1. Find the set of values of x for which

$$\frac{3}{x+3} > \frac{x-4}{x}$$

(7)

$$\frac{3(x)^2(x+3)^{\cancel{2}}}{\cancel{x+3}} > \frac{(x-4)(x^{\cancel{2}})(x+3)^2}{\cancel{x}}$$

$\Rightarrow \; 3x^2(x+3) - x(x-4)(x+3)^2 > 0$

$\Rightarrow \; x(x+3)[3x-(x-4)(x+3)] > 0$

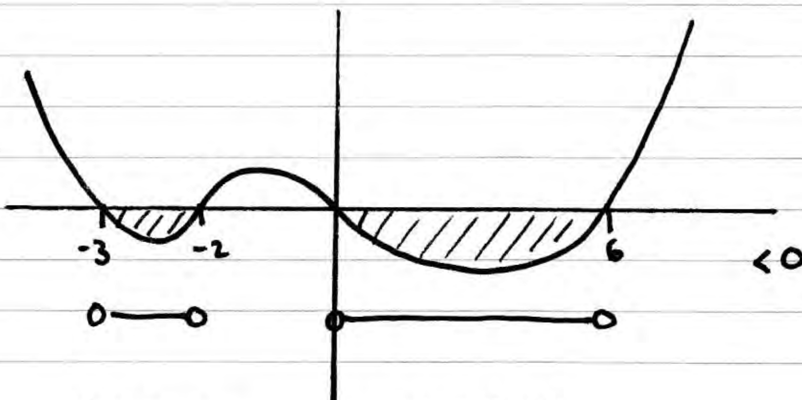
$x(x+3)[3x-x^2+x+12] > 0$

$-x(x+3)[x^2-4x-12] > 0$

$-x(x+3)(x-6)(x+2) > 0$

$x(x+3)(x-6)(x+2) < 0$

$0 \quad -3 \quad 6 \quad -2$



$-3 < x < -2$ or $0 < x < 6$

2.

$$\frac{d^2y}{dx^2} = e^x\left(2y\frac{dy}{dx} + y^2 + 1\right)$$

(a) Show that

$$\frac{d^3y}{dx^3} = e^x\left[2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + ky\frac{dy}{dx} + y^2 + 1\right],$$

where k is a constant to be found.

(3)

Given that, at $x = 0$, $y = 1$ and $\frac{dy}{dx} = 2$,

(b) find a series solution for y in ascending powers of x, up to and including the term in x^3.

(4)

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[e^x\left(2y\frac{dy}{dx} + y^2 + 1\right)\right]$$

$$= e^x\left(2y\frac{dy}{dx} + y^2 + 1\right) + e^x\left[2\frac{dy}{dx}\frac{dy}{dx} + 2y\frac{d^2y}{dx^2} + 2y\frac{dy}{dx}\right]$$

$$= e^x\left[2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 4y\frac{dy}{dx} + y^2 + 1\right] \quad \therefore k = 4$$

b) $x_0 = 0 \quad y_0 = 1 \quad y_0' = 2$

$$y'' = e^x(2yy' + y^2 + 1) = 2(1)(2) + (1)^2 + 1 = 6$$

$$y''' = e^x[2yy'' + 2(y')^2 + 4yy' + y^2 + 1] = 2(1)(6) + 2(2)^2 + 4(1)(2) + (1)^2 + 1$$

$$= 12 + 8 + 8 + 1 + 1$$

$$= 30$$

$$y = y_0 + y_0'x + \frac{y_0''x^2}{2} + \frac{y_0'''x^3}{6}$$

$$\Rightarrow y = 1 + 2x + 3x^2 + 5x^3 \ldots$$

3. Find the general solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x}+5y=\frac{\ln x}{x}, \qquad x>0$$

giving your answer in the form $y=\mathrm{f}(x)$.

(8)

$\frac{dy}{dx}+\frac{5}{x}y=\frac{\ln x}{x^2}$ IF $f(x)=e^{\int \frac{5}{x}dx}=e^{5\ln x}=(e^{\ln x})^5$
$=x^5$

$x^5\frac{dy}{dx}+5x^4y=x^3\ln x \Rightarrow \frac{d}{dx}(x^5y)=x^3\ln x$

$\therefore x^5y=\int x^3\ln x\,dx$ $\left\{ \begin{array}{ll} u=\ln x & v=\frac{1}{4}x^4 \\ u'=\frac{1}{x} & v'=x^3 \end{array} \right\}$

$\Rightarrow x^5y=\frac{1}{4}x^4\ln x-\int\frac{1}{4}x^3dx=\frac{1}{4}x^4\ln x-\frac{1}{16}x^4+c$

$\therefore y=\frac{\ln x}{4x}-\frac{1}{16x}+\frac{c}{x^5}$

4. Given that

$$(2r+1)^3 = Ar^3 + Br^2 + Cr + 1,$$

(a) find the values of the constants A, B and C.

(2)

(b) Show that

$$(2r+1)^3 - (2r-1)^3 = 24r^2 + 2$$

(2)

(c) Using the result in part (b) and the method of differences, show that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

(5)

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(2r+1)^3 = (2r)^3 + 3(2r)^2(1) + 3(2r)(1)^2 + (1)^3$

$= 8r^3 + 12r^2 + 6r + 1$ $\qquad A=8, B=12, C=6$

b) $(2r+1)^3 - (2r-1)^3 = 8r^3 + 12r^2 + 6r + 1 - 8r^3 - 12r^2 + 6r - 1$

$= 24r^2 + 2$

c) $\sum_{1}^{n} 24r^2 + 2 = 24\sum_{1}^{n} r^2 + 2n = (2n+1)^3 - 1$

$r=1 \quad (\cancel{3})^3 - (1)^3 \; +$

$r=2 \quad (\cancel{5})^3 - (\cancel{3})^3 \; +$

$r=3 \quad (\cancel{7})^3 - (\cancel{5})^3 \; +$

$\vdots$

$r=n-1 \quad (\cancel{2n-1})^3 - (\cancel{2n-3})^3 \; +$

$r=n \quad (2n+1)^3 - (\cancel{2n-1})^3 \; +$

$\Rightarrow 24\sum_{1}^{n} r^2 + 2n = 8n^3 + 12n^2 + 6n + \cancel{1} - \cancel{1}$

$\Rightarrow 24\sum_{1}^{n} r^2 = 8n^3 + 12n^2 + 4n$

$\Rightarrow 24\sum_{1}^{n} r^2 = 4n(2n^2 + 3n + 1)$

$\Rightarrow \sum_{1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

5. The point P represents the complex number z on an Argand diagram, where

$$|z - \mathrm{i}| = 2$$

The locus of P as z varies is the curve C.

(a) Find a cartesian equation of C.

(2)

(b) Sketch the curve C.

(2)

A transformation T from the z-plane to the w-plane is given by

$$w = \frac{z + \mathrm{i}}{3 + \mathrm{i}z}, \quad z \neq 3\mathrm{i}$$

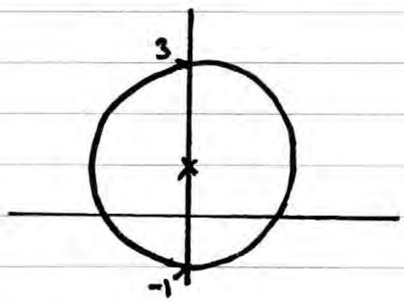
The point Q is mapped by T onto the point R. Given that R lies on the real axis,

(c) show that Q lies on C.

(5)

a) $|x + (y-1)i| = 2 \Rightarrow x^2 + (y-1)^2 = 2^2$

b) Circle $c(0,1)$ $r=2$



c) $z = x + iy \Rightarrow w = \dfrac{(x+iy)+i}{3+i(x+iy)} = \dfrac{x+(y+1)i}{(3-y)+ix}$

$\Rightarrow w = \dfrac{x+(y+1)i}{(3-y)+ix} \times \dfrac{[(3-y)-ix]}{[(3-y)-ix]} = \dfrac{x(3-y)+x(y+1)+i[(y+1)(3-y)-x^2]}{(3-y)^2+x^2}$

If R lies on real axis in w-plane $\Rightarrow$ Imaginary Part $= 0$

$\Rightarrow (y+1)(3-y) - x^2 = 0 \Rightarrow -y^2+2y+3-x^2=0$

$\Rightarrow x^2+y^2-2y=3 \Rightarrow x^2+(y-1)^2=4$

$\therefore$ Q lies on C.

6.

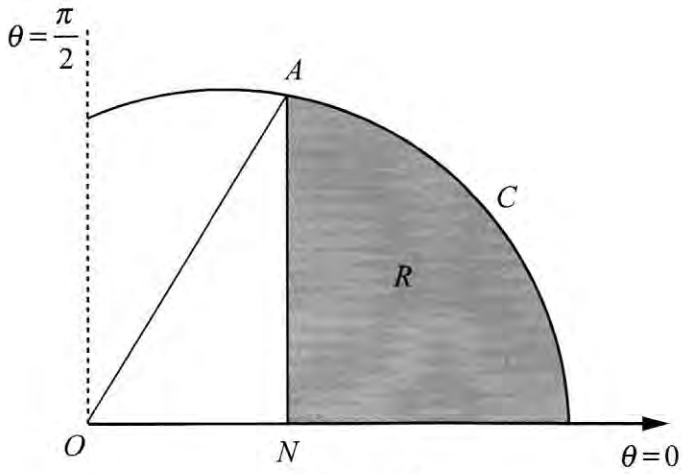


Figure 1

The curve C shown in Figure 1 has polar equation

$$r = 2 + \cos\theta, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

At the point A on C, the value of r is $\frac{5}{2}$.

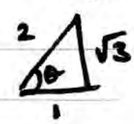
The point N lies on the initial line and AN is perpendicular to the initial line.

The finite region R, shown shaded in Figure 1, is bounded by the curve C, the initial line and the line AN.

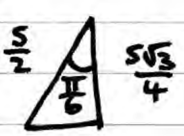
Find the exact area of the shaded region R.

(9)

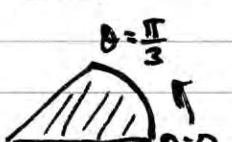
at A $r = \frac{5}{2}$ $\quad \frac{5}{2} = 2 + \cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$



$y = r\sin\theta = \frac{5}{2} \times \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{4} = AN$



Area $= \frac{1}{2}\left(\frac{5}{2}\right)\left(\frac{5\sqrt{3}}{4}\right)\sin\frac{\pi}{6} = \frac{25\sqrt{3}}{32}$



$= \frac{1}{2}\int_0^{\frac{\pi}{3}} (2+\cos\theta)^2 \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} 4 + 4\cos\theta + \left(\frac{1}{2}\cos 2\theta + \frac{1}{2}\right) d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{3}} \frac{9}{2} + 4\cos\theta + \frac{1}{2}\cos 2\theta \, d\theta = \frac{1}{4}\int_0^{\frac{\pi}{3}} 9 + 8\cos\theta + \cos 2\theta \, d\theta$

$= \frac{1}{4}\left[9\theta + 8\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{3}} = \frac{1}{4}\left[3\pi + 4\sqrt{3} + \frac{\sqrt{3}}{4} - 0\right] = \frac{1}{4}\left[3\pi + \frac{17\sqrt{3}}{4}\right]$

Shaded $= \frac{3\pi}{4} + \frac{17\sqrt{3}}{16} - \frac{25\sqrt{3}}{32} = \frac{3\pi}{4} + \frac{9\sqrt{3}}{32}$

7. (a) Use de Moivre's theorem to show that

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

(5)

Hence, given also that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$,

(b) find all the solutions of

$$\sin 5\theta = 5\sin 3\theta,$$

in the interval $0 \leqslant \theta < 2\pi$. Give your answers to 3 decimal places.

(6)

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

a) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

equating Imaginary Parts $\Rightarrow$

$\sin 5\theta \equiv 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$
$\equiv 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$
$\equiv 5(1-2\sin^2\theta+\sin^4\theta)\sin\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$
$\equiv 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 11\sin^5\theta$
$\equiv 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$

b) $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta = 15\sin\theta - 20\sin^3\theta$
$16\sin^5\theta - 10\sin\theta = 0$
$2\sin\theta(8\sin^4\theta - 5) = 0$

$\sin\theta = 0$ $\quad$ $\sin\theta = \sqrt[4]{\frac{5}{8}}$ $\quad$ $\sin\theta = -\sqrt[4]{\frac{5}{8}}$

$\theta = 0, \pi$ $\quad$ $\theta = 1.095^c, 2.046^c$ $(\pi -)$ $\quad$ $\theta = \cancel{-1.095^c}$

$\theta = 4.237^c, 5.188^c$

8. The differential equation

$$\frac{d^2x}{dt^2}+6\frac{dx}{dt}+9x=\cos 3t, \quad t \geqslant 0$$

describes the motion of a particle along the x-axis.

(a) Find the general solution of this differential equation.

(8)

(b) Find the particular solution of this differential equation for which, at $t=0$,

$x=\frac{1}{2}$ and $\frac{dx}{dt}=0$.

(5)

On the graph of the particular solution defined in part (b), the first turning point for $t>30$ is the point A.

(c) Find approximate values for the coordinates of A.

(2)

$x = a\cos 3t + b\sin 3t$

$x' = -3a\sin 3t + 3b\cos 3t$

$x'' = -9a\cos 3t - 9b\sin 3t$

$9x = 9a\cos 3t + 9b\sin 3t$

$+\ 6x' = 18b\cos 3t - 18a\sin 3t$

$+\ x'' = -9a\cos 3t - 9b\sin 3t$

$\cos 3t \equiv 18b\cos 3t - 18a\sin 3t$

$x_{PI} = \frac{1}{18}\sin 3t$

$\therefore a=0 \quad b=\frac{1}{18}$

$x = Ae^{mt}$

$x' = Ame^{mt}$

$x'' = Am^2e^{mt}$

$x''+6x'+9x=0$

$\Rightarrow \underbrace{Ae^{mt}}_{\neq 0}\underbrace{(m^2+6m+9)}_{=0} = 0 \Rightarrow (m+3)^2=0$

$m=-3$ RR

$x_{cf} = (A+Bt)e^{-3t}$

$\therefore x = (A+Bt)e^{-3t} + \frac{1}{18}\sin 3t$

$t=0, x=\frac{1}{2} \Rightarrow \frac{1}{2}=A$

$x = (\frac{1}{2}+Bt)e^{-3t} + \frac{1}{18}\sin 3t$

$x' = -3(\frac{1}{2}+Bt)e^{-3t} + Be^{-3t} + \frac{1}{6}\cos 3t$

$t=0\ x'=0$

$0 = -\frac{3}{2} + B + \frac{1}{6} \Rightarrow B = \frac{4}{3}$

$\therefore x = (\frac{1}{2}+\frac{4}{3}t)e^{-3t} + \frac{1}{18}\sin 3t$

Z

For large t $\quad e^{-3t} \to 0$

$\therefore \quad x \approx \frac{1}{18} \sin 3t$

Max x is when $\sin 3t = 1$

$$3t = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$t = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots \frac{59\pi}{6}$$

larger than 30

$\therefore A\left(\frac{59\pi}{6}, -\frac{1}{18}\right)$