FP2 Paper 4 adapted 2004

1. (a) Show that $(r+1)^3 - (r-1)^3 \equiv Ar^2 + B$, where A and B are constants to be found.

(2)

(b) Prove by the method of differences that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$, $n > 1$.

(6)(Total 8 marks)

$$(r+1)^3 = r^3 + 3r^2 + 3r + 1$$
$$-\ (r-1)^3 = r^3 - 3r^2 + 3r - 1$$
$$\underline{6r^2 + 2}$$

$$\sum 6r^2 + 2 = 6\sum r^2 + 2n$$

$$r=1\ (\cancel{2}^3 - 0^3) + \overset{r=2}{(\cancel{3}^3 - 1^3)} + \overset{r=3}{(\cancel{4}^3 - \cancel{2}^3)} + \ldots + \overset{r=n-2}{((\cancel{n-1})^3 - (\cancel{n-3})^3)} + \overset{r=n-1}{(n^3 - \cancel{(n-2)}^3)} + \overset{r=n}{((n+1)^3 - \cancel{(n-1)}^3)}$$

$$6\sum r^2 + 2n = n^3 + (n+1)^3 - 1 = n^3 + n^3 + 3n^2 + 3n + \cancel{1} - \cancel{1}$$

$$\Rightarrow 6\sum r^2 = 2n^3 + 3n^2 + n = n(2n^2 + 3n + 1) = n(n+1)(2n+1)$$

$$\therefore \sum r^2 = \frac{1}{6}n(n+1)(2n+1)$$

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2.

$$\frac{dy}{dx} + y\left(1 + \frac{3}{x}\right) = \frac{1}{x^2}, \quad x > 0.$$

(a) Verify that x^3e^x is an integrating factor for the differential equation. **(3)**

(b) Find the general solution of the differential equation. **(4)**

(c) Given that $y = 1$ at $x = 1$, find y at $x = 2$. **(3)(Total 10 marks)**

a) $\text{IF } f(x) = e^{\int 1 + \frac{3}{x}} = e^{x + 3\ln x} = e^x \times (e^{\ln x})^3 = e^x \times x^3 = x^3 e^x$

b) $x^3e^x \frac{dy}{dx} + x^3e^x\left(1 + \frac{3}{x}\right)y = xe^x$

$\Rightarrow \frac{d}{dx}(x^3e^x y) = xe^x \Rightarrow x^3e^x y = \int xe^x \, dx$ $\quad u = x \quad v = e^x$ $\quad u' = 1 \quad v' = e^x$

$\Rightarrow x^3e^x y = xe^x - \int e^x dx = xe^x - e^x + c$

$\therefore y = \frac{1}{x^2} - \frac{1}{x^3} + \frac{c}{x^3}e^{-x}$

$(1,1) \quad 1 = 1 - 1 + ce^{-1} \Rightarrow 1 = \frac{c}{e} \quad \therefore c = e^1$

$y = \frac{x-1}{x^3} + \frac{e^{1-x}}{x^3} = \frac{x - 1 + e^{1-x}}{x^3}$

$x = 2 \quad y = \frac{1 + e^{-1}}{8}$

3. (a) Sketch, on the same axes, the graph of $y = |(x-2)(x-4)|$, and the line with equation $y = 6 - 2x$.

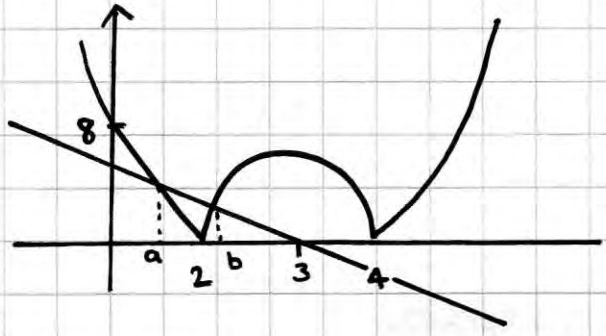
(4)

(b) Find the exact values of x for which $|(x-2)(x-4)| = 6 - 2x$.

(5)

(c) Hence solve the inequality $|(x-2)(x-4)| < 6 - 2x$.

(2)(Total 11 marks)



$x^2 - 6x + 8 = 6 - 2x$

$x^2 - 4x + 2 = 0$

$(x-2)^2 = 2$

$x = 2 \pm \sqrt{2}$

$\cancel{3.4}$ 0.6

$x^2 - 6x + 8 = 2x - 6$

$x^2 - 8x + 14 = 0$

$(x-4)^2 = 2$

$x = 4 \pm \sqrt{2}$

$\cancel{5.4}$ 2.6

$x = 2 - \sqrt{2}, \ 4 - \sqrt{2}$

$2 - \sqrt{2} < x < 4 - \sqrt{2}$

4.

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 65 \sin 2x,\ x > 0.$$

(a) Find the general solution of the differential equation. **(9)**

(b) Show that for large values of x this general solution may be approximated by a sine function and find this sine function. **(3)(Total 12 marks)**

$y = Ae^{mx}$ $\quad y'' + 4y' + 5y' = 0$ $\quad y_{cf} = Pe^{(-2+i)x} + Qe^{(-2-i)x}$

$y' = Ame^{mx}$ $\quad Ae^{mx}(m^2+4m+5) = 0$ $\quad y_{cf} = e^{-2x}(A\cos x + B\sin x)$

$y'' = Am^2e^{mx}$ $\quad \neq 0 \quad (m+2)^2 = -1$

$m = -2 \pm i$

$y = a\sin 2x + b\cos 2x$ $\qquad 5y = 5a\sin 2x + 5b\cos 2x$

$y' = 2a\cos 2x - 2b\sin 2x$ $\qquad + 4y' = -8b\sin 2x + 8a\cos 2x$

$y'' = -4a\sin 2x - 4b\cos 2x$ $\qquad + y'' = -4a\sin 2x - 4b\cos 2x$

$65\sin 2x \equiv (a-8b)\sin 2x + (b+8a)\cos 2x$

$a - 8b = 65 \quad b + 8a = 0$

$b = -8a \Rightarrow 8b = -64a$

$\therefore 65a = 65$

$a = 1 \quad b = -8$ $\qquad \therefore y_{PI} = \sin 2x - 8\cos 2x$

$\therefore y = \sin 2x - 8\cos 2x + e^{-2x}(A\cos x + B\sin x)$

b) If x is large e^{-2x} will be close to zero

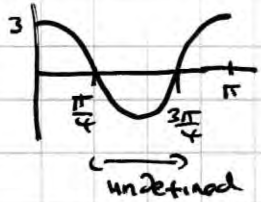
$R\sin(2x - \alpha) = R\sin 2x\cos\alpha - R\cos 2x\sin\alpha$

$1\sin 2x - 8\cos 2x$

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{8}{1}$ $\quad \tan\alpha = 8$ $\quad \alpha = 1.446$ $\quad R = \sqrt{65}$

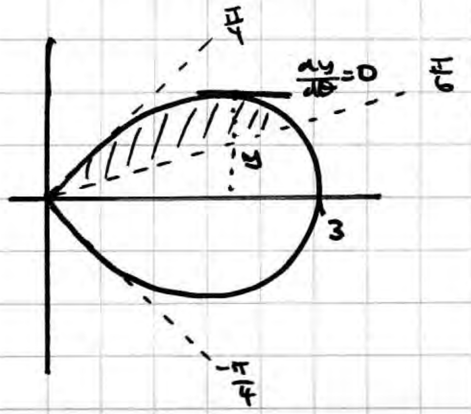
$y \simeq \sqrt{65}\sin(2x - 1.47)$

5. (a) Sketch the curve with polar equation $r = 3\cos 2\theta,\quad -\frac{\pi}{4} \le \theta < \frac{\pi}{4}$ **(2)**



(b) Find the area of the smaller finite region enclosed between the curve and the half-line $\theta = \frac{\pi}{6}$ **(6)**

(c) Find the exact distance between the two tangents which are parallel to the initial line. **(8)(Total 16 marks)**



$$\text{Area} = \frac{1}{2}\int_{\frac{\pi}{6}}^{\pi/4} 9\cos^2 2\theta\, d\theta = \frac{9}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \frac{9}{4}\int_{\frac{\pi}{6}}^{\pi/4} 1 + \cos 4\theta\, d\theta = \frac{9}{4}\left[\theta + \frac{1}{4}\sin 4\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \frac{9}{4}\left[\left(\frac{\pi}{4}\right) - \left(\frac{\pi}{6} + \frac{\sqrt{3}}{8}\right)\right] = \frac{9}{4}\left(\frac{2\pi}{24} - \frac{3\sqrt{3}}{24}\right) =$$

$$= \frac{3}{32}\left(2\pi - 3\sqrt{3}\right)$$

b) $y = r\sin\theta = 3\cos 2\theta \sin\theta$

$$\frac{dy}{d\theta} = -6\sin 2\theta \sin\theta + 3\cos 2\theta\cos\theta = 0$$

$$\Rightarrow 12\sin^2\theta\cancel{\cos\theta} = 3(1 - 2\sin^2\theta)\cancel{\cos\theta} \Rightarrow 4\sin^2\theta = 1 - 2\sin^2\theta$$

$$\Rightarrow \sin^2\theta = \frac{1}{6} \Rightarrow \sin\theta = \pm\sqrt{\frac{1}{6}} \Rightarrow 0.4205\ldots \quad r = 3(1 - 2\sin^2\theta)$$

$$r = 3\left(1 - 2\left(\tfrac{1}{6}\right)\right) = 2$$

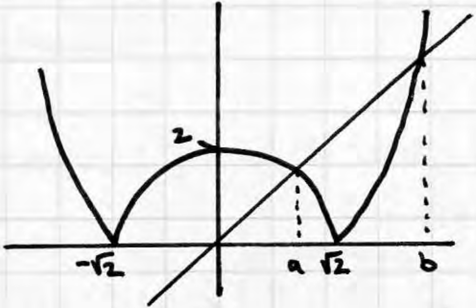
$$\therefore y = r\sin\theta = 2\left(\frac{1}{\sqrt{6}}\right) = \frac{\sqrt{6}}{3}$$

$\therefore$ distance between Pair = $\frac{2\sqrt{6}}{3}$.

6. Find the complete set of values of x for which

$$|x^2 - 2| > 2x.$$

(Total 7 marks)



$x^2 - 2 = 2x$

$x^2 - 2x - 2 = 0$

$(x-1)^2 = 3$

$x = 1 \pm \sqrt{3}$

$b = 1 + \sqrt{3}$

$x^2 - 2 = -2x$

$x^2 + 2x - 2 = 0$

$(x+1)^2 = 3$

$x = -1 \pm \sqrt{3}$

$\therefore a = -1 + \sqrt{3}$

$\therefore \; x < \underline{-1+\sqrt{3}}$ or $x > \underline{1+\sqrt{3}}$

7. (a) Find the general solution of the differential equation

$$\frac{dy}{dx} + 2y = x.$$

(5)

Given that $y = 1$ at $x = 0$,

(b) find the exact values of the coordinates of the minimum point of the particular solution curve,

(4)

(c) draw a sketch of this particular solution curve.

(2)(Total 11 marks)

$\text{IF } f(x) = e^{\int 2\,dx} = e^{2x}$

$e^{2x}\frac{dy}{dx} + 2e^{2x}y = xe^{2x}$

$\therefore \frac{d}{dx}(ye^{2x}) = xe^{2x} \Rightarrow ye^{2x} = \int xe^{2x}\,dx$

$u = x \quad v = \frac{1}{2}e^{2x}$

$u' = 1 \quad v' = e^{2x}$

$\Rightarrow ye^{2x} = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x} + c$

$\therefore ye^{2x} = -\frac{1}{4}e^{2x} + \frac{1}{2}xe^{2x} + c \quad \therefore y = \frac{1}{2}x - \frac{1}{4} + ce^{-2x}$

$(0,1) \quad 1 = 0 - \frac{1}{4} + c \quad \therefore c = \frac{5}{4} \quad \therefore y = \frac{1}{2}x - \frac{1}{4} + \frac{5}{4}e^{-2x}$

b) min Point $\Rightarrow \frac{dy}{dx} = 0 \Rightarrow \frac{1}{2} - \frac{5}{2}e^{-2x} = 0 \Rightarrow 1 = 5e^{-2x} \Rightarrow e^{-2x} = \frac{1}{5}$

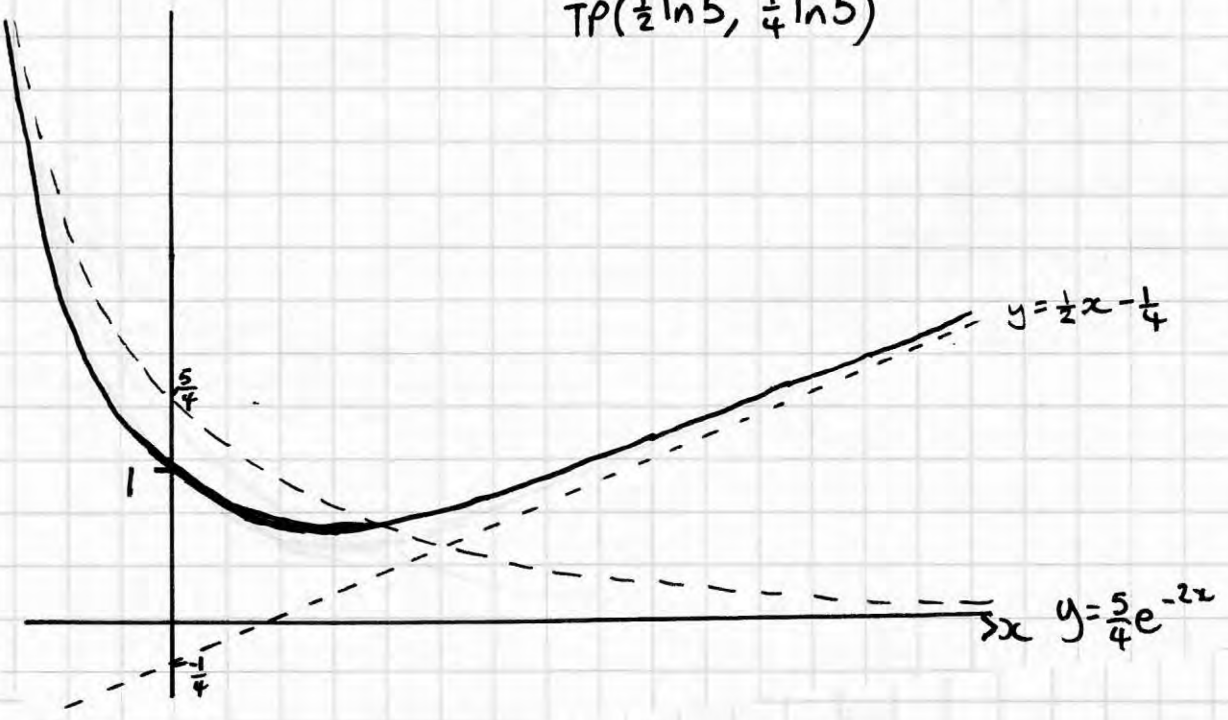
$\Rightarrow -2x = \ln(\frac{1}{5}) \Rightarrow -2x = -\ln 5 \quad \therefore x = \frac{1}{2}\ln 5$

$y = \frac{1}{4}\ln 5 - \frac{1}{4} + \frac{5}{4}e^{\ln(\frac{1}{5})} = \frac{1}{4}\ln 5 - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} \quad (\frac{1}{2}\ln 5, \frac{1}{4}\ln 5)$

c) $y = \frac{5}{4}e^{-2x} + \frac{1}{2}x - \frac{1}{4}$ as $x \to \infty$ $\frac{5}{4}e^{-2x} \to 0 \therefore y \to \frac{1}{2}x - \frac{1}{4}$

as $x \to -\infty$ $\frac{1}{2}x \to 0$ $y \to \frac{5}{4}e^{-2x} - \frac{1}{4}$

$TP(\frac{1}{2}\ln 5, \frac{1}{4}\ln 5)$



8. (a) Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 2e^{-t}.$$

(6)

(b) Find the particular solution that satisfies $y = 1$ and $\frac{dy}{dt} = 1$ at $t = 0$.

(6)(Total 12 marks)

$y = Ae^{mt}$
$y' = Ame^{mt}$
$y'' = Am^2e^{mt}$

$y'' + 2y' + 2y = 0$
$Ae^{mt}(m^2 + 2y + 2) = 0$
$\neq 0 \quad = 0$

$(m+1)^2 = -1 \Rightarrow m = -1 \pm i$

$y_{cf} = Pe^{(-1+i)t} + Qe^{(-1-i)t}$

$\therefore y_{cf} = e^{-t}(A\cos t + B\sin t)$

$y = \lambda e^{-t}$
$y' = -\lambda e^{-t}$
$y'' = \lambda e^{-t}$

$\cancel{2y = 2\lambda e^{-t}}$
$\cancel{2y' = -2\lambda e^{-t}}$
$y'' = \lambda e^{-t}$
$2e^{-t} = \lambda e^{-t} \quad \therefore \lambda = 2$

$y_{PI} = 2e^{-t}$

$\therefore$ GS $y = e^{-t}(A\cos t + B\sin t + 2)$

$(t=0, y=1) \quad 1 = A + 2 \Rightarrow A = -1$

$y = e^{-t}(A\cos t + B\sin t + 2)$

$y' = e^{-t}(-A\sin t + B\cos t) - e^{-t}(A\cos t + B\sin t + 2)$

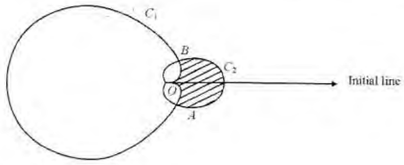
$(t=0, y'=1)$
$1 = B - A - 2 \quad \therefore B = 2$

$\therefore y = e^{-t}(-\cos t + 2\sin t + 2)$

9. The diagram is a sketch of the two curves C_1 and C_2 with polar equations

$C_1 : r = 3a(1 - \cos\theta), -\pi \le \theta < \pi$

$C_2 : r = a(1 + \cos\theta), -\pi \le \theta < \pi.$



The curves meet at the pole O, and at the points A and B.

(a) Find, in terms of a, the polar coordinates of the points A and B. (4)

(b) Show that the length of the line AB is $\frac{3\sqrt{3}}{2}a$. (2)

The region inside C_2 and outside C_1 is shown shaded in the diagram above.

(c) Find, in terms of a, the area of this region. (7)

A badge is designed which has the shape of the shaded region.

Given that the length of the line AB is 4.5 cm,

(d) calculate the area of this badge, giving your answer to three significant figures. (3)

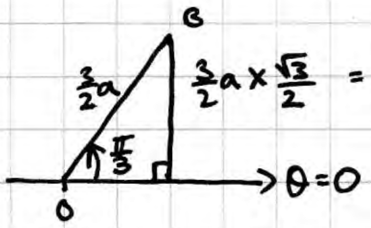
(Total 16 marks)

$A\left(\frac{3}{2}a, -\frac{\pi}{3}\right)$ $B\left(\frac{3}{2}a, \frac{\pi}{3}\right)$

a) $3\cancel{a}(1-\cos\theta) = \cancel{a}(1+\cos\theta) \Rightarrow 3 - 3\cos\theta = 1 + \cos\theta$

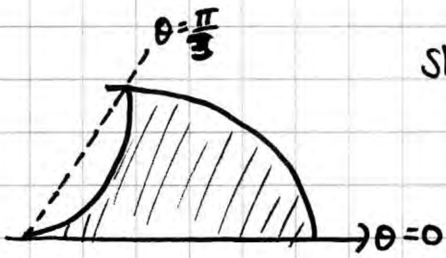
$\Rightarrow 4\cos\theta = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, -\frac{\pi}{3}$ $\quad r = a(1+\frac{1}{2}) = \frac{3}{2}a$

b)



$\frac{3}{2}a \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}a \quad \therefore AB = 2 \times \frac{3\sqrt{3}}{4}a = \frac{3\sqrt{3}}{2}a.$

c)



$$\text{Shaded} = \frac{1}{2}\int_0^{\frac{\pi}{3}} a^2(1+\cos\theta)^2 d\theta - \frac{1}{2}\int_0^{\frac{\pi}{3}} 9a^2(1-\cos\theta)^2 d\theta$$

$$= \frac{1}{2}a^2\int_0^{\frac{\pi}{3}} (1+2\cos\theta+\cos^2\theta) - 9(1-2\cos\theta+\cos^2\theta)\, d\theta$$

$$= \frac{1}{2}a^2\int_0^{\frac{\pi}{3}} -8+20\cos\theta-8\left(\frac{1}{2}+\frac{1}{2}\cos2\theta\right)d\theta$$

$$= \frac{1}{2}a^2\int_0^{\frac{\pi}{3}} -12+20\cos\theta-4\cos2\theta\, d\theta$$

$$= 2a^2\int_0^{\frac{\pi}{3}} -3+5\cos\theta-\cos2\theta\, d\theta$$

$$= 2a^2\left[-3\theta+5\sin\theta-\frac{1}{2}\sin2\theta\right]_0^{\frac{\pi}{3}}$$

$$= 2a^2\left[\left(-\pi+\frac{5\sqrt{3}}{2}-\frac{\sqrt{3}}{4}\right)\right] = 2a^2\left(\frac{9\sqrt{3}}{4}-\pi\right) = \frac{1}{2}a^2(9\sqrt{3}-4\pi)$$

$\therefore$ Area $= a^2(9\sqrt{3}-4\pi)$

d) $\frac{3\sqrt{3}}{2}a = 4.5 \Rightarrow a = \frac{9}{3\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$

$\Rightarrow$ badge $= 3(9\sqrt{3}-4\pi) = 27\sqrt{3}-12\pi \simeq 9.07\text{ cm}^2$

10. Given that $y = \tan x$,

(a) find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$.

(3)

(b) Find the Taylor series expansion of $\tan x$ in ascending powers of $\left(x - \frac{\pi}{4}\right)$ up to and including the term in $\left(x - \frac{\pi}{4}\right)^3$.

(3)

(c) Hence show that $\tan \frac{3\pi}{10} \approx 1 + \frac{\pi}{10} + \frac{\pi^2}{200} + \frac{\pi^3}{3000}$.

(2)

(Total 8 marks)

a)
$y = \tan x$ $\qquad f\left(\frac{\pi}{4}\right) = 1$

$y' = \sec^2 x$ $\qquad f'\left(\frac{\pi}{4}\right) = 2$

$y'' = 2\sec x \times \sec x \tan x = 2\sec^2 \tan x$ $\qquad f''\left(\frac{\pi}{4}\right) = 4$

$y''' = 4\sec^2 x \tan^2 x + 2\sec^4 x$ $\qquad f'''\left(\frac{\pi}{4}\right) = 8 + 2(2)^2 = 16$

b) $\tan x \simeq 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \ldots$

c) $\tan\left(\frac{3\pi}{10}\right) \simeq 1 + 2\left(\frac{6\pi}{20} - \frac{5\pi}{20}\right) + 2\left(\frac{6\pi}{20} - \frac{5\pi}{20}\right)^2 + \frac{8}{3}\left(\frac{6\pi}{20} - \frac{5\pi}{20}\right)^3$

$\simeq 1 + \frac{2\pi}{20} + \frac{2\pi^2}{400} + \frac{8\pi^3}{3(8000)}$

$\simeq 1 + \frac{\pi}{10} + \frac{\pi^2}{200} + \frac{\pi^3}{3000}$

11. (b) Hence find the Maclaurin series expansion of $e^x \cos x$, in ascending powers of x, up to and including the term in x^4.

(3)

(Total 11 marks)

$y = e^x \cos x$

$y' = e^x \cos x - e^x \sin x \quad = e^x(\cos x - \sin x)$

$y'' = e^x(\cos x - \sin x) + e^x(-\sin x - \cos x) = -2e^x \sin x$

$y''' = -2e^x \sin x - 2e^x \cos x \quad = -2e^x(\sin x - \cos x)$

$y'''' = -2e^x(\sin x - \cos x) - 2e^x(\cos x + \sin x) = -4e^x(\sin x - \cos x)$

$f(0) = 1 \quad f'(0) = 1 \quad f''(0) = 0 \quad f'''(0) = 2 \quad f''''(0) = -4$

$\therefore e^x \cos x \simeq 1 + x + \frac{1}{3}x^3 - \frac{1}{6}x^4 \ldots$

12. The transformation T from the complex z-plane to the complex w-plane is given by

$$w = \frac{z+1}{z+i}, \quad z \neq -i.$$

(a) Show that T maps points on the half-line $\arg(z) = \frac{\pi}{4}$ in the z-plane into points on the circle $|w| = 1$ in the w-plane. **(4)**

(b) Find the image under T in the w-plane of the circle [redacted] = 1 in the z-plane. **(6)**

(c) Sketch on separate diagrams the circle [redacted] = 1 in the z-plane and its image under T in the w-plane. **(2)**

(d) Mark on your sketches the point P, where $z = i$, and its image Q under T in the w-plane. **(2)**

(Total 14 marks)

a) $wz + wi = z + 1 \Rightarrow wz - z = 1 - wi \Rightarrow z(w-1) = 1 - wi$

$\therefore z = \frac{1-wi}{w-1} = \frac{1-i(u+iv)}{(u-1)+iv} = \frac{(v+1)-iu}{(u-1)+iv} \times \frac{[(u-1)-iv]}{[(u-1)-iv]}$

$= \frac{(v+1)(u-1)-uv}{(u-1)^2+v^2} + i\left(\frac{u(1-u)-v(v+1)}{(u-1)^2+v^2}\right)$

$= \frac{u-v-1}{(u-1)^2+v^2} + i\frac{(u-u^2-v^2-v)}{(u-1)^2+v^2}$

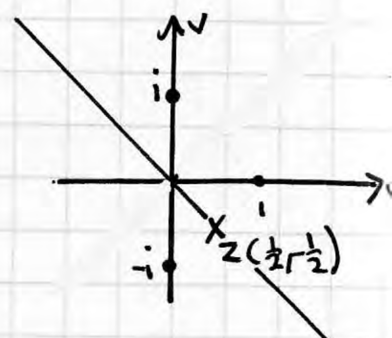
mapping of half line $\arg(z) = \frac{\pi}{4} \Rightarrow x = y$ if $x, y > 0$

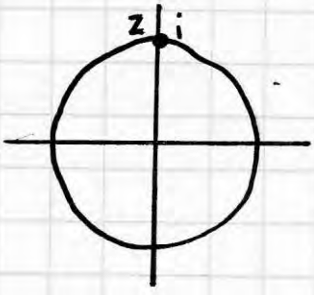
$\Rightarrow \cancel{u} - \cancel{v} - 1 = \cancel{u} - u^2 - v^2 - \cancel{v} \Rightarrow u^2 + v^2 = 1 \therefore$ circle $(0,0)$ $r=1$

in w-plane $\therefore |w| = 1$

b) $\left|\frac{1-wi}{w-1}\right| = 1 \Rightarrow \left|\frac{w+i}{w-1}\right| = 1 \Rightarrow |w+i| = |w-1|$

$\therefore u = -v$





$z = i \quad w = \frac{i+1}{2i} = \frac{1}{2} + \frac{1}{2i} \times \frac{i}{i}$

$= \frac{1}{2} - \frac{1}{2}i$