FP2 June 10

1. (a) Express $\dfrac{3}{(3r-1)(3r+2)}$ in partial fractions. **(2)**

(b) Using your answer to part (a) and the method of differences, show that

$$\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} = \frac{3n}{2(3n+2)}$$ **(3)**

(c) Evaluate $\displaystyle\sum_{r=100}^{1000} \frac{3}{(3r-1)(3r+2)}$, giving your answer to 3 significant figures. **(2)**

$$\frac{3}{(3r-1)(3r+2)} = \frac{A}{3r-1} + \frac{B}{3r+2} \Rightarrow 3 = A(3r+2) + B(3r-1) \quad r=\tfrac{1}{3}\ A=1$$

$$r=-\tfrac{2}{3}\ B=-1$$

$$= \frac{1}{3r-1} + \frac{-1}{3r+2}$$

b) $$\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} = \left(\frac{1}{2} - \frac{1}{5}\right) \quad r=1$$

$$+ \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{8}}\right) \quad r=2$$

$$+ \left(\cancel{\frac{1}{8}} - \frac{1}{11}\right)_{+\dots} \quad r=3 \qquad + \left(\cancel{\frac{1}{3n-4}} - \cancel{\frac{1}{3n-1}}\right) \quad r=n-1$$

$$+ \cancel{\frac{1}{3n-1}} - \frac{1}{3n+2} \quad r=n$$

$$= \frac{1}{2} - \frac{1}{3n+2} = \frac{3n+\cancel{2}-\cancel{2}}{2(3n+2)} = \underline{\underline{\frac{3n}{2(3n+2)}}}$$

c) $$\sum_{100}^{1000} \frac{3}{(3r-1)(3r+2)} = \frac{3000}{2(3002)} - \frac{3\times 99}{2(3\times 99+2)} = \underline{\underline{0.00301}}$$

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2. The displacement x metres of a particle at time t seconds is given by the differential equation

$$\frac{d^2x}{dt^2} + x + \cos x = 0$$

When $t = 0$, $x = 0$ and $\frac{dx}{dt} = \frac{1}{2}$.

Find a Taylor series solution for x in ascending powers of t, up to and including the term in t^3.

(5)

$$\frac{d}{dt}\left(\frac{d^2x}{dt^2}\right) + \frac{d}{dt}(x) + \frac{d}{dt}(\cos x) = 0$$

$$\Rightarrow \frac{d^3x}{dt^3} + \frac{dx}{dt} + -\sin x\frac{dx}{dt} = 0 \quad \Rightarrow \frac{d^3x}{dt^3} + (1-\sin x)\frac{dx}{dt} = 0$$

$t=0 \quad x_0 = 0 \quad x_0' = \frac{1}{2}$

$$x'' + x + \cos x = 0$$

$$x'' + 0 + 1 = 0 \quad \Rightarrow x'' = -1$$

$$x''' + (1-\sin x)x' = 0$$

$$x''' + (1-0)\tfrac{1}{2} = 0 \quad \Rightarrow x''' = -\tfrac{1}{2}$$

$$\therefore \quad x = \tfrac{1}{2}t - \tfrac{1}{2}t^2 - \tfrac{1}{12}t^3 \ldots$$

2

3. (a) Find the set of values of x for which

$$x + 4 > \frac{2}{x+3}$$

(6)

(b) Deduce, or otherwise find, the values of x for which

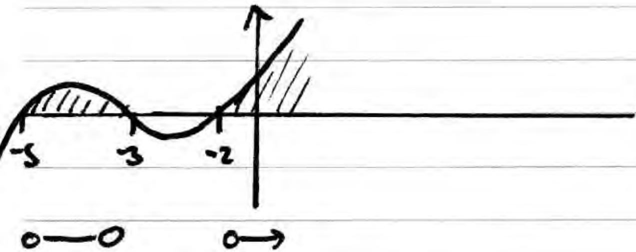
$$x + 4 > \frac{2}{|x+3|}$$

(1)

$(x+3)^2(x+4) > \frac{2(x+3)^{\cancel{2}}}{\cancel{x+3}}$ $\Rightarrow (x+3)^2(x+4) - 2(x+3) > 0$

$\Rightarrow (x+3)[(x+3)(x+4)-2] > 0$ $\Rightarrow (x+3)[x^2+7x+12-2] > 0$

$\Rightarrow (x+3)(x+5)(x+2) > 0$



$x > -2$ or $-5 < x < -3$

b) $\underline{x > -2}$

4. $$z = -8 + (8\sqrt{3})\mathrm{i}$$

(a) Find the modulus of z and the argument of z.

(3)

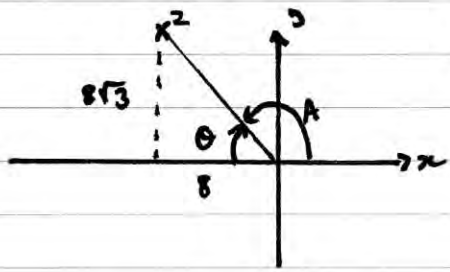
Using de Moivre's theorem,

(b) find z^3,

(2)

(c) find the values of w such that $w^4 = z$, giving your answers in the form $a + \mathrm{i}b$, where $a, b \in \mathbb{R}$.

(5)



$\tan\theta = \left(\frac{8\sqrt{3}}{8}\right) \Rightarrow \theta = \frac{\pi}{3} \Rightarrow A = \frac{2\pi}{3} = \arg(z)$

$|z| = \sqrt{8^2 + (8\sqrt{3})^2} = \underline{16}.$

b) $z^3 = \left(16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right)^3 = 4096\left(\cos(2\pi) + i\sin(2\pi)\right)$
$= \underline{4096}$

c) $w = z^{\frac{1}{4}} = 16^{\frac{1}{4}}\left(\cos\left(\frac{2\pi}{3} + 2k\pi\right) + i\sin\left(\frac{2\pi}{3} + 2k\pi\right)\right)^{\frac{1}{4}}$

$w = 2\left[\cos\left(\frac{6k+2}{3}\right)\pi + i\sin\left(\frac{6k+2}{3}\right)\pi\right]^{\frac{1}{4}}$

$w = 2\left[\cos\left(\frac{3k+1}{6}\right)\pi + i\sin\left(\frac{3k+1}{6}\right)\pi\right] =$

$k=-2$ $\quad w = 2\left[\cos\left(\frac{-5\pi}{6}\right) + i\sin\left(\frac{-5\pi}{6}\right)\right] = -\sqrt{3} + -1i$

$k=-1$ $\quad w = 2\left[\cos\left(\frac{-2\pi}{6}\right) + i\sin\left(\frac{-2\pi}{6}\right)\right] = 1 + -\sqrt{3}i$

$k=0$ $\quad w = 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right] = \sqrt{3} + i$

$k=-1$ $\quad w = 2\left[\cos\left(\frac{4\pi}{6}\right) + i\sin\left(\frac{4\pi}{6}\right)\right] = -1 + \sqrt{3}i$

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5.

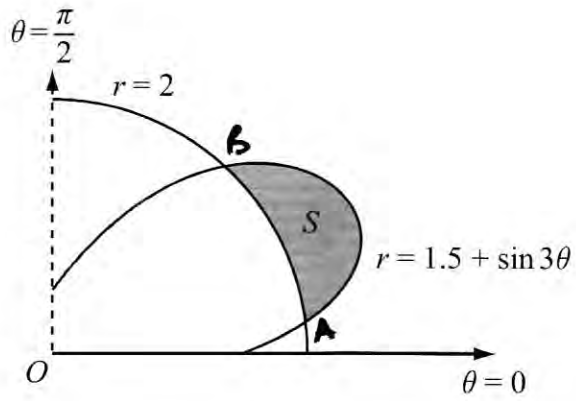


Figure 1

Figure 1 shows the curves given by the polar equations

$$r = 2, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2},$$

and $$r = 1.5 + \sin 3\theta, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}.$$

(a) Find the coordinates of the points where the curves intersect. **(3)**

The region S, between the curves, for which $r > 2$ and for which $r < (1.5 + \sin 3\theta)$, is shown shaded in Figure 1.

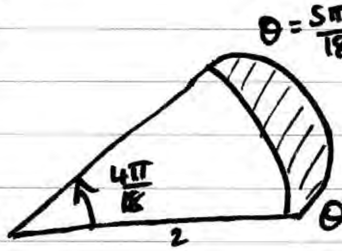
(b) Find, by integration, the area of the shaded region S, giving your answer in the form $a\pi + b\sqrt{3}$, where a and b are simplified fractions. **(7)**

a) $2 = 1.5 + \sin 3\theta \Rightarrow \sin 3\theta = \frac{1}{2} \Rightarrow 3\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\theta = \frac{\pi}{18}, \frac{5\pi}{18}$

$A(2, \frac{\pi}{18})$; $B(2, \frac{5\pi}{18})$

b)



$S = \frac{1}{2}\int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} (1.5+\sin 3\theta)^2 \, d\theta$ - Sector

$S = \frac{1}{2}\int_{\pi/18}^{5\pi/18} \frac{9}{4} + 3\sin 3\theta + \sin^2 3\theta \, d\theta - \frac{1}{2}(2)^2\left(\frac{4\pi}{18}\right)$

$\cos 6\theta = 1 - 2\sin^2 3\theta$

$S = \frac{1}{2}\int_{\pi/18}^{5\pi/18} \frac{9}{4} + 3\sin 3\theta + \left(\frac{1}{2} - \frac{1}{2}\cos 6\theta\right) d\theta - \frac{4\pi}{9}$

$= \frac{1}{2}\left[\frac{11}{4}\theta - \cos 3\theta - \frac{1}{12}\sin 6\theta\right]_{\pi/18}^{5\pi/18} - \frac{4\pi}{9} = \frac{1}{2}\left[\left(\frac{55}{72}\pi + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{24}\right) - \left(\frac{11}{72}\pi - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{24}\right)\right] - \frac{4\pi}{9}$

$= \frac{13\sqrt{3}}{24} - \frac{5\pi}{36}.$

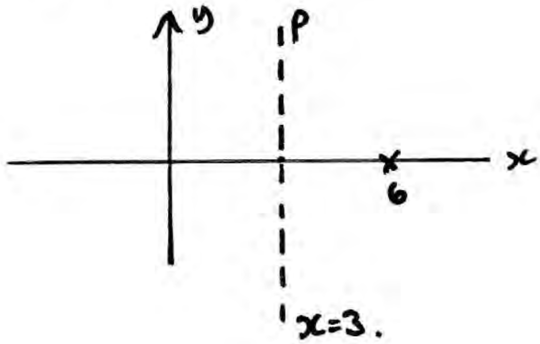
6. A complex number z is represented by the point P in the Argand diagram.

(a) Given that $|z-6| = |z|$, sketch the locus of P. **(2)**

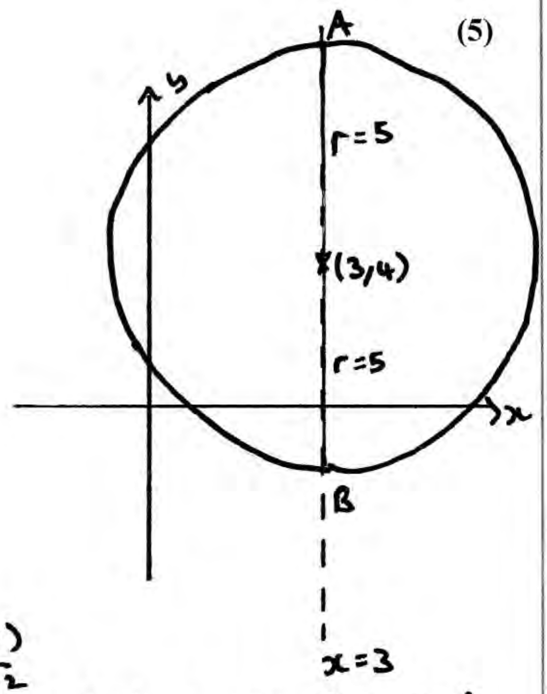
(b) Find the complex numbers z which satisfy both $|z-6| = |z|$ and $|z-3-4i| = 5$. **(3)**

The transformation T from the z-plane to the w-plane is given by $w = \dfrac{30}{z}$.

(c) Show that T maps $|z-6| = |z|$ onto a circle in the w-plane and give the cartesian equation of this circle. **(5)**



b)



(A) $3+9i$ (B) $3-i$

c) $z = \dfrac{30}{w} \Rightarrow 3 = \dfrac{30}{u+iv}$

$3 = \dfrac{30(u-iv)}{(u+iv)(u-iv)} = \dfrac{30u}{u^2+v^2} + i\dfrac{(-30v)}{u^2+v^2}$

real part $=3 \Rightarrow 3 = \dfrac{30u}{u^2+v^2}$

$\Rightarrow 3u^2+3v^2 = 30u \Rightarrow 3u^2-30u+3v^2=0 \Rightarrow u^2-10u+v^2=0$

$\Rightarrow (u-5)^2+v^2 = 25$ $\quad x=3 \Rightarrow$ maps to a circle centre $(5,0)$ $r=5$ in the w-plane.

alt

$|z-6| = |z| \Rightarrow \left|\dfrac{30}{w}-6\right| = \left|\dfrac{30}{w}\right| \Rightarrow \left|\dfrac{30-6w}{w}\right| = \left|\dfrac{30}{w}\right|$

$\Rightarrow |30-6w| = |30| \Rightarrow 6|5-w| = 6|5| \Rightarrow |5-w| = |5|$

$\Rightarrow |w-5| = |5| \Rightarrow |u+iv-5| = |5| = |(u-5)+iv| = |5|$

$\Rightarrow (u-5)^2+v^2 = 25$

2

7. (a) Show that the transformation $z = y^{\frac{1}{2}}$ transforms the differential equation

$$\frac{dy}{dx} - 4y \tan x = 2y^{\frac{1}{2}} \qquad \text{(I)}$$

into the differential equation

$$\frac{dz}{dx} - 2z \tan x = 1 \qquad \text{(II)}$$

(5)

(b) Solve the differential equation (II) to find z as a function of x.

(6)

(c) Hence obtain the general solution of the differential equation (I).

(1)

$z = y^{\frac{1}{2}} \Rightarrow y = z^2 \Rightarrow \frac{dy}{dx} = 2z\frac{dz}{dx}$

$\Rightarrow 2z\frac{dz}{dx} - 4z^2 \tan x = 2z \qquad \div 2z$

$\Rightarrow \frac{dz}{dx} - 2z\tan x = 1$

b) $\frac{dz}{dx} - (2\tan x)z = 1 \qquad \underline{\text{IF}}\ f(x) = e^{\int -2\tan x} = e^{-2\ln|\sec x|}$

$= (e^{\ln|\sec x|})^{-2} = (\sec x)^{-2}$

$= \underline{\cos^2 x}$

$\cos^2 x \frac{dz}{dx} - \cos^2 x(2\tan x)z = \cos^2 x$

$\Rightarrow \frac{d}{dx}(z\cos^2 x) = \cos^2 x \qquad \Rightarrow z\cos^2 x = \int \cos^2 x \, dx$

$\Rightarrow z\cos^2 x = \frac{1}{2}\int \cos 2x + 1 = \frac{1}{4}\sin 2x + \frac{1}{2}x + c$

$z = \frac{\sin 2x}{4\cos^2 x} + \frac{x}{2\cos^2 x} + \frac{c}{\cos^2 x}$

c) $z = y^{\frac{1}{2}} \quad \therefore \quad y = \left(\frac{\sin 2x}{4\cos^2 x} + \frac{x}{2\cos^2 x} + \frac{c}{\cos^2 x}\right)^2$

$\sin 2x = 2\sin x\cos x$

$\Rightarrow y = \left(\frac{1}{2}\tan x + \frac{1}{2}x\sec^2 x + c\sec^2 x\right)^2$

8. (a) Find the value of λ for which $y = \lambda x \sin 5x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 25y = 3\cos 5x$$

(4)

(b) Using your answer to part (a), find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 25y = 3\cos 5x$$

(3)

Given that at $x = 0$, $y = 0$ and $\frac{dy}{dx} = 5$,

(c) find the particular solution of this differential equation, giving your solution in the form $y = f(x)$.

(5)

(d) Sketch the curve with equation $y = f(x)$ for $0 \leqslant x \leqslant \pi$.

(2)

$y = \lambda x \sin 5x$

$y' = 5\lambda x \cos 5x + \lambda \sin 5x$

$y'' = -25\lambda x \sin 5x + 5\lambda \cos 5x + 5\lambda \cos 5x = -25\lambda x \sin 5x + 10\lambda \cos 5x$

$y'' + 25y = \cancel{-25\lambda x \sin 5x} + 10\lambda \cos 5x \cancel{+ 25\lambda x \sin 5x} = 3\cos 5x$

$\therefore 10\lambda = 3 \quad \lambda = \frac{3}{10}$

$y_{PI} = \frac{3}{10} x \sin 5x$

$y = Ae^{mt} \qquad y'' + 25y = 0$

$y' = Ame^{mt} \qquad Ae^{mt}(m^2 + 25) = 0$

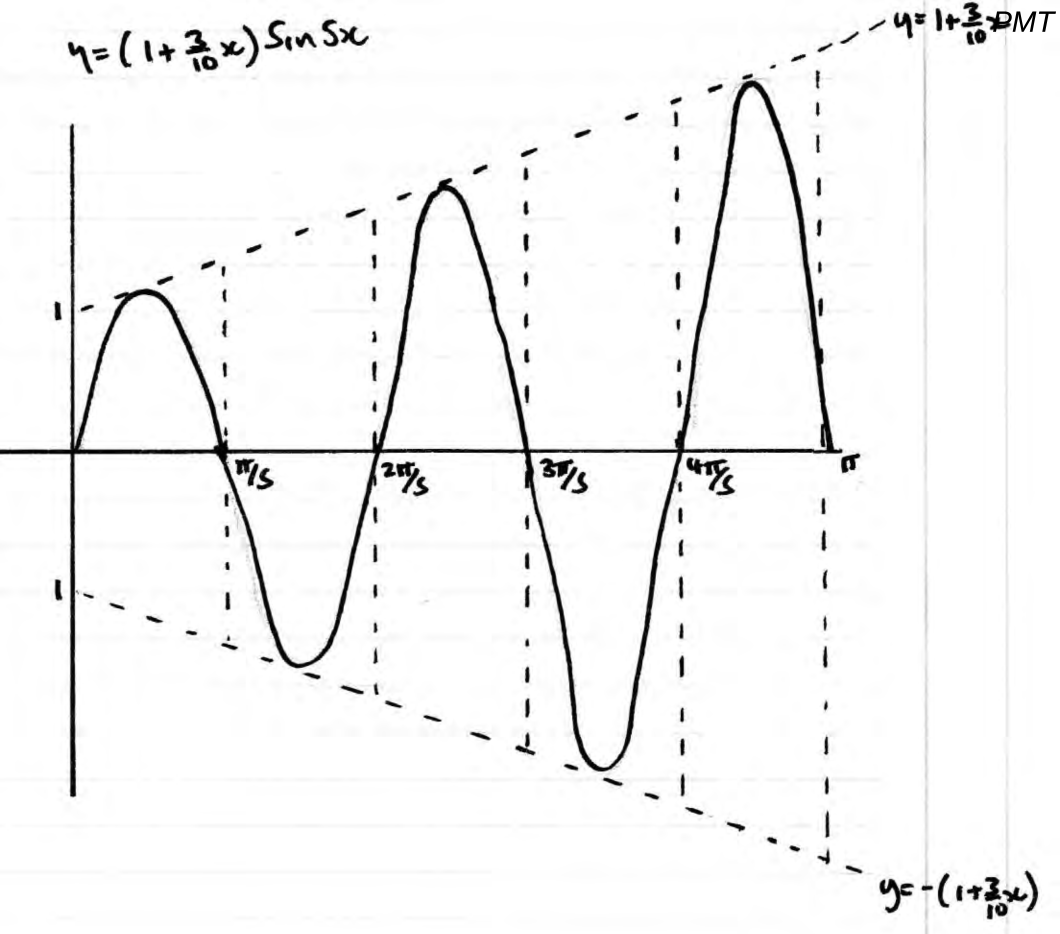
$y'' = Am^2e^{mt} \qquad \neq 0 \quad = 0 \quad \Rightarrow m = \pm 5i$

$y_{cf} = A\cos 5x + B\sin 5x \qquad \therefore y = A\cos 5x + (B + \frac{3}{10}x)\sin 5x$

$x = 0, y = 0 \Rightarrow 0 = A \qquad y = (B + \frac{3}{10}x)\sin 5x$

$x = 0\ y' = 5 \Rightarrow 5 = 5B \Rightarrow \underline{B = 1} \qquad y' = 5(B + \frac{3}{10}x)\cos 5x + \frac{3}{10}\sin 5x$

$\therefore\ y = (1 + \frac{3}{10}x)\sin 5x$



$y=(1+\frac{3}{10}x)\sin 5x$
$y=1+\frac{3}{10}x$
PMT
1
1
$\pi/5$
$2\pi/5$
$3\pi/5$
$4\pi/5$
π
$y=-(1+\frac{3}{10}x)$