FP2 2007 Adapted

1. Obtain the general solution of the differential equation

$$x\frac{dy}{dx} + 2y = \cos x, \quad x > 0,$$

giving your answer in the form $y = f(x)$.

(Total 8 marks)

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos x}{x}$$

$$e^{\int \frac{2}{x}dx} \Rightarrow (e^{\ln x})^2 = x^2$$

$$x^2\frac{dy}{dx} + 2xy = x\cos x \Rightarrow \frac{d}{dx}(x^2y) = x\cos x \Rightarrow x^2y = \int x\cos x\,dx$$

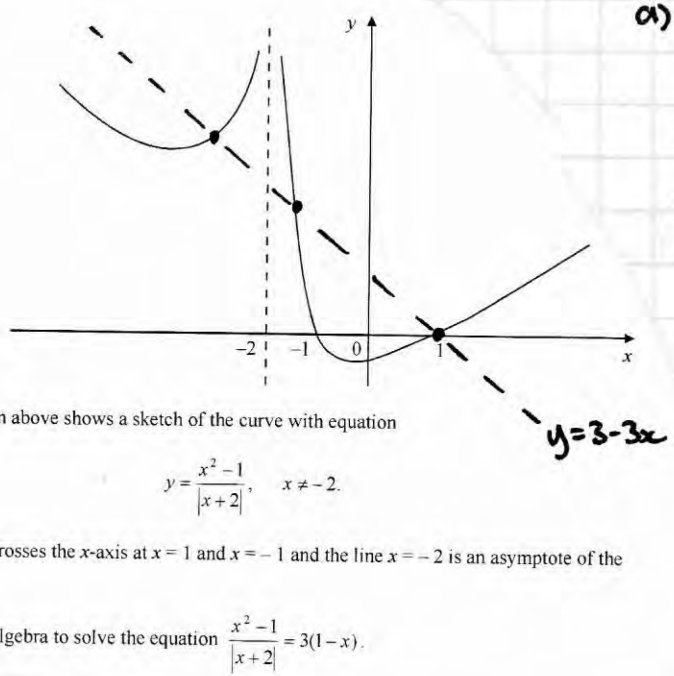
$u = x \quad v = \sin x$

$u' = 1 \quad v' = \cos x$

$$\Rightarrow x^2y = x\sin x - \int \sin x\,dx$$

$$\Rightarrow x^2y = x\sin x + \cos x + c \quad \therefore y = \frac{\sin x}{x} + \frac{\cos x}{x^2} + \frac{c}{x^2}$$

2.



The diagram above shows a sketch of the curve with equation

$$y = \frac{x^2 - 1}{|x+2|}, \quad x \neq -2.$$

The curve crosses the x-axis at $x = 1$ and $x = -1$ and the line $x = -2$ is an asymptote of the curve.

(a) Use algebra to solve the equation $\frac{x^2-1}{|x+2|} = 3(1-x)$.

(6)

(b) Hence, or otherwise, find the set of values of x for which

$$\frac{x^2-1}{|x+2|} < 3(1-x).$$

(3)
(Total 9 marks)

a) $(x^2-1) = 3|(x+2)|(1-x)$

$\Rightarrow x^2 - 1 = 3(x+2)(1-x)$

$\Rightarrow x^2 - 1 = -3x^2 - 3x + 6$

$\Rightarrow 4x^2 + 3x - 7 = 0$

$\Rightarrow (4x+7)(x-1) = 0$

$x = -\frac{7}{4} \quad x = 1$

$\Rightarrow x^2 - 1 = -3(x+2)(1-x)$

$x^2 - 1 = +3x^2 + 3x - 6$

$0 = 2x^2 + 3x - 5$

$0 = (2x+5)(x-1)$

$x = -\frac{5}{2} \quad x = 1$

$x < -\frac{5}{2}$ or $-\frac{7}{4} < x < 1$

3. A scientist is modelling the amount of a chemical in the human bloodstream. The amount x of the chemical, measured in mg l^{-1}, at time t hours satisfies the differential equation

$$2x\frac{d^2x}{dt^2} - 6\left(\frac{dx}{dt}\right)^2 = x^2 - 3x^4, \quad x > 0.$$

(a) Show that the substitution $y = \frac{1}{x^2}$ transforms this differential equation into

$$\frac{d^2y}{dt^2} + y = 3. \qquad \boxed{I}$$

(5)

(b) Find the general solution of differential equation $\boxed{I}$.

(4)

Given that at time $t = 0$, $x = \frac{1}{2}$ and $\frac{dx}{dt} = 0$,

(c) find an expression for x in terms of t,

(4)

(d) write down the maximum value of x as t varies.

(1)

(Total 14 marks)

$x^2y = 1$

$\frac{d}{dt}(x^2y) = 0$

$\left[\frac{d}{dt}(x^2)\right]y + x^2\left(\frac{d}{dt}y\right) = 0$

$2x\frac{dx}{dt}y + x^2\frac{dy}{dt} = 0$

$2xy\frac{dx}{dt} + x^2\frac{dy}{dt} = 0$

$\Rightarrow \frac{dx}{dt} = -\frac{x}{2y}\frac{dy}{dt}$

$\Rightarrow \frac{dx}{dt} = -\frac{x^3}{2}\frac{dy}{dt}$

$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(-\frac{x^3}{2}\frac{dy}{dt}\right) = -\frac{x^3}{2}\frac{d^2y}{dt^2} - \frac{3x^2}{2}\frac{dx}{dt}\frac{dy}{dt}$

$\therefore \frac{d^2x}{dt^2} = -\frac{x^3}{2}\frac{d^2y}{dt^2} + \frac{3x^5}{4}\left(\frac{dy}{dt}\right)^2$

$2x\frac{d^2x}{dt^2} - 6\left(\frac{dx}{dt}\right)^2 = x^2 - 3x^4$

$2x\left(-\frac{x^3}{2}\frac{d^2y}{dt^2} + \frac{3x^5}{4}\left(\frac{dy}{dt}\right)^2\right) - 6\left(-\frac{x^3}{2}\frac{dy}{dt}\right)^2 = x^2 - 3x^4$

$-x^4\frac{d^2y}{dt^2} + \cancel{\frac{3x^6}{2}\left(\frac{dy}{dt}\right)^2} - \cancel{\frac{3x^6}{2}\left(\frac{dy}{dt}\right)^2} = x^2 - 3x^4$

$\div -x^2$ $\quad x^2\frac{dy}{dt^2} = 3x^2 - 1 \quad \Rightarrow \quad \frac{1}{y}\frac{dy}{dt^2} = \frac{3}{y} - 1$

$\times y$ $\quad \frac{dy}{dt} = 3 - y \quad \therefore \frac{dy}{dt} + y = 3$ ✱ $\quad$ hmmm!

alt $\quad y = x^{-2} \quad \frac{dy}{dt} = -2x^{-3}\frac{dx}{dt} \quad \Rightarrow \frac{dx}{dt} = -\frac{x^3}{2}\frac{dy}{dt}$

$\frac{d^2x}{dt^2} = -\frac{3x^2}{2}\frac{dx}{dt}\frac{dy}{dt} - \frac{x^3}{2}\frac{d^2y}{dt^2} \Rightarrow \frac{d^2x}{dt^2} = \frac{3x^5}{4}\left(\frac{dy}{dt}\right)^2 - \frac{x^3}{2}\frac{d^2y}{dt^2}$

then sub in as previously.

$$\frac{d^2y}{dt^2}+y=3 \qquad \begin{aligned} y&=Ae^{mt} \\ y'&=Ame^{mt} \\ y''&=Am^2e^{mt}\end{aligned} \qquad \begin{aligned} &y''+y=0 \\ &\underset{\neq 0}{Ae^{mt}}\underset{=0}{(m^2+1)}=0 \Rightarrow m=\pm i \end{aligned} \qquad \therefore y_{CF}=A\cos t+B\sin t$$

$$\begin{aligned} y&=\lambda \\ y'&=0 \\ y''&=0 \end{aligned} \qquad \begin{aligned} y''+y&=3 \\ \Rightarrow \quad \lambda&=3 \end{aligned} \qquad \therefore y_{PI}=3$$

$$\therefore y=A\cos t+B\sin t+3$$

c) $\frac{1}{x^2}=A\cos t+B\sin t+3$

$$\Rightarrow x=\sqrt{\frac{1}{A\cos t+B\sin t+3}} \qquad x=\tfrac{1}{2}, t=0 \quad \tfrac{1}{2}=\sqrt{\frac{1}{A+3}} \quad \therefore A=1$$

$$\therefore y=\cos t+B\sin t+3$$

$$\frac{dy}{dt}=-\sin t+B\cos t \quad \Rightarrow \frac{-2}{x^3}\frac{dx}{dt}=-\sin t+B\cos t$$

$$x=\tfrac{1}{2}, t=0, \frac{dx}{dt}=0 \qquad \Rightarrow -16\times 0=B \quad \therefore B=0$$

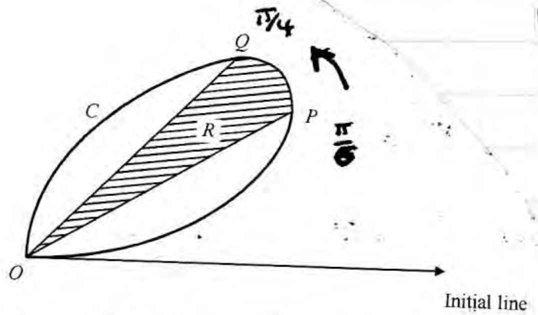
$$\therefore x=\sqrt{\frac{1}{\cos t+3}} \qquad \max_x \Rightarrow \frac{dx}{dt}=0 \to \frac{dy}{dt}=0 \Rightarrow \sin t=0$$

$$\Rightarrow t=0, \pi, \ldots$$

$\therefore$ max when $\cos t=-1$ (at $t=\pi$)

$$\Rightarrow x=\sqrt{\frac{1}{2}} \qquad \therefore \max x=\frac{\sqrt{2}}{2}$$

4.



tangent perp to Initial line $\Rightarrow \frac{dx}{d\theta} = 0$

$x = r\cos\theta \quad x = 4\sin\theta\cos^3\theta$

$\frac{dx}{d\theta} = 4\cos^4\theta - 12\sin^2\theta\cos^2\theta$

$12(1-\cos^2\theta)\cos^2\theta = 4\cos^4\theta$

$12\cos^2\theta = 16\cos^4\theta$

$4\cos^2\theta(4\cos^2\theta - 3) = 0$

$\cos\theta = 0 \qquad \cos\theta = \pm\frac{\sqrt{3}}{2}$

$\theta = \cancel{\frac{\pi}{2}}, \cancel{\frac{3\pi}{2}} \qquad \theta = \frac{\pi}{6}, \cancel{\frac{5\pi}{6}}$

The diagram above shows a sketch of the curve C with polar equation

$$r = 4\sin\theta\cos^2\theta, \qquad 0 \le \theta < \frac{\pi}{2}.$$

The tangent to C at the point P is perpendicular to the initial line.

(a) Show that P has polar coordinates $\left(\frac{3}{2}, \frac{\pi}{6}\right)$.

(6)

$r = 4\sin\theta\cos^2\theta \quad \therefore \cos\theta = \frac{\sqrt{3}}{2}$

$r = 4\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^2 \quad \therefore \sin\theta = \frac{1}{2}$

$r = \frac{3}{2} \quad \therefore P\left(\frac{3}{2}, \frac{\pi}{6}\right)$

The point Q on C has polar coordinates $\left(\sqrt{2}, \frac{\pi}{4}\right)$.

The shaded region R is bounded by OP, OQ and C, as shown in the diagram above.

(b) Show that the area of R is given by

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\sin^2 2\theta\cos 2\theta + \frac{1}{2} - \frac{1}{2}\cos 4\theta\right) d\theta$$

(3)

(c) Hence, or otherwise, find the area of R, giving your answer in the form $a + b\pi$, where a and b are rational numbers.

(5)

(Total 14 marks)

$R = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 16\sin^2\theta\cos^4\theta \, d\theta = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (2\sin\theta\cos\theta)^2\left(\frac{1}{2}\cos 2\theta + \frac{1}{2}\right) d\theta$

$R = \int_{\pi/6}^{\pi/4} \sin^2 2\theta(\cos 2\theta + 1)\, d\theta = \int_{\pi/6}^{\pi/4} \sin^2 2\theta\cos 2\theta + \sin^2 2\theta \, d\theta$

$R = \int_{\pi/6}^{\pi/4} \sin^2 2\theta\cos 2\theta + \frac{1}{2} - \frac{1}{2}\cos 4\theta \, d\theta$

$\therefore R = \left[\frac{1}{6}\sin^3 2\theta + \frac{1}{2}\theta - \frac{1}{8}\sin 4\theta\right]_{\pi/6}^{\pi/4} = \left[\left(\frac{1}{6} + \frac{\pi}{8}\right) - \left(\frac{1}{6}\left(\frac{\sqrt{3}}{2}\right)^3 + \frac{\pi}{12} - \frac{\sqrt{3}}{16}\right)\right] = \frac{1}{6} + \frac{\pi}{24}$

5. Find the set of values of x for which

$$\frac{x+1}{2x-3} < \frac{1}{x-3}$$

(Total 7 marks)

$(x+1)(2x-3)(x-3)^2 < (x-3)(2x-3)^2$

$\Rightarrow (x+1)(2x-3)(x-3)^2 - (x-3)(2x-3)^2 < 0$

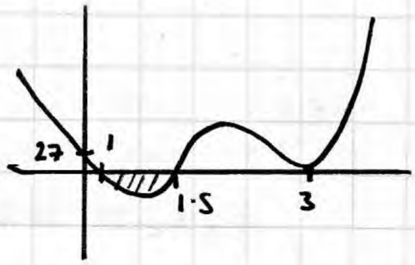
$\Rightarrow (x-3)(2x-3)[(x+1)(x-3)-(2x-3)] < 0$

$\Rightarrow (x-3)(2x-3)[x^2-2x-3-2x+6] < 0$

$\Rightarrow (x-3)(2x-3)[x^2-4x+3] < 0$

$\Rightarrow (x-3)^2(2x-3)(x-1) < 0$

$3,3 \quad 1.5 \quad 1$



$1 < x < 1.5$

6. $$\frac{dy}{dx} - y\tan x = 2\sec^3 x.$$

Given that $y = 3$ at $x = 0$, find y in terms of x

(Total 7 marks)

$\text{IF } f(x) = e^{-\int \tan x\,dx} = e^{\int -\frac{\sin x}{\cos x}dx} = e^{\ln\cos x} = \cos x$

$\cos x\frac{dy}{dx} - y\cos x\tan x = 2\sec^2 x \Rightarrow \frac{d}{dx}(y\cos x) = 2\sec^2 x$

$\Rightarrow y\cos x = 2\int \sec^2 x\,dx = 2\tan x + c \quad \therefore y = \frac{2\tan x + c}{\cos x}$

$(0,3) \quad 3 = \frac{2+c}{1} \quad \therefore c = 1 \quad \Rightarrow y = \frac{2\tan x + 1}{\cos x}$

7. For the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 2x(x+3),$$

find the solution for which at $x = 0$, $\frac{dy}{dx} = 1$ and $y = 1$.

(Total 12 marks)

$y = Ae^{mx}$ $\quad y'' + 3y' + 2y = 0$ $\quad y_{cf} = Ae^{-x} + Be^{-2x}$

$y' = Ame^{mx}$ $\quad Ae^{mx}(m^2+3m+2) = 0$

$y'' = Am^2e^{mx}$ $\quad \neq 0 \quad = 0 \quad (m+2)(m+1) = 0$

$-2 \quad -1$

$y = a + bx + cx^2$ $\quad y'' = 2c$

$y' = b + 2cx$ $\quad + 3y' = 3b + 6cx$ $\quad y_{PI} = x^2 - 1$

$y'' = 2c$ $\quad + 2y = 2a + 2bx + 2cx^2$

$2x^2 + 6x = (2a+3b+2c) + (2b+6c)x + 2cx^2$

$\therefore c = 1 \quad 2b + 6 = 6 \therefore b = 0 \quad 2a + 2 = 0 \therefore a = -1$

$y' = -Ae^{-x} - 2Be^{-2x} + 2x$ $\quad \therefore y = Ae^{-x} + Be^{-2x} + x^2 - 1$

$x=0\ y'=1 \quad 1 = -A - 2B \quad \Rightarrow \quad A + 2B = -1$ $\quad x=0, y=1 \quad 1 = A + B - 1 \therefore A + B = 2$

$A + B = 2 \; -$

$B = -3 \therefore A = 5$

$\therefore y = 5e^{-x} - 3e^{-2x} + x^2 - 1$

8. (a) Sketch the curve C with polar equation

$$r = 5 + \sqrt{3}\cos\theta, \qquad 0 \le \theta \le 2\pi.$$

(2)

(b) Find the polar coordinates of the points where the tangents to C are parallel to the initial line $\theta = 0$. Give your answers to 3 significant figures where appropriate.

(6)

(c) Using integration, find the area enclosed by the curve C, giving your answer in terms of π.

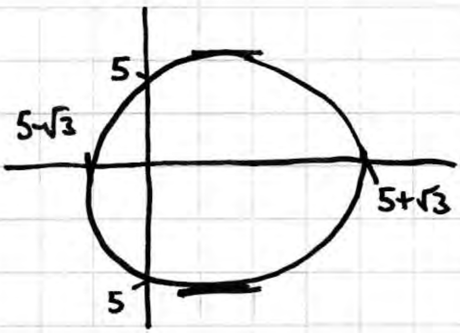
(6)

(Total 14 marks)

$\theta = 0 \quad r_{max} = 5+\sqrt{3} \qquad \theta = \pi \quad r_{min} = 5-\sqrt{3}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad r = 05$

Parallel to initial line $\frac{dy}{d\theta} = 0$



$y = r\sin\theta = 5\sin\theta + \sqrt{3}\sin\theta\cos\theta$

$\frac{dy}{d\theta} = 5\cos\theta + \sqrt{3}\cos^2\theta + -\sqrt{3}\sin^2\theta$

$5\cos\theta + \sqrt{3}\cos^2\theta - \sqrt{3}(1-\cos^2\theta) = 0$

$2\sqrt{3}\cos^2\theta + 5\cos\theta - \sqrt{3} = 0$

$(2\sqrt{3}\cos\theta - 1)(\cos\theta + \sqrt{3}) = 0$

$\therefore \cos\theta = \frac{1}{2\sqrt{3}} \qquad \cos\theta = -\sqrt{3}$ no solution

$\theta = \underline{1.28^c}, \underline{5.01^c}$

$\cos\theta = \frac{1}{2\sqrt{3}} \Rightarrow r = 5 + \frac{\sqrt{3}}{2\sqrt{3}} \therefore r = 5.5 \qquad (5.5, 1.28); (5.5, 5.01)$

$\text{Area} = 2 \times \frac{1}{2}\int_0^{\pi} (5+\sqrt{3}\cos\theta)^2 \, d\theta = \int_0^{\pi} 25 + 10\sqrt{3}\cos\theta + 3\cos^2\theta \, d\theta$

$= \int_0^{\pi} 25 + 10\sqrt{3}\cos\theta + 3(\tfrac{1}{2}\cos 2\theta + \tfrac{1}{2}) \, d\theta = \int_0^{\pi} \frac{53}{2} + 10\sqrt{3}\cos\theta + \frac{3}{2}\cos 2\theta \, d\theta$

$= \frac{1}{2}\int_0^{\pi} 53 + 20\sqrt{3}\cos\theta + 3\cos 2\theta \, d\theta$

$= \frac{1}{2}\left[53\theta + 20\sqrt{3}\sin\theta + \frac{3}{2}\sin 2\theta\right]_0^{\pi} = \frac{1}{2}[53\pi] = \frac{53}{2}\pi$

10.

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+2y=0.$$

At $x=0$, $y=2$ and $\frac{dy}{dx}=-1$.

(a) Find the value of $\frac{d^3y}{dx^3}$ at $x=0$.

(3)

(b) Express y as a series in ascending powers of x, up to and including the term in x^3.

(4)
(Total 7 marks)

$$\frac{d}{dx}\left[(1-x^2)\frac{d^2y}{dx^2}\right]-\frac{d}{dx}\left(x\frac{dy}{dx}\right)+\frac{d}{dx}(2y)=0 \qquad (1-2x)\frac{d^3y}{dx^2}-2x\frac{d^2y}{dx^2}-x\frac{d^2y}{dx^2}-\frac{dy}{dx}+2\frac{dy}{dx}=0$$

$$\Rightarrow (1-2x)\frac{d^3y}{dx^3}-3x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0 \qquad (1)(y'')-0+2(2)=0 \therefore y_0''=-4$$

$$(1)y'''-0-1=0 \quad \therefore y_0'''=1$$

$$\therefore y=2-x-2x^2+\frac{1}{6}x^3$$

11. (a) Given that $z=\cos\theta+i\sin\theta$, use de Moivre's theorem to show that

$$z^n+\frac{1}{z^n}=2\cos n\theta.$$

(2)

$$z^n=\cos n\theta+i\sin n\theta$$
$$z^{-n}=\cos(-n\theta)+i\sin(-n\theta)$$
$$z^{-n}=\cos(n\theta)-i\sin(n\theta)$$
$$\therefore z^n+\frac{1}{z^n}=\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta$$
$$2\cos n\theta$$

(b) Express $32\cos^6\theta$ in the form $p\cos6\theta+q\cos4\theta+r\cos2\theta+s$, where p, q, r and s are integers.

(5)

(c) Hence find the exact value of

$$\int_0^{\frac{\pi}{3}}\cos^6\theta\,d\theta.$$

(4)
(Total 11 marks)

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20

$$z^6+\frac{1}{z^6}=2\cos6\theta \qquad \left(z+\frac{1}{z}\right)^6=(2\cos\theta)^6=32\cos^6\theta$$

$$\left(z+\frac{1}{z}\right)^6=z^6+6z^5\left(\frac{1}{z}\right)+15z^4\left(\frac{1}{z^2}\right)+20z^3\left(\frac{1}{z^3}\right)+15z^2\left(\frac{1}{z^4}\right)+6z\left(\frac{1}{z^5}\right)+\left(\frac{1}{z^6}\right)$$

$$\left(z+\frac{1}{z}\right)^6=\left(z^6+\frac{1}{z^6}\right)+6\left(z^4+\frac{1}{z^4}\right)+15\left(z^2+\frac{1}{z^2}\right)+20$$

$$\therefore 32\cos^6\theta=2\cos6\theta+12\cos4\theta+30\cos2\theta+20$$

c) $$\int_0^{\frac{\pi}{3}}\cos^6\theta\,d\theta=\frac{1}{16}\int_0^{\frac{\pi}{3}}\cos6\theta+6\cos4\theta+15\cos2\theta+10\,d\theta$$

$$=\frac{1}{16}\left[\frac{1}{6}\sin6\theta+\frac{3}{2}\sin4\theta+\frac{15}{2}\sin2\theta+10\theta\right]_0^{\frac{\pi}{3}}$$

$$=\frac{1}{16}\left(-\frac{3\sqrt{3}}{4}+\frac{15\sqrt{3}}{4}+\frac{10\pi}{3}\right)=\frac{3\sqrt{13}}{16}+\frac{5\pi}{24}$$

12. The transformation T from the z-plane, where $z = x + iy$, to the w-plane, where $w = u + iv$, is given by

$$w = \frac{z + i}{z}, \quad z \neq 0.$$

(a) The transformation T maps the points on the line with equation $y = x$ in the z-plane, other than $(0, 0)$, to points on a line l in the w-plane. Find a cartesian equation of l.

(5)

(b) Show that the image, under T, of the line with equation $x + y + 1 = 0$ in the z-plane is a circle C in the w-plane, where C has cartesian equation

$$u^2 + v^2 - u + v = 0.$$

(7)

(c) On the same Argand diagram, sketch l and C.

(3)

(Total 15 marks)

$wz = z + i \Rightarrow wz - z = i \Rightarrow z(w-1) = i \Rightarrow z = \frac{i}{w-1}$

$z = \left(\frac{i}{(u-1)+iv}\right)\left(\frac{(u-1)-iv}{(u-1)-iv}\right) = \frac{v + i(u-1)}{(u-1)^2+v^2}$

$y = x \Rightarrow \frac{v}{(u-1)^2+v^2} = \frac{u-1}{(u+1)^2+v^2} \Rightarrow v = u-1$

b) $y = x - 1 \Rightarrow \frac{u-1}{(u+1)^2+v^2} = \frac{v}{(u-1)^2+v^2} - \frac{1((u-1)^2+v^2)}{(u-1)^2+v^2}$

$\Rightarrow (u-1) = -v - (u-1)^2 - v^2$

$\Rightarrow u - 1 = -v - u^2 + 2u - 1 - v^2$

$\Rightarrow u^2 - u + v^2 + v = 0$ #

$\Rightarrow \left(u - \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{2}$ $\quad C\left(\frac{1}{2}, -\frac{1}{2}\right)$ $\quad r = \frac{\sqrt{2}}{2}$

$u = 0, v = 0$

$u = 0 \quad \left(v + \frac{1}{2}\right)^2 = \frac{1}{4}$

$v + \frac{1}{2} = \pm\frac{1}{2} \Rightarrow v = 0, v = 1$