June 09 FP2



1. (a) Express $\dfrac{1}{r(r+2)}$ in partial fractions.

**(1)**

(b) Hence show that $\displaystyle\sum_{r=1}^{n} \frac{4}{r(r+2)} = \frac{n(3n+5)}{(n+1)(n+2)}$.

**(5)**

a)

$$\frac{1}{r(r+2)} = \frac{A}{r} + \frac{B}{r+2} \Rightarrow 1 = A(r+2) + B(r) \qquad r=0 \Rightarrow A = \tfrac{1}{2}$$
$$r=-2 \Rightarrow B = -\tfrac{1}{2}$$

$$= \frac{1}{2r} - \frac{1}{2(r+2)}$$

b) $\displaystyle\sum_{1}^{n} \frac{4}{r(r+2)} = 4\sum_{1}^{n} \frac{1}{r(r+2)}$

$r=1 \quad \left(\frac{1}{2} - \cancel{\frac{1}{6}}\right) +$ $\qquad r=n-2 \quad \left(\cancel{\frac{1}{2(n-2)}} - \cancel{\frac{1}{2n}}\right) +$

$r=2 \quad \left(\frac{1}{4} - \cancel{\frac{1}{8}}\right) +$ $\qquad r=n-1 \quad \left(\cancel{\frac{1}{2(n-1)}} - \frac{1}{2(n+1)}\right) +$

$r=3 \quad \left(\cancel{\frac{1}{6}} - \cancel{\frac{1}{10}}\right) +$ $\qquad r=n \quad \left(\cancel{\frac{1}{2n}} - \frac{1}{2(n+2)}\right)$

$r=4 \quad \left(\cancel{\frac{1}{8}} - \cancel{\frac{1}{12}}\right) +$

$$\therefore 4\sum_{1}^{n} \frac{1}{r(r+2)} = 4\left[\frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}\right]$$

$$= \cancel{4}\left[\frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{\cancel{4}(n+1)(n+2)}\right]$$

$$= \frac{3n^2 + 9n + \cancel{6} - 2n - \cancel{4} - 2n - \cancel{2}}{(n+1)(n+2)} = \frac{n(3n+5)}{(n+1)(n+2)}$$

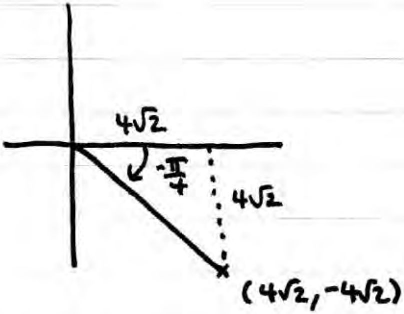
☆

**2.** Solve the equation

$$z^3 = 4\sqrt{2} - 4\sqrt{2}\mathrm{i},$$

giving your answers in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

**(6)**




$$r = \sqrt{(4\sqrt{2})^2 + (4\sqrt{2})^2} = 8$$

$$\therefore z^3 = 8\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$$

$$\Rightarrow z = 2\left(\cos\left(\frac{-\pi}{4} + 2k\pi\right) + i\sin\left(\frac{-\pi}{4} + 2k\pi\right)\right)^{\frac{1}{3}}$$

$$\Rightarrow z = 2\left[\cos\left(\frac{8k-1}{4}\right)\pi + i\sin\left(\frac{8k-1}{4}\right)\pi\right]^{\frac{1}{3}}$$

$$\Rightarrow z = 2\left[\cos\left(\frac{8k-1}{12}\right)\pi + i\sin\left(\frac{8k-1}{12}\right)\pi\right]$$

$k = -1 \quad z = 2\left[\cos\frac{-9\pi}{12} + i\sin\frac{-9\pi}{12}\right]$

$k = 0 \quad z = 2\left[\cos\frac{-\pi}{12} + i\sin\frac{-\pi}{12}\right]$

$k = 1 \quad z = 2\left[\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right]$

3. Find the general solution of the differential equation

$$\sin x \frac{dy}{dx} - y\cos x = \sin 2x \sin x,$$

giving your answer in the form $y = f(x)$.

**(8)**

$\frac{dy}{dx} - \left(\frac{\cos x}{\sin x}\right)y = \sin 2x \Rightarrow \frac{dy}{dx} - (\cot x)y = \sin 2x$

IF $f(x) = e^{-\int \cot x \, dx} = e^{-\ln|\sin x|} = \left(e^{\ln|\sin x|}\right)^{-1} = (\sin x)^{-1} = \operatorname{cosec} x$

$\Rightarrow \operatorname{cosec} x \frac{dy}{dx} - (\operatorname{cosec} x \cot x)y = \sin 2x \operatorname{cosec} x$

$\Rightarrow \frac{d}{dx}(y \operatorname{cosec} x) = \frac{2\cancel{\sin x}\cos x}{\cancel{\sin x}} \Rightarrow y \operatorname{cosec} x = 2\int \cos x \, dx$

$\Rightarrow y = \frac{2\sin x + c}{\operatorname{cosec} x} \Rightarrow y = 2\sin^2 x + c\sin x$

**4.**

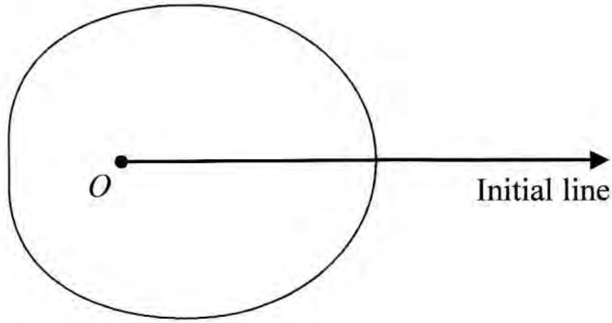



**Figure 1**

Figure 1 shows a sketch of the curve with polar equation

$$r = a + 3\cos\theta, \quad a > 0, \quad 0 \leqslant \theta < 2\pi$$

The area enclosed by the curve is $\frac{107}{2}\pi$.

Find the value of $a$.

**(8)**

$$\text{Area} = \cancel{2} \times \cancel{\tfrac{1}{2}}\int_0^{\pi} (a+3\cos\theta)^2 \, d\theta = \int_0^{\pi} a^2 + 6a\cos\theta + 9\cos^2\theta \, d\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1 \Rightarrow \frac{9}{2}\cos 2\theta + \frac{9}{2} = 9\cos^2\theta$$

$$\int_0^{\pi} a^2 + 6a\cos\theta + \frac{9}{2}\cos 2\theta + \frac{9}{2} \, d\theta = \int_0^{\pi} \left(a^2+\frac{9}{2}\right) + 6a\cos\theta + \frac{9}{2}\cos 2\theta \, d\theta$$

$$= \left[\left(a^2+\frac{9}{2}\right)\theta + 6a\sin\theta + \frac{9}{4}\sin 2\theta\right]_0^{\pi} = \left(a^2+\frac{9}{2}\right)\pi = \left(\frac{2a^2+9}{2}\right)\pi$$

$$\therefore \frac{2a^2+9}{2} = \frac{107}{2} \Rightarrow 2a^2 = 98 \Rightarrow a^2 = 49 \quad \therefore \underline{a = 7}$$

**5.**

$$y = \sec^2 x$$

(a) Show that $\dfrac{d^2y}{dx^2} = 6\sec^4 x - 4\sec^2 x$.

**(4)**

(b) Find a Taylor series expansion of $\sec^2 x$ in ascending powers of $\left(x - \dfrac{\pi}{4}\right)$, up to and including the term in $\left(x - \dfrac{\pi}{4}\right)^3$.

**(6)**

$y = (\sec x)^2$

$y' = 2(\sec x)^1 \times \sec x \tan x = 2\sec^2 x \tan x$

$y'' = 4(\sec x)' \sec x \tan^2 x + 2\sec^4 x = 4\sec^2 x \tan^2 x + 2\sec^4 x$

$\dfrac{\sin^2 x}{\cos^2 x} + \dfrac{\cos^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} \Rightarrow \tan^2 x + 1 = \sec^2 x$

$y'' = 4\sec^2 x(\sec^2 x - 1) + 2\sec^4 x = 6\sec^4 x - 4\sec^2 x$

$y''' = 24\sec^4 x \tan x - 8\sec^2 x \tan x$

$y\left(\frac{\pi}{4}\right) = \dfrac{1}{\left(\cos\left(\frac{\pi}{4}\right)\right)^2} = (\sqrt{2})^2 = 2$

$y'\left(\frac{\pi}{4}\right) = 2(\sqrt{2})^2(1) = 4$

$y''\left(\frac{\pi}{4}\right) = 4(\sqrt{2})^2(1)^2 + 2(\sqrt{2})^4 = 8 + 8 = 16$

$y'''\left(\frac{\pi}{4}\right) = 24(\sqrt{2})^4(1) - 8(\sqrt{2})^2(1) = 96 - 16 = 80$

$\therefore y = 2 + 4\left(x - \frac{\pi}{4}\right) + 8\left(x - \frac{\pi}{4}\right)^2 + \frac{40}{3}\left(x - \frac{\pi}{4}\right)^3$

**6.** A transformation $T$ from the $z$-plane to the $w$-plane is given by

$$w = \frac{z}{z + \mathrm{i}}, \quad z \neq -\mathrm{i}$$

The circle with equation $|z| = 3$ is mapped by $T$ onto the curve $C$.

(a) Show that $C$ is a circle and find its centre and radius.

**(8)**

The region $|z| < 3$ in the $z$-plane is mapped by $T$ onto the region $R$ in the $w$-plane.

(b) Shade the region $R$ on an Argand diagram.

**(2)**

$wz + iw = z \Rightarrow z - wz = iw \Rightarrow |z||(1-w)| = |iw|$

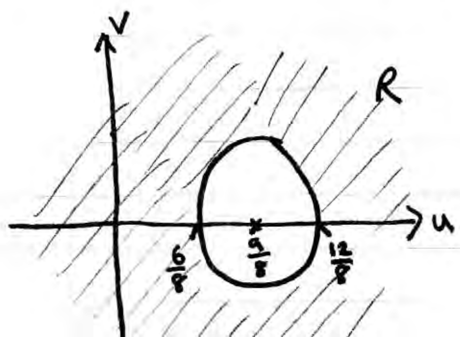
$\Rightarrow 3|w-1| = |\cancel{i}w| \Rightarrow 3|(u-1)+iv| = |u+iv|$

$\Rightarrow 9[(u-1)^2+v^2] = u^2+v^2 \Rightarrow 9u^2-18u+9+9v^2 = u^2+v^2$

$\Rightarrow 8u^2-18u+8v^2 = -9 \Rightarrow u^2-\frac{9}{4}u+v^2 = -\frac{9}{8}$

$\Rightarrow \left(u-\frac{9}{8}\right)^2+v^2 = -\frac{9}{8}+\frac{81}{64} = \frac{9}{64}$ Circle $c\left(\frac{9}{8},0\right)$ $r = \frac{3}{8}$

b)



**7.** (a) Sketch the graph of $y = |x^2 - a^2|$, where $a > 1$, showing the coordinates of the points where the graph meets the axes.

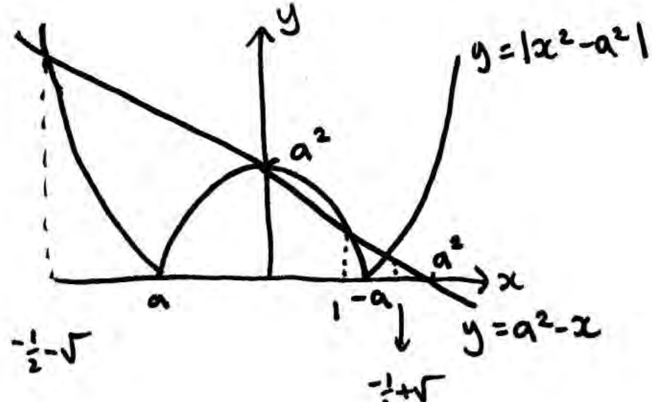
**(2)**

(b) Solve $|x^2 - a^2| = a^2 - x$, $a > 1$.

**(6)**

(c) Find the set of values of $x$ for which $|x^2 - a^2| > a^2 - x$, $a > 1$.

**(4)**




$x^2 - a^2 = a^2 - x$

$x^2 + x - 2a^2 = 0$

$(x + \frac{1}{2})^2 = 2a^2 + \frac{1}{4}$

$x + \frac{1}{2} = \pm\sqrt{2a^2 + \frac{1}{4}}$

$x = -\frac{1}{2} \pm \sqrt{2a^2 + \frac{1}{4}}$

$x^2 - \cancel{a^2} = x - \cancel{a^2}$

$x^2 - x = 0$

$x(x - 1) = 0$

$\underline{x = 0}$ $\underline{x = 1}$

$x < -\frac{1}{2} - \sqrt{2a^2 + \frac{1}{4}}$

or

$0 < x < 1$

or

$x > -\frac{1}{2} + \sqrt{2a^2 + \frac{1}{4}}$

**8.**

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 2e^{-t}$$

Given that $x = 0$ and $\frac{dx}{dt} = 2$ at $t = 0$,

(a) find $x$ in terms of $t$.

**(8)**

The solution to part (a) is used to represent the motion of a particle $P$ on the $x$-axis. At time $t$ seconds, where $t > 0$, $P$ is $x$ metres from the origin $O$.

(b) Show that the maximum distance between $O$ and $P$ is $\frac{2\sqrt{3}}{9}$ m and justify that this distance is a maximum.

**(7)**

$x = Ae^{mt}$ $\quad x'' + 5x' + 6x = 0$

$x' = Ame^{mt}$ $\quad Ae^{mt}(m^2 + 5m + 6) = 0$

$x'' = Am^2e^{mt}$ $\quad \neq 0 \quad = 0 \Rightarrow (m+3)(m+2) = 0$

$m = -3 \quad m = -2$

$\therefore x_{CF} = Ae^{-2t} + Be^{-3t}$

$x = \lambda e^{-t}$ $\quad x'' + 5x' + 6x = 2e^{-t}$

$x' = -\lambda e^{-t}$ $\quad \lambda e^{-t} - 5\lambda e^{-t} + 6\lambda e^{-t} = 2e^{-t}$

$x'' = \lambda e^{-t}$ $\quad 2\lambda e^{-t} = 2e^{-t} \quad \therefore \lambda = 1$

$x_{PI} = e^{-t} \quad \therefore x = Ae^{-2t} + Be^{-3t} + e^{-t}$

$x = 0, t = 0 \Rightarrow 0 = A + B + 1 \Rightarrow A + B = -1$

$x' = -2Ae^{-2t} - 3Be^{-3t} - e^{-t}$

$x = 0, x' = 2, t = 0 \Rightarrow 2 = -2A - 3B - 1 \Rightarrow 2A + 3B = -3$

$2A + 2B = -2$

$\underline{B = -1} \quad \underline{A = 0}$

$\therefore x = e^{-t} - e^{-3t}$

b) max distance when $x' = 0$ $\quad x' = 3e^{-3t} - e^{-t} \Rightarrow 3e^{-3t} = e^{-t}$

$\Rightarrow 3 = \frac{e^{-t}}{e^{-3t}} \Rightarrow 3 = e^{2t} \Rightarrow \ln 3 = 2t \Rightarrow t = \frac{1}{2}\ln 3 \quad \therefore t = \ln\sqrt{3}$

$x_{max} = e^{-\ln\sqrt{3}} - e^{-3\ln\sqrt{3}} = (e^{\ln\sqrt{3}})^{-1} - (e^{\ln\sqrt{3}})^{-3} = \frac{1}{\sqrt{3}} - \frac{1}{(\sqrt{3})^3}$

$$= \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{3-1}{3\sqrt{3}} = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}$$

$$x'' = e^{-t} - 9e^{-3t} \qquad t = \ln\sqrt{3}$$

$$x'' = (e^{\ln\sqrt{3}})^{-1} - 9(e^{\ln\sqrt{3}})^{-3} = \frac{1}{\sqrt{3}} - \frac{9}{3\sqrt{3}} = \frac{-6}{3\sqrt{3}}$$

$\therefore x'' < 0$ $\frown$ $\Rightarrow$ $x$ is a maximum