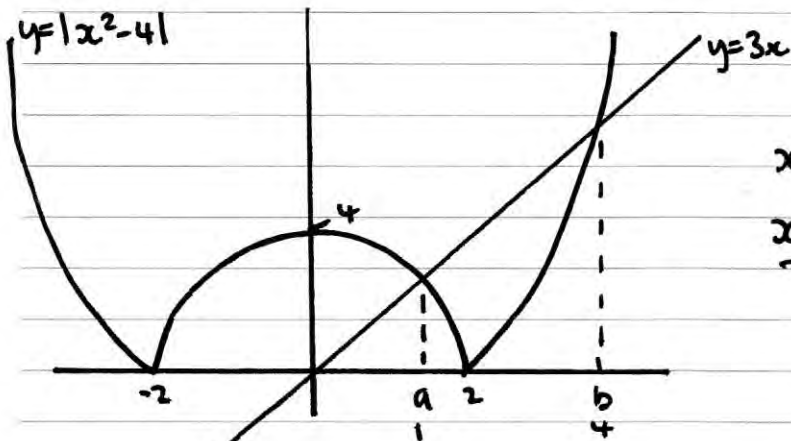


1. Find the set of values of x for which

$$|x^2 - 4| > 3x$$

(5)



$$x < a \text{ or } x > b$$

$$\underline{x < 1} \text{ or } \underline{x > 4}$$

$$\begin{aligned} x^2 - 4 &= 3x \\ x^2 - 3x - 4 &= 0 \\ (x - 4)(x + 1) &= 0 \\ \underline{x = 4} \quad \cancel{x = -1} \end{aligned}$$

$$\begin{aligned} x^2 - 4 &= -3x \\ x^2 + 3x - 4 &= 0 \\ (x + 4)(x - 1) &= 0 \\ \cancel{x = -4} \quad \underline{x = 1} \end{aligned}$$

2. The curve C has polar equation

$$r = 1 + 2 \cos \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

At the point P on C , the tangent to C is parallel to the initial line.

Given that O is the pole, find the exact length of the line OP .

(7)

parallel to initial line $\Rightarrow \frac{dy}{dx} = 0$

$$y = r \sin \theta = (1 + 2 \cos \theta) \sin \theta$$

$$\frac{dy}{d\theta} = (1 + 2 \cos \theta) \cos \theta - 2 \sin \theta \times \sin \theta$$

$$\Rightarrow \cos \theta + 2 \cos^2 \theta - 2 \sin^2 \theta = 0$$

$$\Rightarrow 2 \cos^2 \theta + \cos \theta - 2 + 2 \cos^2 \theta = 0$$

$$\Rightarrow 4 \cos^2 \theta + \cos \theta - 2 = 0$$

$$\Rightarrow \cos \theta = \frac{-1 \pm \sqrt{1 + 32}}{8} = \frac{-1 \pm \sqrt{33}}{8} \quad \therefore \theta = 0.936^\circ$$

$$r = 1 + 2 \cos(0.936 \dots) = 2.186 \dots \quad (\text{exact!})$$

$$r = 1 + 2 \left[\frac{-1 + \sqrt{33}}{8} \right] = 1 + \frac{-1 + \sqrt{33}}{4} = \frac{3 + \sqrt{33}}{4} = OP$$

2

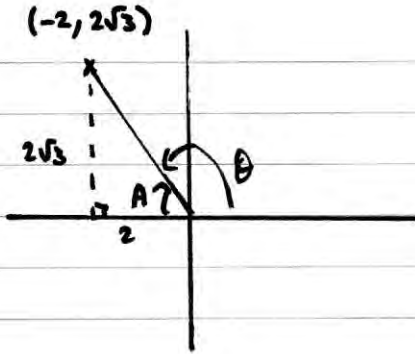
3. (a) Express the complex number $-2 + (2\sqrt{3})i$ in the form $r(\cos\theta + i\sin\theta)$, $-\pi < \theta \leq \pi$.

(b) Solve the equation

$$z^4 = -2 + (2\sqrt{3})i$$

giving the roots in the form $r(\cos\theta + i\sin\theta)$, $-\pi < \theta \leq \pi$.

(5)



$$A = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$\therefore \theta = \frac{2\pi}{3} \quad r = \sqrt{2^2 + (2\sqrt{3})^2} = 4$$

$$= 4 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$$

$$z^4 = 4 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$$

$$\Rightarrow z = 4^{\frac{1}{4}} \left(\cos\left(\frac{2\pi}{3} + 2k\pi\right) + i \sin\left(\frac{2\pi}{3} + 2k\pi\right) \right)^{\frac{1}{4}}$$

$$\Rightarrow z = \sqrt{2} \left[\cos\left(\frac{(6k+2)\pi}{3}\right) + i \sin\left(\frac{(6k+2)\pi}{3}\right) \right]^{\frac{1}{4}}$$

$$\Rightarrow z = \sqrt{2} \left[\cos\left(\frac{(6k+2)\pi}{12}\right) + i \sin\left(\frac{(6k+2)\pi}{12}\right) \right]$$

$$k = -2 \Rightarrow z = \sqrt{2} \left[\cos \frac{-10\pi}{12} + i \sin \frac{-10\pi}{12} \right]$$

$$k = -1 \Rightarrow z = \sqrt{2} \left[\cos \frac{-4\pi}{12} + i \sin \frac{-4\pi}{12} \right]$$

$$k = 0 \Rightarrow z = \sqrt{2} \left[\cos \frac{2\pi}{12} + i \sin \frac{2\pi}{12} \right]$$

$$k = 1 \Rightarrow z = \sqrt{2} \left[\cos \frac{8\pi}{12} + i \sin \frac{8\pi}{12} \right]$$

4. Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2\cos t - \sin t \quad (9)$$

$$x = a \cos t + b \sin t$$

$$x'' + 5x' + 6x = 2\cos t - \sin t$$

$$x' = -a \sin t + b \cos t$$

$$x'' = -a \cos t + -b \sin t$$

$$-a \cos t - b \sin t$$

$$+ 5b \cos t - 5a \sin t$$

$$+ \underline{6a \cos t + 6b \sin t}$$

$$(5a+5b)\cos t + (5b-5a)\sin t = 2\cos t - \sin t$$

$$\Rightarrow 5a+5b=2$$

$$5a+5b=1$$

$$b=1$$

$$a = \frac{1}{10}$$

$$b = \frac{3}{10}$$

$$\frac{1}{2} + 5b = 2 \Rightarrow 5b = \frac{3}{2}$$

$$\therefore x_{PE} = \frac{3}{10} \cos t + \frac{1}{10} \sin t$$

$$x = Ae^{mt}$$

$$x'' + 5x' + 6x = 0$$

$$x' = Ame^{mt}$$

$$\Rightarrow Ae^{mt}(m^2 + 5m + 6) = 0$$

$$x'' = Am^2e^{mt}$$

$$\neq 0$$

$$= 0$$

$$\Rightarrow (m+3)(m+2) = 0$$

$$m = -3 \quad m = -2$$

$$\therefore x_{CF} = Ae^{-2t} + Be^{-3t}$$

$$\therefore x = \frac{3}{10} \cos t + \frac{1}{10} \sin t + Ae^{-2t} + Be^{-3t}$$

5.

$$x \frac{dy}{dx} = 3x + y^2$$

(a) Show that

$$x \frac{d^2y}{dx^2} + (1-2y) \frac{dy}{dx} = 3 \quad (2)$$

Given that $y = 1$ at $x = 1$,(b) find a series solution for y in ascending powers of $(x-1)$, up to and including the term in $(x-1)^3$.

(8)

$$\frac{d}{dx} \left(x \frac{dy}{dx} \right) = \frac{d}{dx} (3x) + \frac{d}{dx} (y^2)$$

$$\Rightarrow x \frac{d^2y}{dx^2} + \frac{dy}{dx} = 3 + 2y \frac{dy}{dx} \quad \Rightarrow x \frac{d^2y}{dx^2} + (1-2y) \frac{dy}{dx} = 3$$

$$x=1, y=1 \quad xy' = 3x + y^2 \Rightarrow (1)y' = 3(1) + (1)^2$$

$$\therefore y' = 4$$

$$xy'' + (1-2y)y' = 3 \Rightarrow (1)y'' + (1-2(1))4 = 3$$

$$y'' = 3 + 4 \therefore y'' = 7$$

$$y = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3$$

$$\therefore y = 1 + 4(x-1) + \frac{7}{2}(x-1)^2 + \frac{f'''(1)}{6}(x-1)^3$$

$$\frac{d}{dx} \left(x \frac{d^2y}{dx^2} \right) + \frac{d}{dx} \left[(1-2y) \frac{dy}{dx} \right] = \frac{d}{dx} (3)$$

$$x \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} \frac{dy}{dx} + (1-2y) \frac{d^2y}{dx^2} = 0$$

$$\Rightarrow xy''' - 2(y')^2 + (1-2y)y'' = 0 \quad (1)y''' - 2(4)^2 + (1-2(1))7 = 0$$

$$y''' = 32 \Rightarrow f'''(1) = 32$$

$$\therefore y = 1 + 4(x-1) + \frac{7}{2}(x-1)^2 + \frac{16}{3}(x-1)^3 \dots$$

6. (a) Express $\frac{1}{r(r+2)}$ in partial fractions.

(2)

(b) Hence prove, by the method of differences, that

$$\sum_{r=1}^n \frac{1}{r(r+2)} = \frac{n(an+b)}{4(n+1)(n+2)}$$

where a and b are constants to be found.

(6)

(c) Hence show that

$$\sum_{r=n+1}^{2n} \frac{1}{r(r+2)} = \frac{n(4n+5)}{4(n+1)(n+2)(2n+1)}$$

(3)

$$\begin{aligned} \text{a) } \frac{1}{r(r+2)} &= \frac{A}{r} + \frac{B}{r+2} \Rightarrow 1 = A(r+2) + B(r) & r=0 &\Rightarrow A = \frac{1}{2} \\ & & r=-2 &\Rightarrow B = -\frac{1}{2} \end{aligned}$$

$$\therefore \frac{1}{r(r+2)} = \frac{1}{2r} - \frac{1}{2(r+2)}$$

$$\begin{aligned} \text{b) } \sum_{r=1}^n \frac{1}{r(r+2)} &= \left(\frac{1}{2} - \frac{1}{6} \right) + \left(\frac{1}{4} - \frac{1}{8} \right) + \left(\frac{1}{6} - \frac{1}{10} \right) + \dots \\ &\dots + \left(\frac{1}{2(n-1)} - \frac{1}{2(n+1)} \right) + \left(\frac{1}{2n} - \frac{1}{2(n+2)} \right) \end{aligned}$$

$$\therefore \sum_{r=1}^n \frac{1}{r(r+2)} = \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} = \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)}$$

$$= \frac{3n^2 + 9n + 6 - 2n - 4 - 2n - 2}{4(n+1)(n+2)} = \frac{n(3n+5)}{4(n+1)(n+2)}$$

$$\text{c) } \sum_{r=n+1}^{2n} \frac{1}{r(r+2)} = \sum_{r=1}^{2n} \frac{1}{r(r+2)} - \sum_{r=1}^n \frac{1}{r(r+2)}$$

$$= \frac{2n(6n+5)}{4(2n+1)(2n+2)} - \frac{n(3n+5)}{4(n+1)(n+2)} = \frac{n(6n+5)}{4(2n+1)(n+1)} - \frac{n(3n+5)}{4(n+1)(n+2)}$$

$$= \frac{n(6n+5)(n+2) - n(3n+5)(2n+1)}{4(n+1)(2n+1)(n+2)} = \frac{n[6n^2 + 17n + 10 - 6n^2 - 13n - 5]}{4(n+1)(n+2)(2n+1)}$$

$$= \frac{n(4n+5)}{4(n+1)(n+2)(2n+1)} \quad \#$$

7. (a) Show that the substitution $y = vx$ transforms the differential equation

$$3xy^2 \frac{dy}{dx} = x^3 + y^3 \quad (I)$$

into the differential equation

$$3v^2x \frac{dv}{dx} = 1 - 2v^3 \quad (II) \quad (3)$$

(b) By solving differential equation (II), find a general solution of differential equation (I) in the form $y = f(x)$. (6)

Given that $y = 2$ at $x = 1$,

(c) find the value of $\frac{dy}{dx}$ at $x = 1$ (2)

$$y = vx \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(vx) = x \frac{dv}{dx} + v$$

$$\Rightarrow 3x(vx)^2 \left(x \frac{dv}{dx} + v \right) = x^3 + (vx)^3$$

$$= 3x^4 v^2 \frac{dv}{dx} + 3x^3 v^3 = x^3 + v^3 x^3 \quad (\div x^3)$$

$$= 3xv^2 \frac{dv}{dx} + 3v^3 = 1 + v^3 \quad \therefore 3v^2x \frac{dv}{dx} = 1 - 2v^3$$

$$b) \int \frac{3v^2}{1-2v^3} dv = \int \frac{1}{x} dx \Rightarrow \ln x + C = -\frac{1}{2} \int \frac{-6v^2}{1-2v^3} dv$$

$$\Rightarrow \ln x + C = -\frac{1}{2} \ln |1-2v^3| \Rightarrow \ln |1-2v^3| = -2 \ln x - 2C$$

$$\Rightarrow \ln |1-2v^3| = \ln x^{-2} - 2C \Rightarrow 1-2v^3 = e^{\ln x^{-2} - 2C} = Ax^{-2} \quad A = e^{-2C}$$

$$\Rightarrow 2v^3 = 1 - Ax^{-2} \Rightarrow \frac{1}{2} + \frac{B}{x^2} = v^3 \quad B = -\frac{A}{2}$$

$$\Rightarrow \frac{y^3}{x^3} = \frac{1}{2} + \frac{B}{x^2} \Rightarrow y^3 = \frac{1}{2}x^3 + Bx \quad \therefore y = \sqrt[3]{\frac{x^3 + Cx}{2}} \quad C = -2B$$

$$c) (1, 2) \quad 2 = \sqrt[3]{\frac{1+C}{2}} \Rightarrow 8 = \frac{1+C}{2} \Rightarrow C = 15 \quad \therefore y = \sqrt[3]{\frac{x^3 + 15x}{2}} \quad (\text{not needed})$$

$$3xy^2 \frac{dy}{dx} = x^3 + y^3 \Rightarrow 3(1)(2)^2 \frac{dy}{dx} = 1^3 + 2^3 \Rightarrow 12 \frac{dy}{dx} = 9$$

$$\therefore \frac{dy}{dx} = \frac{3}{4}$$

8. The point P represents a complex number z on an Argand diagram such that

$$|z - 6i| = 2|z - 3|$$

(a) Show that, as z varies, the locus of P is a circle, stating the radius and the coordinates of the centre of this circle.

(6)

The point Q represents a complex number z on an Argand diagram such that

$$\arg(z - 6) = -\frac{3\pi}{4}$$

(b) Sketch, on the same Argand diagram, the locus of P and the locus of Q as z varies.

(4)

(c) Find the complex number for which both $|z - 6i| = 2|z - 3|$ and $\arg(z - 6) = -\frac{3\pi}{4}$

(4)

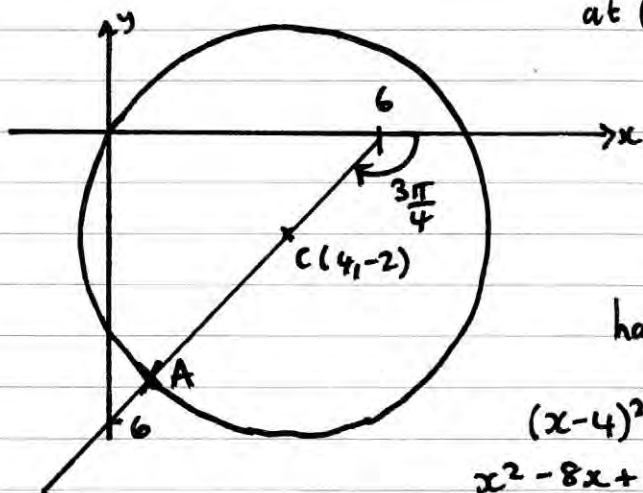
$$a) |x + (y-6)i| = 2|(x-3) + yi|$$

$$\Rightarrow x^2 + (y-6)^2 = 4[(x-3)^2 + y^2] \Rightarrow x^2 + y^2 - 12y + 36 = 4x^2 - 24x + 36 + 4y^2$$

$$\Rightarrow 3x^2 - 24x + 3y^2 + 12y = 0 \Rightarrow x^2 - 8x + y^2 + 4y = 0$$

$$\Rightarrow (x-4)^2 + (y+2)^2 = 20 \quad \text{Circle Centre } (4, -2) \quad r = \sqrt{20} \\ \approx 4.5$$

b)



at $(0,0)$ $(-4)^2 + (2)^2 = 20$
 \therefore passes through origin.

half line \searrow
 will pass through $(4, -2)$

$$\text{half line } \Rightarrow y = x - 6 \quad x \leq 6$$

$$(x-4)^2 + ((x-6)+2)^2 = 20$$

$$x^2 - 8x + 16 + x^2 - 8x + 16 = 20$$

$$x^2 - 8x + 16 = 10 \Rightarrow x^2 - 8x + 6 = 0$$

$$\Rightarrow (x-4)^2 = 10 \Rightarrow x = 4 \pm \sqrt{10}$$

$$\therefore x = 4 - \sqrt{10} \quad \text{Since } x \leq 6$$

$$y = x - 6 \Rightarrow y = -2 - \sqrt{10}$$

$$A \Rightarrow (4 - \sqrt{10}) + i(-2 - \sqrt{10})$$