}$$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\therefore 4\sin^2\theta = 2 - 2\cos 2\theta$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 6 - 8\sin\theta - 2\cos 2\theta \, d\theta + \frac{2\pi}{3}$$

$$= \left[6\theta + 8\cos\theta - \sin(2\theta) \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} + \frac{2\pi}{3} = \left[(3\pi + 0 + 0) - \left(\pi + 4\sqrt{3} - \frac{\sqrt{3}}{2}\right) \right] + \frac{2\pi}{3}$$

$$= \left(2\pi - \frac{7\sqrt{3}}{2} \right) + \frac{2\pi}{3} = \frac{8\pi}{3} - \frac{7\sqrt{3}}{2}$$