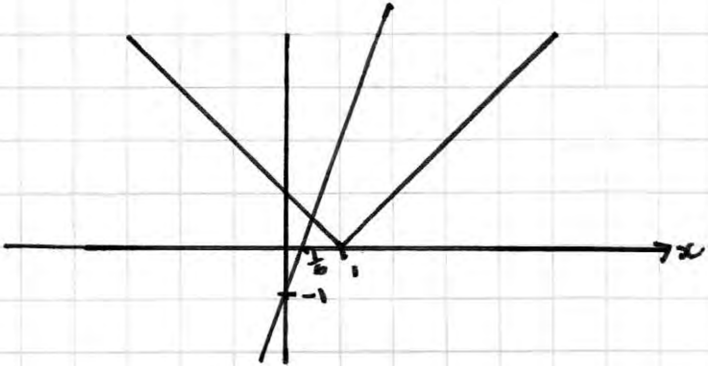
FP2 2002

1. Find the set of values for which

$$|x-1| > 6x-1.$$

(5)



Intersection with reflected part of modulus

$\therefore x-1 = 1-6x$

$7x = 2 \quad x = \frac{2}{7}$

$\therefore x < \frac{2}{7}$

2. (*a*) Find the general solution of the differential equation $t\frac{dv}{dt} - v = t, \quad t > 0$

and hence show that the solution can be written in the form $v = t(\ln t + c)$, where c is an arbitrary cnst. **(6)**

(*b*) This differential equation is used to model the motion of a particle which has speed v m s^{-1} at time t s. When $t = 2$ the speed of the particle is 3 m s^{-1}. Find, to 3 sf, the speed of the particle when $t = 4$. **(4)**

$\frac{dv}{dt} - \frac{1}{t}v = 1 \qquad \text{IF } f(t) = e^{\int -\frac{1}{t}dt} = (e^{\ln t})^{-1} = \frac{1}{t}$

$\frac{1}{t}\frac{dv}{dt} - \frac{1}{t^2}v = \frac{1}{t} \Rightarrow \frac{d}{dt}\left(\frac{v}{t}\right) = \frac{1}{t} \Rightarrow \frac{v}{t} = \int \frac{1}{t}dt \Rightarrow \frac{v}{t} = \ln t + c$

$\therefore v = t\ln t + ct$

$t = 2, v = 3 \qquad 3 = 2\ln 2 + 2c \qquad c = \frac{3}{2} - \ln 2$

$\therefore v = t\ln t - t\ln 2 + \frac{3}{2}t = t\ln\left(\frac{t}{2}\right) + \frac{3}{2}t$

$t = 4 \quad v = 4\ln 2 + 6 \qquad \simeq 8.77$

3. (*a*) Show that $y = \frac{1}{2}x^2e^x$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x. \qquad \textbf{(4)}$$

(*b*) Solve the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$,

given that at $x = 0$, $y = 1$ and $\frac{dy}{dx} = 2$. **(9)**

$y = \frac{1}{2}x^2e^x$

$y' = \frac{1}{2}x^2e^x + xe^x = (\frac{1}{2}x^2 + x)e^x$

$y'' = (\frac{1}{2}x^2 + x)e^x + (x+1)e^x = (\frac{1}{2}x^2 + 2x + 1)e^x$

$y'' = (\frac{1}{2}x^2 + 2x + 1)e^x$

$-2y' = (-x^2 - 2x)e^x$ +

$+y = (\frac{1}{2}x^2)e^x$

$e^x = e^x$ #

$\therefore y = \frac{1}{2}x^2e^x$ is the particular integral solution to the 2nd order diff equation.

b) $y = Ae^{mx}$ $\quad y'' - 2y' + y = 0$

$y' = Ame^{mx}$ $\quad Ae^{mx}(m^2 - 2m + 1) = 0$

$y'' = Am^2e^{mx}$ $\quad \neq 0 \quad = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1$ RR

$y_{cf} = (A + Bx)e^x$ $\quad \therefore y = (A + Bx + \frac{1}{2}x^2)e^x$

$x = 0, y = 1 \Rightarrow 1 = A$ $\quad y = (1 + Bx + \frac{1}{2}x^2)e^x$

$y' = (1 + Bx + \frac{1}{2}x^2)e^x + (B + x)e^x$

$x = 0 \quad y' = 2$ $\quad 2 = 1 + B \quad \therefore B = 1$

$\therefore y = (1 + x + \frac{1}{2}x^2)e^x$

4. The curve C has polar equation $r = 3a\cos\theta, -\frac{\pi}{2} \le \frac{\pi}{2}$.

The curve D has polar equation $r = a(1 + \cos\theta), -\pi \le \theta < \pi$. Given that a is a positive constant,

(*a*) sketch, on the same diagram, the graphs of C and D, indicating where each curve cuts the initial line. **(4)**

The graphs of C intersect at the pole O and at the points P and Q.

(*b*) Find the polar coordinates of P and Q. **(3)**

(*c*) Use integration to find the exact area enclosed by the curve D and the lines $\theta = 0$ and $\theta = \frac{\pi}{3}$ **(7)**

The region R contains all points which lie outside D and inside C.

Given that the value of the smaller area enclosed by the curve C and the line $\theta = \frac{\pi}{3}$ is

$$\frac{3a^2}{16}(2\pi - 3\sqrt{3}),$$

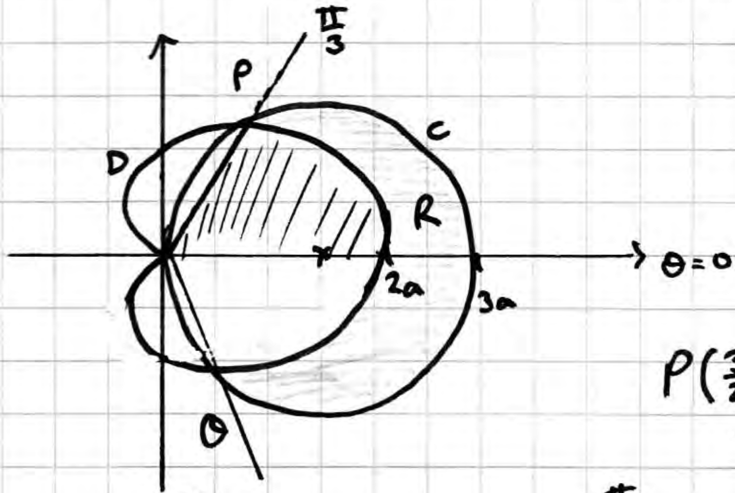
(*d*) show that the area of R is πa^2. **(4)**

$r = a(1+\cos\theta)$

$r_{max} = 2a \quad \theta = 0$

$r_{min} = 0 \quad \theta = \pi$

$r = a$ at $\theta = \frac{\pi}{2}, -\frac{\pi}{2}$



$3a\cos\theta = a(1+\cos\theta)$

$2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, -\frac{\pi}{3}$

$r = a(1+(\frac{1}{2})) = \frac{3}{2}a$

$P(\frac{3}{2}a, \frac{\pi}{3})$ $Q(\frac{3}{2}a, -\frac{\pi}{3})$

c) $\text{Area} = \frac{1}{2}a^2\int_0^{\frac{\pi}{3}}(1+\cos\theta)^2 d\theta = \frac{1}{2}a^2\int_0^{\frac{\pi}{3}} 1+2\cos\theta+(\frac{1}{2}+\frac{1}{2}\cos 2\theta)\, d\theta$

$= \frac{1}{2}a^2\int_0^{\pi/3}\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\, d\theta = \frac{1}{4}a^2\int_0^{\pi/3} 3+4\cos\theta+\cos 2\theta\, d\theta$

$= \frac{1}{4}a^2\left[3\theta+4\sin\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{3}} = \frac{1}{4}a^2\left[(\pi+2\sqrt{3}+\frac{\sqrt{3}}{4})-(0)\right]$

$= \frac{1}{4}a^2\left(\pi+\frac{9\sqrt{3}}{4}\right) = \frac{1}{16}a^2(4\pi+9\sqrt{3})$

d) $R = 2 \times \left[\frac{\pi(\frac{3a}{2})^2}{2} - \frac{1}{16}a^2(4\pi+9\sqrt{3}) - \frac{3}{16}a^2(2\pi-3\sqrt{3}\right]$

$R = \frac{9\pi}{4}a^2 - \frac{1}{8}a^2(4\pi+9\sqrt{3}+6\pi-9\sqrt{3}) = \left(\frac{9\pi}{4}-\frac{10\pi}{8}\right)a^2 = \frac{\pi a^2}{2}$

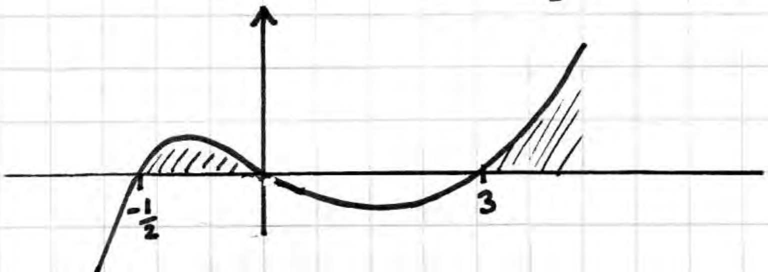
5. Using algebra, find the set of values of x for which $2x-5>\frac{3}{x}$. **(7)**

$(2x-5)x^2 > \frac{3x^2}{x}$ $\Rightarrow$ $2x^3-5x^2-3x>0$

$x(2x^2-5x-3)>0$

$x(2x+1)(x-3)>0$

$0 \quad -\frac{1}{2} \quad 3$



$-\frac{1}{2}<x<0$ or $x>3$

6. (*a*) Find the general solution of the differential equation

$$\cos x \frac{dy}{dx} + (\sin x)y = \cos^3 x. \qquad \textbf{(6)}$$

(*b*) Show that, for $0 \leq x \leq 2\pi$, there are two points on the *x*-axis through which all the solution curves for this differential equation pass. **(2)**

(*c*) Sketch the graph, for $0 \leq x \leq 2\pi$, of the particular solution for which $y = 0$ at $x = 0$. **(3)**

$$\frac{dy}{dx} + (\tan x)y = \cos^2 x \qquad \text{IF } f(x) = e^{\int \tan x \, dx} = e^{\ln|\sec x|} = \sec x$$

$$\sec x \frac{dy}{dx} + (\sec x \tan x)y = \cos x$$

$$\frac{d}{dx}(y \sec x) = \cos x \Rightarrow y \sec x = \int \cos x \, dx = \sin x + c$$

$$\therefore y = \sin x \cos x + c\cos x$$

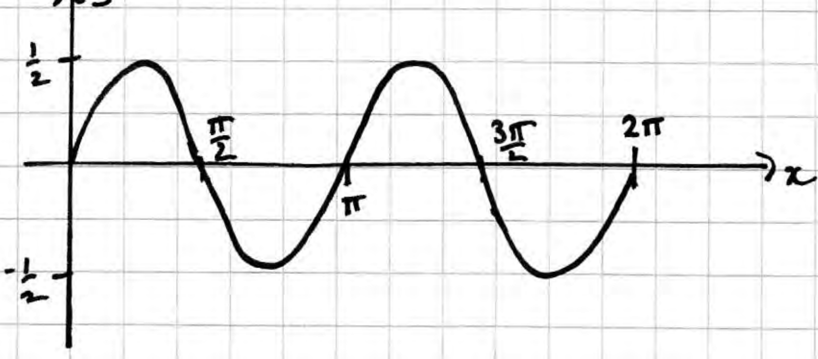
b) $\sin x \cos x + c\cos x = 0 \Rightarrow \cos x(\sin x + c) = 0$

$\cos x = 0$

$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

$(0,0) \Rightarrow 0 = 0 + c \quad \therefore c = 0$

$\therefore y = \sin x \cos x \qquad y = \frac{1}{2}\sin 2x$



7. (*a*) Find the general solution of the differential equation

$$2\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 3y = 3t^2 + 11t. \qquad \textbf{(8)}$$

(*b*) Find the particular solution of this differential equation for which $y = 1$ and $\frac{dy}{dt} = 1$ when $t = 0$. **(5)**

(*c*) For this particular solution, calculate the value of y when $t = 1$. **(1)**

$y = Ae^{mt}$ $\quad 2y'' + 7y' + 3y = 0$ $\quad y_{CF} = Ae^{-3t} + Be^{-\frac{1}{2}t}$

$y' = Ame^{mt}$ $\quad \therefore Ae^{mt}(2m^2 + 7m + 3) = 0$

$y'' = Am^2e^{mt}$ $\quad \neq 0 \quad = 0 \quad (2m+1)(m+3) = 0 \quad m = -\frac{1}{2} \quad m = -3$

$y = a + bt + ct^2$ $\quad + 3y = 3a + 3bt + 3ct^2$

$y' = b + 2ct$ $\quad + 7y' = 7b + 14ct \quad +$

$y'' = 2c$ $\quad 2y'' = 4c$

$3t^2 + 11t \quad (3a + 7b + 4c) + (3b + 14c)t + 3ct^2$

$\therefore 3c = 3 \quad \therefore c = 1 \quad 3b + 14 = 11 \quad \therefore b = -1$

$3a - 7 + 4 = 0 \quad \therefore a = 1 \quad y_{PI} = 1 - t + t^2$

$\therefore y = Ae^{-3t} + Be^{-\frac{1}{2}t} + 1 - t + t^2$

$y' = -3Ae^{-3t} - \frac{1}{2}Be^{-\frac{1}{2}t} - 1 + 2t$

$t = 0, y = 1 \quad 1 = A + B + 1 \quad \therefore A + B = 0$

$t = 0, y' = 1 \quad 1 = -3A - \frac{1}{2}B - 1 \quad 6A + B = -4$

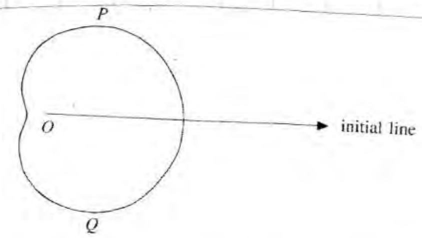
$-5A = 4 \quad A = -\frac{4}{5} \quad b = \frac{4}{5}$

$\therefore y = \frac{4}{5}e^{-\frac{1}{2}t} - \frac{4}{5}e^{-3t} + 1 - t + t^2$

$t = 1 \quad y = \frac{4}{5}e^{-\frac{1}{2}} - \frac{4}{5}e^{-3} + 1 = 1.445..$

8.

Figure 1



The curve *C* shown in Fig. 1 has polar equation

$r = a(3 + \sqrt{5}\cos\theta), \quad -\pi \le \theta < \pi.$

(*a*) Find the polar coordinates of the points *P* and *Q* where the tangents to *C* are parallel to the initial line. **(6)**

The curve *C* represents the perimeter of the surface of a swimming pool. The direct distance from *P* to *Q* is 20 m.

(*b*) Calculate the value of *a*. **(3)**

(*c*) Find the area of the surface of the pool. **(6)**

a) $\frac{dy}{d\theta} = 0 \quad y = r\sin\theta = a(3+\sqrt{5}\cos\theta)\sin\theta$

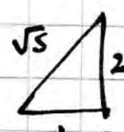
$\frac{dy}{d\theta} = a(3+\sqrt{5}\cos\theta)\cos\theta + a(-\sqrt{5}\sin\theta)\sin\theta = 0$

$\Rightarrow \sqrt{5}\sin^2\theta = 3\cos\theta + \sqrt{5}\cos^2\theta$

$\Rightarrow \sqrt{5}(1-\cos^2\theta) = 3\cos\theta + \sqrt{5}\cos^2\theta$

$\Rightarrow 2\sqrt{5}\cos^2\theta + 3\cos\theta - \sqrt{5} = 0$

$(\sqrt{5}\cos\theta - 1)(2\cos\theta + \sqrt{5}) = 0$



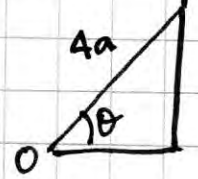
$\sin\theta = \frac{2}{\sqrt{5}}$

$\cos\theta = \frac{1}{\sqrt{5}} \quad \cos\theta = -\frac{\sqrt{5}}{2}$ no solutions

$\theta = 1.107^c, -1.107^c \qquad r = a(3+\sqrt{5}(\frac{1}{\sqrt{5}})) = 4a$

$P(4a, 1.107) \; Q(4a, -1.107)$

b)



$4a\sin\theta = \frac{8}{\sqrt{5}}a$

$\Rightarrow PQ = \frac{16}{\sqrt{5}}a = 20\text{m} \quad \therefore a = \frac{20\sqrt{5}}{16} = \frac{5\sqrt{5}}{4}$

c) $\text{Area} = 2 \times \frac{1}{2}a^2\int_0^{\pi}(3+\sqrt{5}\cos\theta)^2 d\theta = a^2\int_0^{\pi} 9 + 6\sqrt{5}\cos\theta + 5(\frac{1}{2}+\frac{1}{2}\cos 2\theta)\, d\theta$

$= a^2\int_0^{\pi} \frac{23}{2} + 6\sqrt{5}\cos\theta + \frac{5}{2}\cos 2\theta \, d\theta = \frac{1}{2}a^2\int_0^{\pi} 23 + 12\sqrt{5}\cos\theta + 5\cos 2\theta \, d\theta$

$= \frac{1}{2}a^2\left[23\theta + 12\sqrt{5}\sin\theta + \frac{5}{2}\sin 2\theta\right]_0^{\pi} = \frac{23\pi}{2}a^2 \quad \therefore \text{Area} = \frac{23\pi}{2} \times \frac{25 \times 5}{16}$

$= \frac{2875}{32}\pi \approx 282\text{m}^2$

9. (*a*) The point *P* represents a complex number *z* in an Argand diagram. Given that

$$|z - 2i| = 2|z + i|,$$

(i) find a cartesian equation for the locus of *P*, simplifying your answer. **(2)**

(ii) sketch the locus of *P*. **(3)**

(*b*) A transformation *T* from the *z*-plane to the *w*-plane is a translation $-7 + 11i$ followed by an enlargement with centre the origin and scale factor 3.

Write down the transformation *T* in the form

$$w = az + b, \quad a, b \in \mathbb{C}.$$

(2)

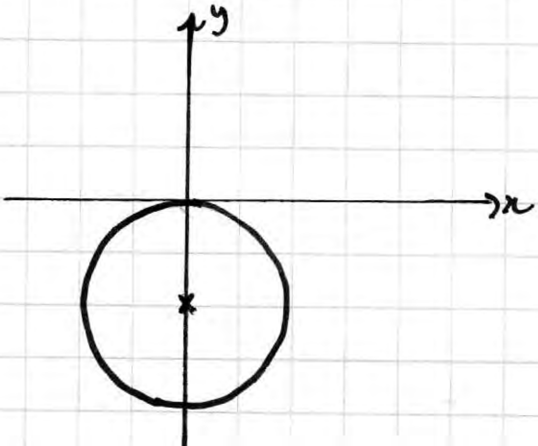
$|x+(y-2)i| = 2|x+(y+1)i|$

$\Rightarrow x^2+(y-2)^2 = 4(x^2+(y+1)^2)$

$\Rightarrow x^2 + y^2 - 4y + 4 = 4x^2 + 4y^2 + 8y + 4$

$\Rightarrow 3x^2 + 3y^2 + 12y = 0 \Rightarrow x^2+y^2+4y=0 \quad x^2+(y+2)^2 = 4$

Circle $C(0,-2)$ $r=2$



$w = 3z - 21 + 33i$

10.

$$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+y=0.$$

(*a*) Find an expression for $\frac{d^3y}{dx^3}$.

(5)

Given that $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$,

(*b*) find the series solution for *y*, in ascending powers of *x*, up to an including the term in x^3.

(5)

(c) Comment on whether it would be sensible to use your series solution to give estimates for *y* at $x = 0.2$ and at $x = 50$.

(2)

$$\frac{d}{dx}\left(y\frac{d^2y}{dx^2}\right)+\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right]+\frac{dy}{dx}=0$$

$$\frac{dy}{dx}\frac{d^2y}{dx^2}+y\frac{d^3y}{dx^3}+2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$$

$$y\frac{d^3y}{dx^3}+3\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$$

$x_0=0 \quad y_0=1 \quad y_0'=1$

$y_0y_0''+(y_0')^2+y_0=0 \Rightarrow y''=-2$

$yy_0''' + 3(y_0')y_0''+y_0' \Rightarrow y'''=5$

$\therefore \; y=1+x-x^2+\frac{5}{6}x^3 \ldots$

c) fine for $x=0.2$ but not fine for $x=50$ as approximation will be divergent for large values of x