FP2 S13 R

1. A transformation T from the z-plane to the w-plane is given by

$$w = \frac{z + 2\mathrm{i}}{\mathrm{i}z} \qquad z \neq 0$$

The transformation maps points on the real axis in the z-plane onto a line in the w-plane.

Find an equation of this line.

(4)

$$wiz = z + 2i$$

$$wiz - z = 2i$$

$$z(wi - 1) = 2i$$

$$z = \frac{2i}{wi-1} = \frac{-2}{-w-i} = \frac{2}{w+i} = \frac{2}{u+(v+1)i} \times \frac{(u-(v+1)i)}{(u-(v+1)i)}$$

$$z = \frac{2u - 2(v+1)i}{u^2+(v+1)^2}$$ $\therefore$ real part maps to $\frac{2u}{u^2+(v+1)^2}$

$\Rightarrow$ Imaginary part is zero

$\Rightarrow \frac{2(v+1)}{u^2+(v+1)^2} = 0 \Rightarrow \underline{v = -1}$

alt $\quad z = x + iy \quad$ points on real axis $\Rightarrow$ Imaginary part is zero $\therefore z = x$

$$w = \frac{z+2i}{iz} \Rightarrow w = \frac{x+2i}{xi} = \frac{1}{i} + \frac{2}{x}$$

$$\Rightarrow w = \frac{2}{x} - i \Rightarrow u + iv \equiv \left(\frac{2}{x}\right) + (-1)i$$

$$\Rightarrow u = \frac{2}{x} \quad \underline{\underline{v = -1}}$$

2. Use algebra to find the set of values of x for which

$$\frac{6x}{3-x} > \frac{1}{x+1}$$

(7)

$$\frac{6x\,(3-x)^{\cancel{2}}(x+1)^2}{\cancel{3-x}} > \frac{(x+1)^{\cancel{2}}(3-x)^2}{\cancel{x+1}}$$

$$6x(3-x)(x+1)^2 - (x+1)(3-x)^2 > 0$$

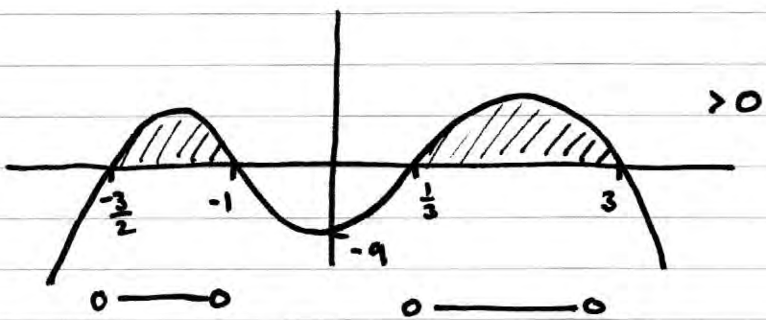
$$(3-x)(x+1)\left[6x(x+1) - (3-x)\right] > 0$$

$$(3-x)(x+1)(6x^2+6x-3+x) > 0$$

$$(3-x)(x+1)(6x^2+7x-3) > 0$$

$$(3-x)(x+1)(3x-1)(2x+3) > 0$$

$3 \qquad -1 \qquad \frac{1}{3} \qquad -\frac{3}{2}$



$$\left(-\frac{3}{2} < x < -1\right) \cup \left(\frac{1}{3} < x < 3\right)$$

3. (a) Express $\dfrac{2}{(r+1)(r+3)}$ in partial fractions.

(2)

(b) Hence show that

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \frac{n(5n+13)}{6(n+2)(n+3)}$$

(4)

(c) Evaluate $\displaystyle\sum_{r=10}^{100} \frac{2}{(r+1)(r+3)}$, giving your answer to 3 significant figures.

(2)

a) $= \dfrac{A}{r+1} + \dfrac{B}{r+3} \Rightarrow A(r+3)+B(r+1) \equiv 2$

$r=-1 \quad 2A=2$
$\underline{A=1}$

$r=-3 \quad -2B=2$
$\underline{B=-1}$

$= \dfrac{1}{r+1} - \dfrac{1}{r+3}$

b) $\displaystyle\sum_{r=1}^{n} \frac{2}{(r+1)(r+3)} = \overset{r=1}{\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)} + \overset{r=2}{\left(\frac{1}{3}-\cancel{\frac{1}{5}}\right)} + \overset{r=3}{\left(\cancel{\frac{1}{4}}-\cancel{\frac{1}{6}}\right)} + \overset{r=4}{\left(\cancel{\frac{1}{5}}-\cancel{\frac{1}{7}}\right)} + \ldots$

$+ \overset{r=n-2}{\left(\cancel{\frac{1}{n-1}}-\cancel{\frac{1}{n+1}}\right)} + \overset{r=n-1}{\left(\cancel{\frac{1}{n}}-\frac{1}{n+2}\right)} + \overset{r=n}{\left(\cancel{\frac{1}{n+1}}-\frac{1}{n+3}\right)}$

$= \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{n+2} - \dfrac{1}{n+3} = \dfrac{5}{6} - \dfrac{1}{n+2} - \dfrac{1}{n+3}$

$= \dfrac{5(n+2)(n+3) - 6(n+3) - 6(n+2)}{6(n+2)(n+3)} = \dfrac{5n^2+25n+30-6n-18-6n-12}{6(n+2)(n+3)}$

$= \dfrac{5n^2+13n}{6(n+2)(n+3)} = \dfrac{n(5n+13)}{6(n+2)(n+3)}$

c) $\displaystyle\sum_{10}^{100} \frac{2}{(r+1)(r+3)} = \frac{100(500+13)}{6(102)(103)} - \frac{9(45+13)}{6(11)(12)} \simeq 0.155 \text{ (3sf)}$

5. (a) Find, in the form $y = f(x)$, the general solution of the equation

$$\frac{dy}{dx} + 2y\tan x = \sin 2x, \qquad 0 < x < \frac{\pi}{2}$$

(6)

Given that $y = 2$ at $x = \frac{\pi}{3}$

(b) find the value of y at $x = \frac{\pi}{6}$, giving your answer in the form $a + k\ln b$, where a and b are integers and k is rational.

(4)

$$\frac{dy}{dx} + (2\tan x)y = \sin 2x$$

$$\text{IF} \Rightarrow f(x) = e^{2\int \tan x} = \left(e^{\ln|\sec x|}\right)^2 = \sec^2 x$$

$$\Rightarrow \sec^2 x \frac{dy}{dx} + \sec^2 x(2\tan x)y = \sec^2 x \sin 2x$$

$$\frac{d}{dx}(y\sec^2 x) = \frac{2\sin x \cancel{\cos x}}{\cos^{\cancel{2}} x} = 2\tan x$$

$$\Rightarrow y\sec^2 x = 2\int \tan x\,dx = 2\ln|\sec x| + c$$

$$\therefore \quad y = \frac{2\ln|\sec x| + c}{\sec^2 x}$$

b) $\left(\frac{\pi}{3}, 2\right)$ $\quad 2 = \dfrac{2\ln\left(\frac{1}{\cos\frac{\pi}{3}}\right) + c}{\left(\frac{1}{\cos\frac{\pi}{3}}\right)^2} \Rightarrow 2 = \dfrac{2\ln 2 + c}{4}$

$$\therefore c = 8 - 2\ln 2.$$

$$\therefore \quad y = \frac{2\ln\left|\frac{\sec x}{2}\right| + 8}{\sec^2 x}$$

$x = \frac{\pi}{6}$ $\quad y = \dfrac{2\ln\left(\frac{1}{2}\frac{1}{\cos\frac{\pi}{6}}\right) + 8}{\left(\frac{1}{\cos\frac{\pi}{6}}\right)^2} = \dfrac{2\ln\left(\frac{1}{\sqrt{3}}\right) + 8}{\left(\frac{4}{3}\right)} = \dfrac{2\ln(3)^{-\frac{1}{2}} + 8}{\left(\frac{4}{3}\right)}$

$$y = 6 - \frac{3}{4}\ln 3$$

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6. The complex number $z = e^{i\theta}$, where θ is real.

(a) Use de Moivre's theorem to show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

where n is a positive integer.

(2)

(b) Show that

$$\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$$

(5)

(c) Hence find all the solutions of

$$\cos 5\theta + 5\cos 3\theta + 12\cos\theta = 0$$

in the interval $0 \leqslant \theta < 2\pi$

(4)

a) $Z = \cos\theta + i\sin\theta$ $\quad Z^n = (\cos\theta + i\sin\theta)^n$

$Z^n = \cos n\theta + i\sin n\theta$

$Z^n + \frac{1}{Z^n} = Z^n + Z^{-n} = (\cos n\theta + i\sin n\theta) + (\cos(-n\theta) + i\sin(-n\theta))$

$= (\cos n\theta + i\sin n\theta) + (\cos(n\theta) - i\sin(n\theta))$

$= 2\cos(n\theta)$

b) $\left(Z + \frac{1}{Z}\right)^5 = (2\cos\theta)^5 = 32\cos^5\theta$

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$\left(Z + \frac{1}{Z}\right)^5 = Z^5 + 5Z^4\left(\frac{1}{Z}\right) + 10Z^3\left(\frac{1}{Z^2}\right) + 10Z^2\left(\frac{1}{Z^3}\right) + 5Z\left(\frac{1}{Z^4}\right) + \left(\frac{1}{Z^5}\right)$

$= \left(Z^5 + \frac{1}{Z^5}\right) + 5\left(Z^3 + \frac{1}{Z^3}\right) + 10\left(Z + \frac{1}{Z}\right)$

$\Rightarrow 32\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$

$\therefore \cos\theta = \frac{1}{16}(\cos\theta + 5\cos 3\theta + 10\cos\theta)$

c) $16\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta$ $\quad \therefore 16\cos^5\theta + 2\cos\theta = \cos 5\theta + 5\cos 3\theta + 12\cos\theta$

$\Rightarrow \underbrace{2\cos\theta}_{=0}\underbrace{(8\cos^4\theta + 1)}_{\text{no solution}} = 0$ $\quad \therefore \cos\theta = 0 \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

7. (a) Find the value of λ for which $\lambda t^2 e^{3t}$ is a particular integral of the differential equation

$$\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 6e^{3t}, \qquad t \geqslant 0$$

(5)

(b) Hence find the general solution of this differential equation.

(3)

Given that when $t = 0$, $y = 5$ and $\frac{dy}{dt} = 4$

(c) find the particular solution of this differential equation, giving your solution in the form $y = f(t)$.

(5)

a) $y = \lambda t^2 e^{3t}$

$y' = 3\lambda t^2 e^{3t} + 2\lambda t e^{3t} = (3\lambda t^2 + 2\lambda t)e^{3t}$

$y'' = 3(3\lambda t^2 + 2\lambda t)e^{3t} + (6\lambda t + 2\lambda)e^{3t} = (9\lambda t^2 + 12\lambda t + 2\lambda)e^{3t}$

$y'' - 6y' + 9y = \begin{pmatrix} 9\cancel{\lambda t^2} + 12\cancel{\lambda t} + 2\lambda \\ -18\cancel{\lambda t^2} + -12\cancel{\lambda t} \\ +9\cancel{\lambda t^2} \end{pmatrix} e^{3t} \equiv 6e^{3t} \quad \therefore 2\lambda = 6$

$\lambda = 3$

$y_{PI} = 3t^2 e^{3t}$

b) $y = Ae^{mt}$ $\qquad y'' - 6y' + 9y = 0$

$y' = Ame^{mt}$

$y'' = Am^2 e^{mt}$ $\qquad \underset{\neq 0}{Ae^{mt}}\underset{=0}{(m^2 - 6m + 9)} = 0$

$(m-3)^2 = 0 \quad \therefore m = 3$

$y_{CF} = (A + Bt)e^{3t} \qquad \therefore y_{GS} = (A + Bt)e^{3t} + 3t^2 e^{3t}$

$= (A + Bt + 3t^2)e^{3t}$

$t = 0, y = 5 \Rightarrow A = 5 \qquad y = (5 + Bt + 3t^2)e^{3t}$

$y' = 3(5 + Bt + 3t^2)e^{3t} + (B + 6t)e^{3t}$

$t = 0, y' = 4 \qquad 4 = 15 + B \quad \therefore B = -11$

$\therefore y = (5 - 11t + 3t^2)e^{3t}$

8.

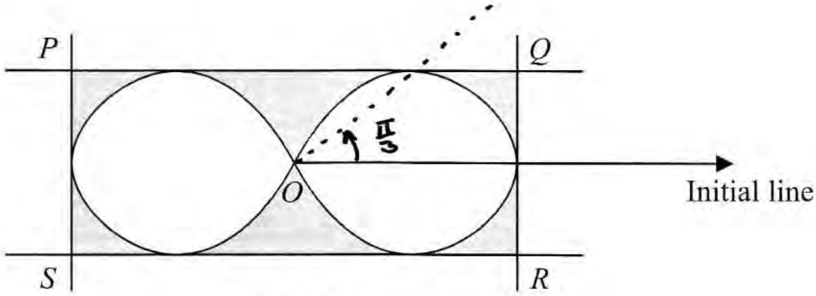


Figure 1

Figure 1 shows a closed curve C with equation

$$r = 3(\cos 2\theta)^{\frac{1}{2}}, \qquad \text{where } -\frac{\pi}{4} < \theta \leqslant \frac{\pi}{4},\ \frac{3\pi}{4} < \theta \leqslant \frac{5\pi}{4}$$

The lines PQ, SR, PS and QR are tangents to C, where PQ and SR are parallel to the initial line and PS and QR are perpendicular to the initial line. The point O is the pole.

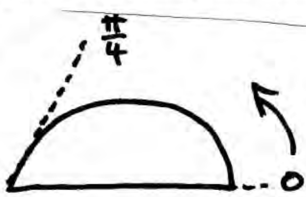
(a) Find the total area enclosed by the curve C, shown unshaded inside the rectangle in Figure 1.

(4)

(b) Find the total area of the region bounded by the curve C and the four tangents, shown shaded in Figure 1.

(9)

a) unshaded = 4 ×



$= 4 \times \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(3(\cos 2\theta)^{\frac{1}{2}}\right)^2 d\theta$

$= 4 \times \frac{9}{2}\int_0^{\pi/4} \cos 2\theta \, d\theta = 18\left[\frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = 18\left(\frac{1}{2} - 0\right) = \underline{9}$

b) at $\theta = 0$ $r = 3$ $\therefore$ $\vec{SR} = 6$.

Along PQ tangent to curve is parallel to initial line, $\frac{dy}{d\theta} = 0$

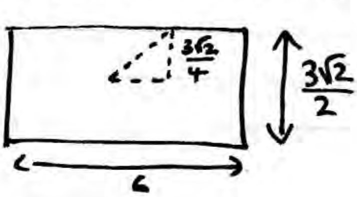
$y = r\sin\theta = 3(\cos 2\theta)^{\frac{1}{2}} \times \sin\theta$

$\frac{dy}{d\theta} = 3(\cos 2\theta)^{\frac{1}{2}} \times \cos\theta - \frac{3}{\cancel{2}}(\cos 2\theta)^{-\frac{1}{2}} \times \cancel{2}\sin 2\theta \times \sin\theta$

$\frac{dy}{d\theta} = 0 \Rightarrow \cancel{3}\cos 2\theta \times \cancel{\cos\theta} = \cancel{3}(2\sin\theta\cancel{\cos\theta})\sin\theta$

$\Rightarrow (1 - 2\sin^2\theta) = 2\sin^2\theta \Rightarrow \sin^2\theta = \frac{1}{4} \Rightarrow \sin\theta = \frac{1}{2}$

$y = 3(\cos 2\theta)^{\frac{1}{2}} \times \sin\theta$ $\quad \theta = \frac{\pi}{6}$ $\quad y = 3\sqrt{\frac{1}{2}} \times \frac{1}{2} = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}$



Rectangle $= 6 \times \frac{3\sqrt{2}}{2} = 9\sqrt{2}$

$\therefore$ ~~un~~shaded $= \underline{9\sqrt{2} - 9}$