1. Given that $3x \sin 2x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 4y = k \cos 2x, \qquad \text{where } k \text{ is a constant,}$$

(a) calculate the value of k, (4)

(b) find the particular solution of the differential equation for which at $x = 0$, $y = 2$, and for which at $x = \frac{\pi}{4}$, $y = \frac{\pi}{2}$.

(4)(Total 8 marks)

$y = 3x\sin 2x$
$y' = 3\sin 2x + 6x\cos 2x$
$y'' = 6\cos 2x + 6\cos 2x + -12x\sin 2x$
$y'' = 12\cos 2x - 12x\sin 2x$

$y'' = -12x\sin 2x + 12\cos 2x$
$+4y \quad 12x\sin 2x$
$k\cos 2x \equiv 12\cos 2x \quad \therefore k = 12$

$y_{PI} = 3x\sin 2x \Rightarrow \underline{k = 12}$

$y = Ae^{mx}$
$y' = Ame^{mx}$
$y'' = Am^2e^{mx}$

$y'' = Am^2e^{mx}$
$+4y \quad 4e^{mx}$
$0 \quad Ae^{mx}(m^2+4)$

$\Rightarrow m = \pm 2i \Rightarrow y_{CF} = A\cos 2x + B\sin 2x$

$\therefore y = A\cos 2x + (B+3x)\sin 2x$

$(0,2) \Rightarrow 2 = A \quad y = 2\cos 2x + (B+3x)\sin 2x$
$y' = -4\sin 2x + 3\sin 2x + 2(B+3x)\cos 2x$ [not required)

$(\frac{\pi}{4}, \frac{\pi}{2}) \Rightarrow \frac{\pi}{2} = (0) + (B + \frac{3\pi}{4})(1) \quad \therefore B = -\frac{\pi}{4}.$

$\therefore y = 2\cos 2x + (3x - \frac{\pi}{4})\sin 2x$

2

2. Given that for all real values of r, $(2r + 1)^3 - (2r - 1)^3 = Ar^2 + B$,

where A and B are constants,

(a) find the value of A and the value of B. (2)

(b) Hence, or otherwise, prove that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$. (5)

(c) Calculate $\sum_{r=1}^{40} (3r-1)^2$. (3)(Total 10 marks)

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
$(2r+1)^3 = (2r)^3 + 3(2r)^2 + 3(2r) + 1$
$(2r+1)^3 = 8r^3 + 12r^2 + 6r + 1$
$-(2r-1)^3 = 8r^3 - 12r^2 + 6r - 1$
$24r^2 + 2 \quad \underline{A = 24\ B = 2}$

$\sum_{r=1}^{n} 24r^2 + 2 = 24\sum_{1}^{n} r^2 + 2n$

$\overset{r=1}{(\cancel{3^3} - 1^3)} + \overset{r=2}{(\cancel{5^3} - \cancel{3^3})} + \overset{r=3}{(\cancel{7^3} - \cancel{5^3})} + \dots + \overset{r=n}{\left((2n+1)^3 - \cancel{(2n-1)^3}\right)}$

$\Rightarrow 24\sum r^2 + 2n = (2n+1)^3 - 1 = 8n^3 + 12n^2 + 6n + 1 - 1$
$\Rightarrow 24\sum r^2 = 8n^3 + 12n^2 + 4n = 4n(2n^2 + 3n + 1)$
$\therefore \sum r^2 = \frac{1}{6}n(2n^2 + 3n + 1) \Rightarrow \sum_{1}^{n} r^2 = \frac{1}{6}n(2n+1)(n+1)$

c) $\sum_{r=1}^{40} (3r-1)^2 = \sum_{1}^{40} 9r^2 - 6\sum_{1}^{40} r + \sum_{1}^{40} 1 = \frac{3}{2}(40)(81)(41) - 6\left(\frac{40 \times 41}{2}\right) + 40$

$= 194380$

2

3. (a) Use algebra to find the exact solutions of the equation

$$|2x^2 + x - 6| = 6 - 3x.$$

(6)

(b) On the same diagram, sketch the curve with equation $y = |2x^2 + x - 6|$ and the line with equation $y = 6 - 3x$.

(3)

(c) Find the set of values of x for which

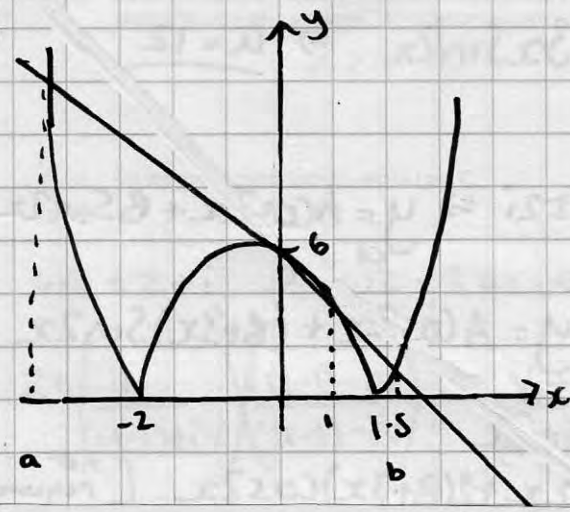
$$|2x^2 + x - 6| > 6 - 3x.$$

(3)(Total 12 marks)

$|(2x-3)(x+2)|$

1.5   −2

b)



$2x^2+x-6=6-3x$

$2x^2+4x-12=0$

$x^2+2x-6=0$

$(x+1)^2=7$

$x=-1\pm\sqrt{7}$

$a=-1-\sqrt{7}$

$b=-1+\sqrt{7}$

$2x^2+x-6=3x-6$

$2x^2-2x=0$

$2x(x-1)=0$

$x=0 \quad x=1$

c) $x<-1-\sqrt{7}$

or $x>-1+\sqrt{7}$

or $0<x<1$

4. During an industrial process, the mass of salt, S kg, dissolved in a liquid t minutes after the process begins is modelled by the differential equation

$$\frac{dS}{dt} + \frac{2S}{120-t} = \frac{1}{4}, \qquad 0 \le t < 120.$$

Given that $S = 6$ when $t = 0$,

(a) find S in terms of t,

(8)

(b) calculate the maximum mass of salt that the model predicts will be dissolved in the liquid at any one time during the process.

(4)(Total 12 marks)

If $f(x) = e^{-2\int \frac{-1}{120-t}dt}$

$=\left[e^{\ln(120-t)}\right]^{-2}$

$=\frac{1}{(120-t)^2}$

$\frac{1}{(120-t)^2}\frac{dS}{dt} + \frac{2S}{(120-t)^3} = \frac{1}{4(120-t)^2} \Rightarrow \frac{d}{dt}\left[\frac{S}{(120-t)^2}\right] = \frac{1}{4(120-t)^2}$

$\Rightarrow \frac{S}{(120-t)^2} = \frac{1}{4}\int (120-t)^{-2}dt = \frac{1}{4}(120-t)^{-1} + c$

$\therefore S = \frac{1}{4}(120-t) + c(120-t)^2$

$S=6, t=0 \quad 6 = 30 + c120^2 \Rightarrow c = \frac{-1}{600}$

$\therefore S = 30 - \frac{1}{4}t - \frac{(120-t)^2}{600}$

b) Max when $\frac{dS}{dt}=0 \Rightarrow -\frac{1}{4} - \frac{(120-t)}{300} = 0 \Rightarrow 120-t=75$

$t=45$

$\therefore \text{Max } S = \frac{1}{4}(120-45) + \frac{(120-45)^2}{-600}$

Max $S = 9.38$

5. (a) Find the Taylor expansion of $\cos 2x$ in ascending powers of $\left(x-\frac{\pi}{4}\right)$ up to and including the term in $\left(x-\frac{\pi}{4}\right)^5$.

(5)

(b) Use your answer to (a) to obtain an estimate of $\cos 2$, giving your answer to 6 decimal places.

(3)(Total 8 marks)

$y = \cos 2x$ $\quad f(\frac{\pi}{4}) = 0$

$y' = -2\sin 2x$ $\quad f'(\frac{\pi}{4}) = -2$

$y'' = -4\cos 2x$ $\quad f''(\pi/4) = 0$

$y''' = 8\sin 2x$ $\quad f'''(\pi/4) = 8$

$y'''' = 16\cos 2x$ $\quad f''''(\pi/4) = 0$

$y''''' = -32\sin 2x$ $\quad f'''''(\pi/4) = -32$

$\therefore f(x) = -2(x-\frac{\pi}{4}) + \frac{4}{3}(x-\frac{\pi}{4})^3 - \frac{4}{15}(x-\frac{\pi}{4})^5$

$f(1) \simeq \cos 2$

$f(1) \simeq -0.416147$

6. (a) Use de Moivre's theorem to show that $\quad$ **$\sin 5\theta = \sin\theta(16\cos^4\theta - 12\cos^2\theta + 1)$.**

(5)

(b) Hence, or otherwise, solve, for $0 \le \theta < \pi$

$$\sin 5\theta + \cos\theta \sin 2\theta = 0.$$

(6)(Total 11 marks)

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

equating Imaginary Parts $\Rightarrow \sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$\Rightarrow \sin 5\theta = \sin\theta[5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-2\cos^2\theta+\cos^4\theta)]$

$\Rightarrow \sin 5\theta = \sin\theta[5\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta]$

$\therefore \sin 5\theta = \sin\theta[16\cos^4\theta - 12\cos^2\theta + 1]$ ✱

b) $\sin 5\theta + \cos\theta\sin 2\theta = 0$

$= \sin\theta[16\cos^4\theta - 12\cos^2\theta + 1] + 2\sin\theta\cos^2\theta = 0$

$= \sin\theta[16\cos^4\theta - 10\cos^2\theta + 1] = 0$

$= \sin\theta(8\cos^2\theta - 1)(2\cos^2\theta - 1) = 0$

$\sin\theta = 0$ $\quad \cos\theta = \sqrt{\frac{1}{8}}$ $\quad \cos\theta = -\sqrt{\frac{1}{8}}$ $\quad \cos\theta = \frac{1}{\sqrt{2}}$ $\quad \cos\theta = -\frac{1}{\sqrt{2}}$

$\therefore \theta = 0$ $\quad \theta = 1.209^c$ $\quad \theta = 1.932^c$ $\quad \theta = \frac{\pi}{4}$ $\quad \theta = \frac{3}{4}\pi$

7. $\frac{d^2x}{dt^2}+3\sin x=0$. At $t=0$, $x=0$ and $\frac{dx}{dt}=0.4$

$t=0$
$x=0$
$x'=0.4$

(b) Find a series solution for x, in ascending powers of t, up to and including the term in t^3. **(4)**

(c) Use your answer to (b) to obtain an estimate of x at $t=0.3$. **(2)(Total 11 marks)**

$$\frac{d^3x}{dt^3}+\frac{d}{dt}(3\sin x)=0 \Rightarrow \frac{d^3x}{dt^3}+(3\cos x)\frac{dx}{dt}=0$$

$$x''+3\sin x=0 \Rightarrow x''+3(0) \Rightarrow x''=0$$

$$x'''+(3\cos(0))0.4=0 \Rightarrow x'''=-1.2$$

$$x=0.4t-0.2t^3 \qquad t=0.3 \quad x\simeq 0.1146$$

8. The point P represents a complex number z on an Argand diagram, where

$$|z-6+3i|=3|z+2-i|.$$

(a) Show that the locus of P is a circle, giving the coordinates of the centre and the radius of this circle. **(7)**

The point Q represents a complex number z on an Argand diagram, where

$$\tan[\arg(z+6)]=\frac{1}{2}.$$

(b) On the same Argand diagram, sketch the locus of P and the locus of Q. **(5)**

(c) On your diagram, shade the region which satisfies both

$$|z-6+3i|>3|z+2-i| \text{ and } \tan[\arg(z+6)]>\frac{1}{2}.$$

(2)(Total 14 marks)

a) $|(x-6)+i(y+3)|=3|(x+2)+i(y-1)|$

$\Rightarrow (x-6)^2+(y+3)^2=9(x+2)^2+9(y-1)^2$

$\Rightarrow x^2-12x+36+y^2+6y+9=9x^2+36x+36+9y^2-18y+9$

$\Rightarrow 8x^2+48x+8y^2-24y=0 \quad (\div 8) \quad x^2+6x+y^2-3y=0$

$(x+3)^2+(y-2)^2=13$ radius $(-3,\frac{3}{2})$ $r=\sqrt{13}$

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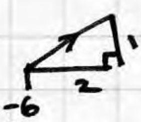
b) $\arg(z+6)=\tan^{-1}(\frac{1}{2}) \Rightarrow$



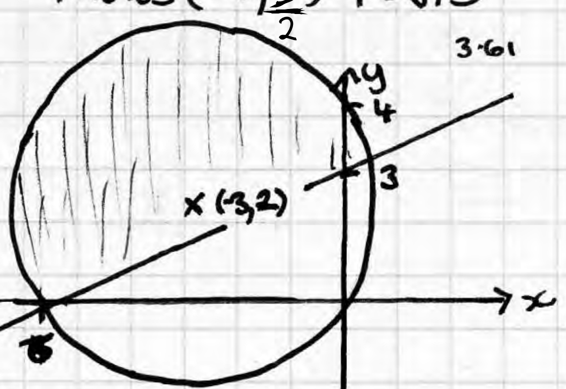
$x=0$ $9+(y-2)^2=13$ $y-2=\pm 2$

$\therefore y=0$ $y=+4$

passes through origin

$y=0$ $(x+3)^2=9$

$x=-3\pm -3$



$\arg(z+6)>\tan^{-1}(\frac{1}{2})$ i.e above angle