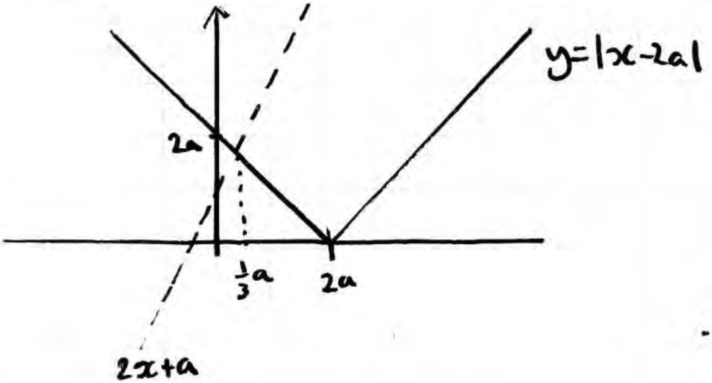
FP2 Paper 5 *adapted 2005

1. (a) Sketch the graph of $y = |x - 2a|$, given that $a > 0$.

(2)

(b) Solve $|x - 2a| > 2x + a$, where $a > 0$.

(3)(Total 5 marks)



2x+a Intersects with reflected part of |x-2a|

$$\therefore -(x-2a) = 2x+a$$
$$3x = a$$
$$x = \tfrac{1}{3}a$$

$$\therefore x < \tfrac{1}{3}a$$

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2. Find the general solution of the differential equation

$$\frac{dy}{dx} + 2y\cot 2x = \sin x, \qquad 0 < x < \frac{\pi}{2},$$

giving your answer in the form $y = f(x)$.

(Total 7 marks)

$$\text{IF } f(x) = e^{2\int \cot 2x}$$
$$= e^{2 \times \frac{1}{2}\ln|\sin 2x|}$$
$$= \sin 2x$$

$$\sin 2x \frac{dy}{dx} + (2\cot 2x \sin 2x)y = \sin 2x \sin x$$

$$\therefore \frac{d}{dx}(y\sin 2x) = 2\sin^2 x \cos x \quad \Rightarrow y\sin 2x = \int 2\sin^2 x \cos x \, dx$$

$$\therefore y\sin 2x = \tfrac{2}{3}\sin^3 x + c \quad \Rightarrow y = \frac{\tfrac{2}{3}\sin^3 x + c}{\sin 2x}$$

7. (a) Show that the transformation $y = xv$ transforms the equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (2+9x^2)y = x^5, \qquad \text{I}$$

into the equation

$$\frac{d^2v}{dx^2} + 9v = x^2. \qquad \text{II}$$

(5)

(b) Solve the differential equation II to find v as a function of x.

(6)

(c) Hence state the general solution of the differential equation I.

(1)

(Total 12 marks)

$y = xv$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\frac{d^2y}{dx^2} = \frac{dv}{dx} + \frac{d}{dx}\left(x\frac{dv}{dx}\right)$

$= \frac{dv}{dx} + \frac{dv}{dx} + x\frac{d^2v}{dx^2}$

$= 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$

$x^2\left(2\frac{dv}{dx} + x\frac{d^2v}{dx^2}\right) - 2x\left(v + x\frac{dv}{dx}\right) + (2+9x^2)xv = x^5$

$= x^3\frac{d^2v}{dx^3} + \cancel{2x^2\frac{dv}{dx}} - \cancel{2x^2\frac{dv}{dx}} - \cancel{2xv} + \cancel{2xv} + 9x^3v = x^5$

$\boxed{\div x^3}$ $\quad \frac{d^2v}{dx^3} + 9v = x^2$

b)

$V = Ae^{Mx}$ $\quad v'' + 9v = 0$ $\quad v = ax^2 + bx + c$ $\quad v'' = 2a$

$v' = Ame^{mx}$ $\quad Ae^{Mx}(m^2+9) = 0$ $\quad v' = 2ax + b$ $\quad +9v = 9c + 9bx + 9ax^2$

$v' = Am^2e^{Mx}$ $\quad \neq 0 \quad = 0 \Rightarrow M = \pm 3i$ $\quad v'' = 2a$ $\quad x^2 = (2a+9c) + 9bx + 9ax^2$

$\therefore V_{CF} = A\cos 3x + B\sin 3x$

$\therefore a = \frac{1}{9} \quad b = 0 \quad 9c = -\frac{2}{9}$

$\therefore V = A\cos 3x + B\sin 3x + \frac{1}{9}x^2 - \frac{2}{81}$ $\quad V_{PI} = \frac{1}{9}x^2 - \frac{2}{81}$ $\quad c = \frac{-2}{81}$

c) $v = \frac{y}{x}$ $\therefore y = Ax\cos 3x + Bx\sin 3x + \frac{1}{9}x^3 - \frac{2}{81}x$

4. The curve C has polar equation $r = 6\cos\theta$, $\quad -\frac{\pi}{2} \le \theta < \frac{\pi}{2}$,

and the line D has polar equation $r = 3\sec\left(\frac{\pi}{3} - \theta\right)$, $\quad -\frac{\pi}{6} < \theta < \frac{5\pi}{6}$.

(a) Find a cartesian equation of C and a cartesian equation of D.

(5)

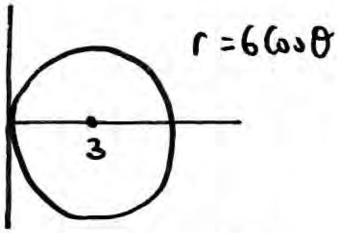
(b) Sketch on the same diagram the graphs of C and D, indicating where each cuts the initial line.

(3)

The graphs of C and D intersect at the points P and Q.

(c) Find the polar coordinates of P and Q.

(5)(Total 13 marks)



$\therefore (x-3)^2 + y^2 = 9$

alt $\quad x = r\cos\theta \Rightarrow x = 6\cos^2\theta \qquad y = r\sin\theta \Rightarrow y = 6\sin\theta\cos\theta$

$\frac{x}{6} = \cos^2\theta \qquad \therefore y^2 = 36\sin^2\theta\cos^2\theta$

$1 - \frac{x}{6} = \sin^2\theta \qquad \Rightarrow y^2 = 36\left(\frac{6-x}{6}\right)\left(\frac{x}{6}\right)$

$\Rightarrow y^2 = (6-x)x \quad \therefore \underline{y^2 = 6x - x^2}$

b) $r\cos\left(\frac{\pi}{3} - \theta\right) = 3 \Rightarrow r\cos\frac{\pi}{3}\cos\theta + r\sin\left(\frac{\pi}{3}\right)\sin\theta = 3$

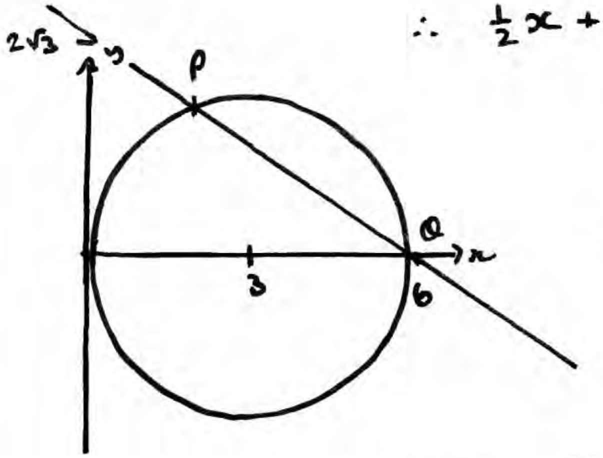
$\Rightarrow \frac{1}{2}r\cos\theta + \frac{\sqrt{3}}{2}r\sin\theta = 3$

$\therefore \frac{1}{2}x + \frac{\sqrt{3}}{2}y = 3 \Rightarrow x + \sqrt{3}y = 6$

$x = 0 \Rightarrow y = \frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$

$y = 0 \quad x = 6$

$\underline{Q(6,0)}$



$6\cos\theta = 3\sec\left(\frac{\pi}{3} - \theta\right)$

$\Rightarrow 6\cos\theta\left[\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right] = 3$

$\Rightarrow \cancel{3}\cos^2\theta + \cancel{3}\sqrt{3}\sin\theta\cos\theta = \cancel{3}$

$\Rightarrow \sqrt{3}\sin\theta = \frac{1 - \cos^2\theta}{\cos\theta} \Rightarrow \sqrt{3}\cancel{\sin}\theta = \frac{\sin^{\cancel{2}}\theta}{\cos\theta}$

$\therefore \tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3} \Rightarrow r = 6\cos\frac{\pi}{3} = 3 \quad \underline{P\left(3, \frac{\pi}{3}\right)}$

5 Find the general solution of the differential equation

$$(x+1)\frac{dy}{dx} + 2y = \frac{1}{x}, \qquad x > 0.$$

giving your answer in the form $y = f(x)$.

(Total 7 marks)

$$\frac{dy}{dx} + \frac{2}{x+1}y = \frac{1}{x(x+1)} \qquad IF = e^{\int \frac{2}{x+1}dx} = e^{2\ln|x+1|} = (x+1)^2$$

$$(x+1)^2\frac{dy}{dx} + 2(x+1)y = \frac{x+1}{x} \qquad \Rightarrow \frac{d}{dx}\left((x+1)^2y\right) = \frac{x+1}{x}$$

$$\Rightarrow (x+1)^2y = \int 1 + \frac{1}{x}\,dx = x + \ln x + c$$

$$\therefore y = \frac{x + \ln x + c}{(x+1)^2}$$

6

(a) On the same diagram, sketch the graphs of $y = |x^2 - 4|$ and $y = |2x - 1|$, showing the coordinates of the points where the graphs meet the axes.

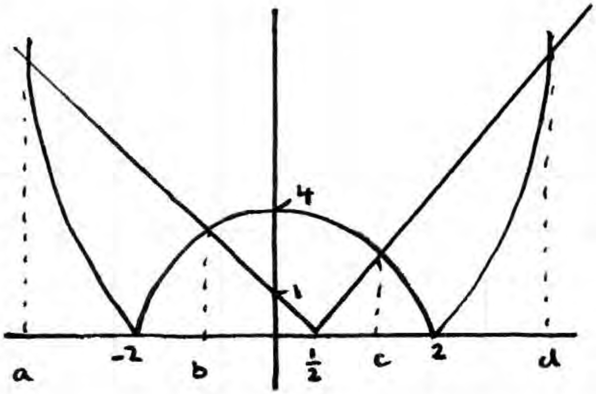
(4)

(b) Solve $|x^2 - 4| = |2x - 1|$, giving your answers in surd form where appropriate.

(5)

(c) Hence, or otherwise, find the set of values of x for which $|x^2 - 4| > |2x - 1|$.

(3)(Total 12 marks)



$x^2 - 4 = 2x - 1$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3 \quad x = -1$

$x^2 - 4 = 1 - 2x$

$x^2 + 2x - 5 = 0$

$(x+1)^2 = 6$

$x = -1 \pm \sqrt{6}$

$a = -1 - \sqrt{6}$

$b = -1$

$c = -1 + \sqrt{6}$

$d = 3$

$x > 3$

or $-1 < x < -1 + \sqrt{6}$

or $x < -1 - \sqrt{6}$

7. (a) Find the general solution of the differential equation

$$2\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 2x = 2t + 9.$$

(6)

$x = Ae^{mt}$ $\quad 2x'' + 5x' + 2x = 0$

$x' = Ame^{mt}$ $\quad Ae^{mt}(2m^2+5m+2) = 0$

$x'' = Am^2e^{mt}$ $\quad \neq 0$

$(2m+1)(m+2) = 0$

$m = -\frac{1}{2} \quad m = -2$

$x_{CF} = Ae^{-\frac{1}{2}t} + Be^{-2t}$

(b) Find the particular solution of this differential equation for which $x = 3$ and $\frac{dx}{dt} = -1$ when $t = 0$.

(4)

The particular solution in part (b) is used to model the motion of a particle P on the x-axis. At time t seconds ($t \geq 0$), P is x metres from the origin O.

(c) Show that the minimum distance between O and P is $\frac{1}{2}(5 + \ln 2)$ m and justify that the distance is a minimum.

(4)(Total 14 marks)

$x = a + bt$ $\quad +2x'' = 0$

$x' = b$ $\quad +5x' = 5b$

$x'' = 0$ $\quad 2x = 2a + 2bt$

$2t + 9 \equiv (2a+5b) + 2bt$

$\therefore b = 1$

$2a + 5 = 9$

$2a = 4$

$a = 2$

$x_{PI} = 2 + t$

$x = Ae^{-\frac{1}{2}t} + Be^{-2t} + t + 2$

$x = 3, t = 0 \Rightarrow 3 = A + B + 2 \Rightarrow A + B = 1$

$x' = -\frac{1}{2}Ae^{-\frac{1}{2}t} - 2Be^{-2t} + 1$

$x' = -1, t = 0 \quad -1 = -\frac{1}{2}A - 2B + 1 \quad \frac{1}{2}A + 2B = 2$

$A + 4B = 4$ −

$A + B = 1$

$3B = 3 \therefore B = 1$

$A = 0$

$\therefore x = e^{-2t} + t + 2$

Min distance $\Rightarrow \frac{dx}{dt} = 0 \Rightarrow -2e^{-2t} + 1 = 0 \Rightarrow e^{-2t} = \frac{1}{2}$

$\Rightarrow -2t = \ln\frac{1}{2} = -\ln 2$

$2t = \ln 2 \Rightarrow t = \frac{1}{2}\ln 2$

$x_{min} = e^{\ln(\frac{1}{2})} + \frac{1}{2}\ln 2 + 2$

$x_{min} = \frac{5}{2} + \frac{1}{2}\ln 2 = \frac{1}{2}(5 + \ln 2)$

8. The curve C which passes through O has polar equation

$$r = 4a(1 + \cos\theta), \quad -\pi < \theta \le \pi.$$

The line l has polar equation

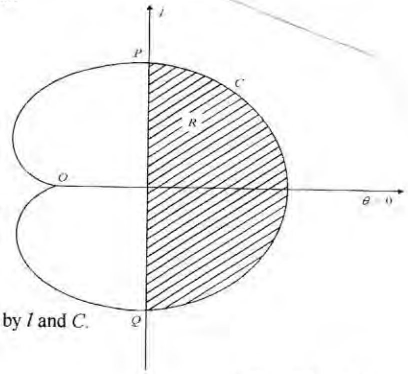
$$r = 3a\sec\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

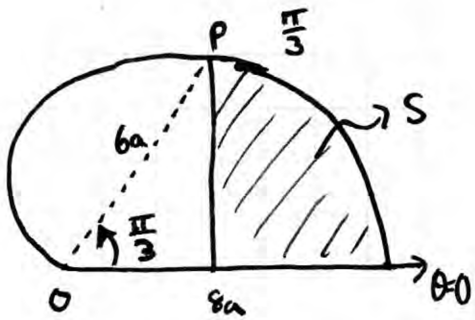
The line l cuts C at the points P and Q, as shown in the diagram.

(a) Prove that $PQ = 6\sqrt{3}a$. (6)

The region R, shown shaded in the diagram, is bounded by l and C.

(b) Use calculus to find the exact area of R.





(7)(Total 13 marks)

$$4\cancel{a}(1+\cos\theta) = 3\cancel{a}\sec\theta$$

$$\Rightarrow 4\cos\theta(1+\cos\theta) = 3 \Rightarrow 4\cos^2\theta + 4\cos\theta - 3 = 0$$

$$(2\cos\theta + 3)(2\cos\theta - 1) = 0$$

$\cos\theta = 1.5$ No Solutions $\cos\theta = \frac{1}{2} \therefore \theta = \frac{\pi}{3}$

$$r = 4a(1+\cos\tfrac{\pi}{3}) \quad \therefore r = 6a$$

$$\therefore PQ = 2 \times 6a\sin\frac{\pi}{3}$$

$$= \cancel{2} \times 6a \times \frac{\sqrt{3}}{\cancel{2}} = \underline{6\sqrt{3}a} \ \#$$

$$\text{area} \quad S = \frac{1}{2}\int_0^{\pi/3} 16a^2(1+\cos\theta)^2\,d\theta \ -$$

$6a\cos\frac{\pi}{3} = 3a$

$$\rightsquigarrow \text{Area} = \frac{1}{2}(6a)(3a)\sin\frac{\pi}{3} = \frac{1}{2}(18a^2)\frac{\sqrt{3}}{2} = \frac{9\sqrt{3}}{2}a^2.$$

$$S = 8a^2\int_0^{\pi/3} 1 + 2\cos\theta + \left(\frac{\cos 2\theta + 1}{2}\right)d\theta - \frac{9\sqrt{3}}{2}a^2$$

$$S = 8a^2\int_0^{\pi/3} \frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\,d\theta - \frac{9\sqrt{3}}{2}a^2$$

$$\Rightarrow S = 4a^2\int_0^{\pi/3} 3 + 4\cos\theta + \cos 2\theta\,d\theta - \frac{9\sqrt{3}}{2}a^2 = 4a^2\left[3\theta + 4\sin\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/3} - \frac{9\sqrt{3}}{2}a^2$$

$$= 4a^2\left[\left(\pi + 2\sqrt{3} + \frac{\sqrt{3}}{4}\right) - (0)\right] - \frac{9\sqrt{3}}{2}a^2 = 4\pi a^2 + 9\sqrt{3}a^2 - \frac{9\sqrt{3}}{2}a^2$$

$$= \left(4\pi + \frac{9\sqrt{3}}{2}\right)a^2$$

(X2)

$$\therefore R = (8\pi + 9\sqrt{3})a^2$$

9. A complex number z is represented by the point P in the Argand diagram. Given that

$$|z - 3i| = 3,$$

(a) sketch the locus of P.

(2)

(b) Find the complex number z which satisfies both $|z - 3i| = 3$ and $\arg(z - 3i) = \frac{3}{4}\pi$.

(4)

The transformation T from the z-plane to the w-plane is given by

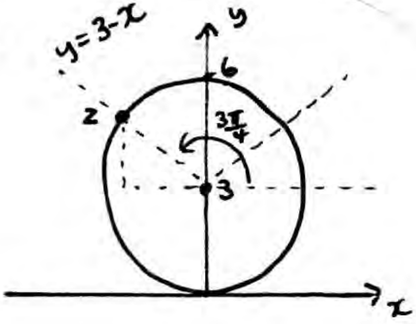
$$w = \frac{2i}{z}.$$

(c) Show that T maps $|z - 3i| = 3$ to a line in the w-plane, and give the cartesian equation of this line.

(5)

(Total 11 marks)

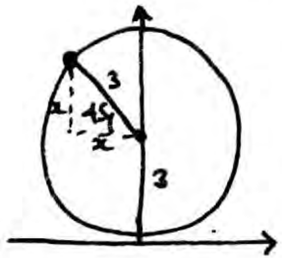
a)



$x^2+(y-3)^2=9$ $\quad y=3-x$ $\quad \Rightarrow x^2+(3-x-3)^2=9$ $\quad \therefore 2x^2=9$

$\Rightarrow x=\pm\frac{3}{\sqrt{2}}=\pm\frac{3\sqrt{2}}{2}$ $\qquad z=-\frac{3\sqrt{2}}{2}+i\left(3+\frac{3\sqrt{2}}{2}\right)$

alt



$\therefore 2x^2=9 \Rightarrow x=\frac{3\sqrt{2}}{2}$

$x=-\frac{3\sqrt{2}}{2}$ $\quad y=3+\frac{3\sqrt{2}}{2}$

c) $z=\frac{2i}{w}$ $\quad |z-3i|=3 \Rightarrow \left|\frac{2i}{w}-3i\right|=3 \Rightarrow \left|\frac{2i-i3w}{w}\right|=3 \Rightarrow \left|\frac{3w-2}{w}\right|=3$

$\Rightarrow (3w-2)^2=9w^2 \Rightarrow \cancel{9w^2}-12w+4=\cancel{9w^2} \Rightarrow 12w=4 \Rightarrow w=\frac{1}{3}$

$u+iv=\frac{1}{3}$ $\therefore$ $u=\frac{1}{3}$ Circle $C(0,3)$ $r=3$ in z-plane maps to line $u=\frac{1}{3}$ in w-plane.

10. (a) Given that $z = e^{i\theta}$, show that

$$z^n - \frac{1}{z^n} = 2i \sin n\theta,$$

where n is a positive integer.

(2)

(b) Show that

$$\sin^5 \theta = \frac{1}{16}(\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta).$$

(5)

(c) Hence solve, in the interval $0 \le \theta < 2\pi$,

$$\sin 5\theta - 5 \sin 3\theta + 6 \sin \theta = 0.$$

(5)(Total 12 marks)

$$z^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i \sin n\theta \qquad = \cos n\theta + i \sin n\theta$$

$$z^{-n} = (e^{i\theta})^{-n} = e^{-in\theta} = \cos(-n\theta) + i \sin(-n\theta) \qquad \cos n\theta - i \sin n\theta \; +$$

$$\therefore z^n + \frac{1}{z^n} \qquad 2\cos n\theta$$

$$b) \left(z - \frac{1}{z}\right)^5 = (2i\sin\theta)^5 = 32i\sin^5\theta$$

$$\cos n\theta + i \sin n\theta \; -$$

$$\cos n\theta - i \sin n\theta$$

$$\therefore z^n - \frac{1}{z^n} = \qquad 2i \sin(n\theta) \; \#$$

$$\left(z - \frac{1}{z}\right)^5 = z^5 - 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) - 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) - \frac{1}{z^5}$$

$$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$$\therefore 32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta \qquad \boxed{\div 32i}$$

$$\Rightarrow \sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta) \; \#$$

$$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$$

$$\Rightarrow 16\sin^5\theta - 4\sin\theta = \sin 5\theta - 5\sin 3\theta + 6\sin\theta = 0$$

$$\Rightarrow 4\sin\theta(4\sin^4\theta - 1) = 0$$

$$\sin\theta = 0 \qquad \sin\theta = \sqrt{\tfrac{1}{2}} = \tfrac{1}{\sqrt{2}} \qquad \sin\theta = -\sqrt{\tfrac{1}{2}} = -\tfrac{1}{\sqrt{2}}$$

$$\theta = 0, \pi \qquad \theta = \frac{\pi}{4}, \frac{3\pi}{4} \qquad \theta = \frac{5\pi}{4}, \frac{7\pi}{4}$$

11. The variable y satisfies the differential equation

$$4(1+x^2)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} = y.$$

At $x = 0$, $y = 1$ and $\frac{dy}{dx} = \frac{1}{2}$.

(a) Find the value of $\frac{d^2y}{dx^2}$ at $x = 0$. **(1)** (c) Find the value of $\frac{d^3y}{dx^3}$ at $x = 0$ **(4)**

(d) Express y as a series, in ascending powers of x, up to and including the term in x^3. **(2)**

(e) Find the value that the series gives for y at $x = 0.1$, giving your answer to 5 decimal places.

(1)(Total 14 marks)

$x_0 = 0 \quad y_0 = 1 \quad y'_0 = \frac{1}{2}$

$4(1+0)y''_0 + \cancel{4(0)(\tfrac{1}{2})} = 1$

$\therefore y''_0 = \frac{1}{4}$

c) $\frac{d}{dx}\left[(4+4x^2)\frac{d^2y}{dx^2}\right] + \frac{d}{dx}\left[4x\left(\frac{dy}{dx}\right)\right] = \frac{dy}{dx}$

$= \quad 8x\frac{d^2y}{dx^2} + (4+4x^2)\frac{d^3y}{dx^3} + 4\frac{dy}{dx} + 4x\frac{d^2y}{dx^2} = \frac{dy}{dx}$

$(4+4x^2)\frac{d^3y}{dx^3} + 12x\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 0$

$\Rightarrow \quad 4y'''_0 + 3(\tfrac{1}{2}) = 0 \Rightarrow 4y'''_0 = -\frac{3}{2} \quad \therefore y'''_0 = -\frac{3}{8}$

$\therefore \quad y = 1 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{16}x^3 \quad ---$

$x = 0.1 \quad y = \underline{1.05119}$