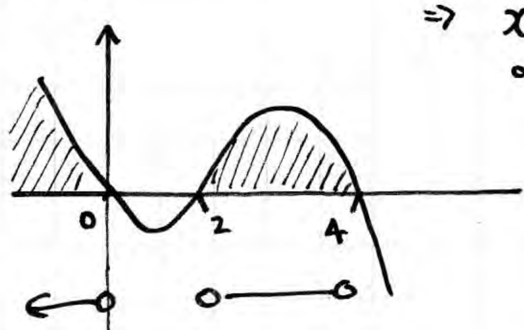
1. Find the set of values of x for which $\frac{x^2}{x-2} > 2x$. **(Total 6 marks)**

$\frac{x^2(x-2)^{\cancel{2}}}{(\cancel{x-2})} > 2x(x-2)^2 \Rightarrow x^2(x-2)-2x(x-2)^2 > 0$

$\Rightarrow x(x-2)[x-2(x-2)] > 0$

$\Rightarrow x(x-2)(4-x) > 0$

0 2 4



$x<0$ or $2<x<4$

2. (a) Find the general solution of the differential equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=0. \qquad (4)$$

(b) Given that $x = 1$ and $\frac{dx}{dt}=1$ at $t = 0$, find the particular solution of the differential equation, giving your answer in the form $x = f(t)$. **(5)**

(c) Sketch the curve with equation $x = f(t)$, $0 \le t \le \pi$, showing the coordinates, as multiples of π, of the points where the curve cuts the x-axis. **(4)(Total 13 marks)**

$x = Ae^{mt}$

$x' = Ame^{mt}$

$x'' = Am^2e^{mt}$

$x'' = Am^2e^{mt}$

$2x' = 2Ame^{mt}$

$5x = 5Ae^{mt}$

$\therefore Ae^{mt}(m^2+2m+5)=0$

$\neq 0 \quad (m+1)^2=-4$

$\therefore m=-1\pm 2i$

$\therefore x = Pe^{(-1+2i)t} + Qe^{(-1-2i)t} = e^{-t}(Pe^{2it}+Qe^{-2it})$

$\therefore x = e^{-t}(A\cos 2t + B\sin 2t)$

$t=0, x=1 \Rightarrow \underline{1=A} \quad \Rightarrow x = e^{-t}(\cos 2t + B\sin 2t)$

$x' = -e^{-t}(\cos 2t + B\sin 2t) + e^{-t}(-2\sin 2t + 2B\cos 2t)$

$t=0, x'=1 \quad 1 = -1(1)+1(2B) \Rightarrow 2B=2 \quad \therefore B=1$

$\therefore x = e^{-t}(\cos 2t + \sin 2t)$

c) $t=0 \quad x=1 \qquad x=0 \Rightarrow \cos 2t = -\sin 2t \Rightarrow \tan 2t = -1$

$t=\pi \quad x=e^{-\pi}$

$x=0.04$

$\Rightarrow 2t = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4} \therefore t=\frac{3\pi}{8}, \frac{7\pi}{8}$

$t=\frac{\pi}{2} \quad x = -e^{-\frac{\pi}{2}} = -0.21$

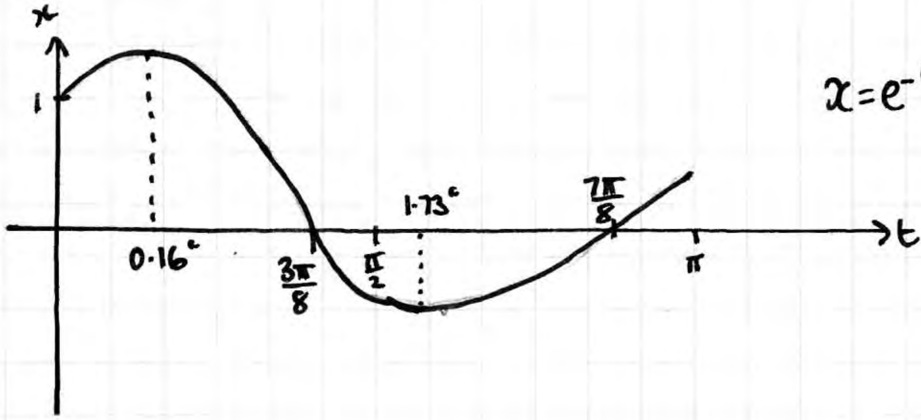
$\frac{dx}{dt} = e^{-t}[2\cos 2t - 2\sin 2t - \sin 2t - \cos 2t]$

$\frac{dx}{dt} = e^{-t}[\cos 2t - 3\sin 2t]$

TP $x'=0 \Rightarrow 3\sin 2t = \cos 2t \Rightarrow \tan 2t = \frac{1}{3} \Rightarrow 2t = 0.322^c, 3.463^c$

$t = 0.16 \quad , 1.73^c$

$x = 1.08 \qquad x=-0.22$



$x = e^{-t}[\cos 2t + \sin 2t]$

alt $\quad x = (\sin 2t + \cos 2t)e^{-t}$

$x = \sqrt{2}\left[\sin\left(2t + \frac{\pi}{4}\right)\right]e^{-t}$

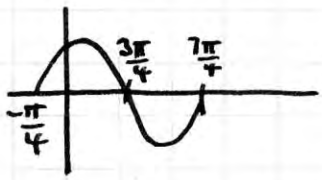
$R\sin(2t+\alpha) = R\sin 2t\cos\alpha + R\cos 2t\sin\alpha$

$\quad 1\sin 2t \quad + 1\cos 2t$

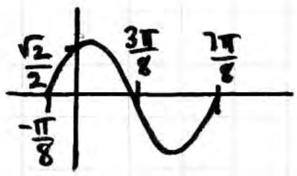
$\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{1} \Rightarrow \tan\alpha = 1$

$\alpha = \frac{\pi}{4}$

$R^2 = 1^2 + 1^2 \therefore R = \sqrt{2}$

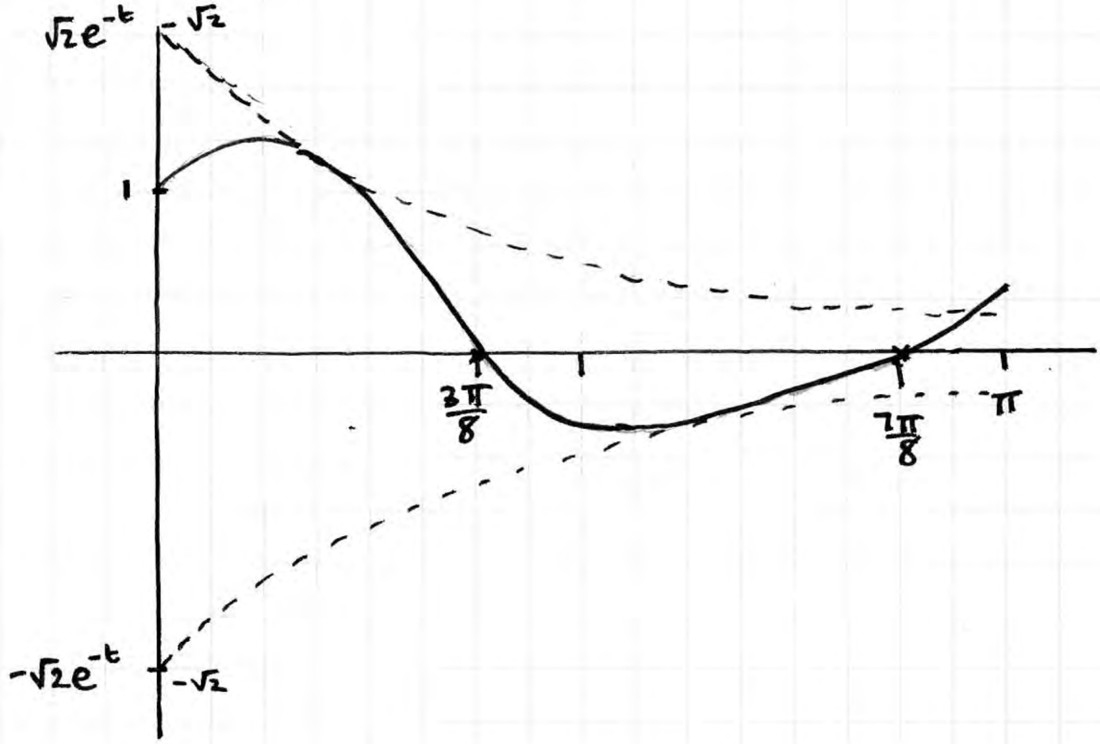


$\sin\left(t + \frac{\pi}{4}\right)$



$\sin\left(2t + \frac{\pi}{4}\right)$

$x = \sqrt{2}e^{-t}\left[\sin\left(2t + \frac{\pi}{4}\right)\right]$



3. (a) Show that the substitution $y = vx$ transforms the differential equation

$$\frac{dy}{dx} = \frac{3x - 4y}{4x + 3y} \quad \text{(I)}$$

into the differential equation

$$x\frac{dy}{dx} = -\frac{3v^2 + 8v - 3}{3v + 4} \quad \text{(II)}. \qquad \textbf{(4)}$$

(b) By solving differential equation (II), find a general solution of differential equation (I). **(5)**

(c) Given that $y = 7$ at $x = 1$, show that the particular solution of differential equation (I) can be written as

$$(3y - x)(y + 3x) = 200.$$

(5)(Total 14 marks)

$$\frac{dy}{dx} = \frac{d}{dx}(vx) = v + x\frac{dv}{dx} \Rightarrow v + x\frac{dv}{dx} = \frac{3x - 4vx}{4x + 3vx} = \frac{\cancel{x}(3-4v)}{\cancel{x}(4+3v)}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{3-4v}{4+3v} - \frac{v(4+3v)}{4+3v} = \frac{3-4v-4v-3v^2}{4+3v}$$

$$\therefore x\frac{dv}{dx} = -\frac{3v^2+8v-3}{3v+4} \quad \#$$

$$\int \frac{3v+4}{3v^2+8v-3}\,dv = \int -\frac{1}{x}\,dx \Rightarrow \frac{1}{2}\int \frac{6v+8}{3v^2+8v-3}\,dv = -\ln x + c$$

$$\Rightarrow \frac{1}{2}\ln|3v^2+8v-3| = -\ln x + c \Rightarrow \ln|(3v-1)(v+3)| = \ln x^{-2} + 2c$$

$$\Rightarrow (3v-1)(v+3) = e^{\ln x^{-2} + 2c} = \frac{A}{x^2} \qquad A = e^{2c}$$

c) $v = \frac{y}{x} \Rightarrow \left(3\frac{y}{x} - 1\right)\left(\frac{y}{x} + 3\right) = \frac{A}{x^2} \Rightarrow \frac{1}{\cancel{x^2}}(3y-x)(y+3x) = \frac{A}{\cancel{x^2}}$

$(1,7) \Rightarrow (21-1)(7+3) = A \quad \therefore A = 200 \qquad \therefore (3y-x)(y+3x) = 200$

8.

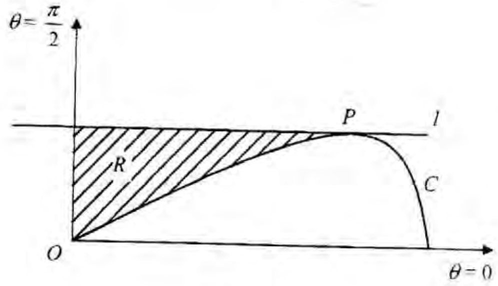


Figure 1

A curve C has polar equation $r^2 = a^2 \cos 2\theta$, $0 \leqslant \theta \leqslant \frac{\pi}{4}$. The line l is parallel to the initial line, and l is the tangent to C at the point P, as shown in Figure 1.

(a) (i) Show that, for any point on C, $r^2 \sin^2\theta$ can be expressed in terms of $\sin\theta$ and a only. **(1)**

(ii) Hence, using differentiation, show that the polar coordinates of P are $\left(\frac{a}{\sqrt{2}}, \frac{\pi}{6}\right)$. **(6)**

The shaded region R, shown in Figure 1, is bounded by C, the line l and the half-line with equation $\theta = \frac{\pi}{2}$.

(b) Show that the area of R is $\frac{a^2}{16}(3\sqrt{3} - 4)$. **(8)**

(Total 15 marks)

a) $r^2 \sin^2\theta$

$= a^2 \cos 2\theta \sin^2\theta$

$= a^2(1 - 2\sin^2\theta)\sin^2\theta$

$= a^2(\sin^2\theta - 2\sin^4\theta)$

ii) at P, tangent is parallel to initial line. $\Rightarrow \frac{dy}{d\theta} = 0$

$y = r\sin\theta$

$y^2 = r^2 \sin^2\theta$

$y^2 = a^2(\sin^2\theta - 2\sin^4\theta)$

$\frac{d}{d\theta} y^2 = a^2 \frac{d}{d\theta}(\sin^2\theta - 2\sin^4\theta)$

$\therefore 2y\frac{dy}{d\theta} = a^2(2\sin\theta\cos\theta - 8\sin^3\theta\cos\theta)$

$\Rightarrow \frac{dy}{d\theta} = 0 \Rightarrow 2\sin\theta\cancel{\cos\theta} = 8\sin^3\theta\cancel{\cos\theta}$

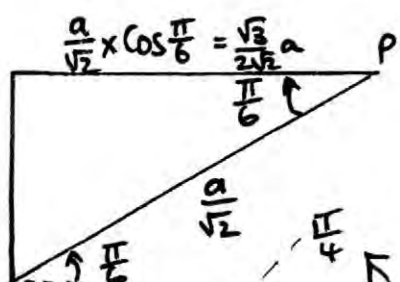
$\Rightarrow \sin^2\theta = \frac{1}{4} \quad \sin\theta = \pm\frac{1}{2}$

$\therefore \theta = \frac{\pi}{6}$

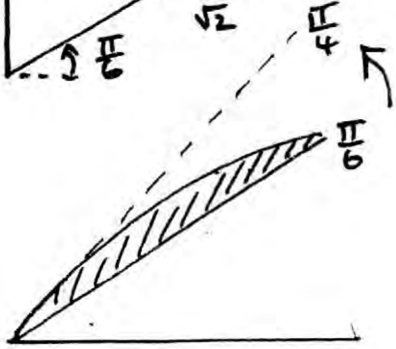
$r^2 = a^2\cos 2\theta \Rightarrow r = a^2 \cos\frac{\pi}{3} = \frac{1}{2}a^2$

$\therefore r = \sqrt{\frac{1}{2}a^2} = \frac{a}{\sqrt{2}} \quad \therefore P\left(\frac{a}{\sqrt{2}}, \frac{\pi}{6}\right)$

c)



Area $= \frac{1}{2}\left(\frac{\sqrt{3}}{2\sqrt{2}}a\right)\left(a\frac{1}{\sqrt{2}}\right)\sin\frac{\pi}{6} = \frac{\sqrt{3}}{16}a^2$



Area $= \frac{1}{2}a^2\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 2\theta \, d\theta = \frac{1}{2}a^2\left[\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \frac{1}{2}a^2\left[\left(\frac{1}{2}\right) - \left(\frac{\sqrt{3}}{4}\right)\right] = \frac{1}{4}a^2 - \frac{\sqrt{3}}{8}a^2$

$\therefore R = \frac{\sqrt{3}}{16}a^2 - a^2\left(\frac{1}{4} - \frac{\sqrt{3}}{8}\right) = \frac{3\sqrt{3}}{16}a^2 - \frac{1}{4}a^2$

$= \frac{a^2}{16}(3\sqrt{3} - 4)$

5 Solve the equation

$$z^5 = \mathrm{i},$$

giving your answers in the form $\cos\theta + \mathrm{i}\sin\theta$.

(Total 5 marks)

$z^5 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$z^5 = \cos\left(\frac{\pi}{2} + 2k\pi\right) + i\sin\left(\frac{\pi}{2} + 2k\pi\right)$

$\Rightarrow z = \left(\cos\left(\frac{4k+1}{2}\right)\pi + i\sin\left(\frac{4k+1}{2}\right)\pi\right)^{\frac{1}{5}}$

$\Rightarrow z = \left(\cos\left(\frac{4k+1}{10}\right)\pi + i\sin\left(\frac{4k+1}{10}\right)\pi\right)$

$k=-2 \quad \cos\left(\frac{-7\pi}{10}\right) + i\sin\left(-\frac{7\pi}{10}\right)$ $\qquad k=0 \quad \cos\left(\frac{\pi}{10}\right) + i\sin\left(\frac{\pi}{10}\right)$

$k=-1 \quad \cos\left(\frac{-3\pi}{10}\right) + i\sin\left(-\frac{3\pi}{10}\right)$ $\qquad k=1 \quad \cos\left(\frac{5\pi}{10}\right) + i\sin\left(\frac{5\pi}{10}\right)$

$k=2 \quad \cos\left(\frac{9\pi}{10}\right) + i\sin\left(\frac{9\pi}{10}\right)$

7

$$(1 + 2x)\frac{dy}{dx} = x + 4y^2.$$

(a) Show that

$$(1 + 2x)\frac{d^2y}{dx^2} = 1 + 2(4y - 1)\frac{dy}{dx}. \qquad (1)$$

(2)

(b) Differentiate equation (1) with respect to x to obtain an equation involving

$$\frac{d^3y}{dx^3}, \frac{d^2y}{dx^2}, \frac{dy}{dx}, x \text{ and } y.$$

(3)

Given that $y = \frac{1}{2}$ at $x = 0$,

(c) find a series solution for y, in ascending powers of x, up to and including the term in x^3.

(6)

(Total 11 marks)

$\frac{d}{dx}\left[(1+2x)\frac{dy}{dx}\right] = \frac{d}{dx}(x) + \frac{d}{dx}(4y^2)$

$= (1+2x)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 1 + 8y\frac{dy}{dx}$

$\Rightarrow (1+2x)\frac{d^2y}{dx^2} = 1 + 2(4y-1)\frac{dy}{dx}$

b) $\frac{d}{dx}\left[(1+2x)\frac{d^2y}{dx^2}\right] = \frac{d}{dx}\left[(8y-2)\frac{dy}{dx}\right]$

$\Rightarrow (1+2x)\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = (8y-2)\frac{d^2y}{dx^2} + 8\left(\frac{dy}{dx}\right)^2$

$\Rightarrow (1+2x)\frac{d^3y}{dx^3} = 4(2y-1)\frac{d^2y}{dx^2} + 8\left(\frac{dy}{dx}\right)^2$

c) $x_0 = 0 \quad y_0 = \frac{1}{2}$ $\qquad (1+2x)y' = x + 4y^2 \Rightarrow (1)y' = 4\left(\frac{1}{2}\right)^2 \Rightarrow y' = 1$

$(1+2x)y'' = 1 + 2(4y-1)y' \Rightarrow (1)y'' = 1 + 2(1)(1) \Rightarrow y'' = 3$

$(1+2x)y''' = 4(2y-1)y'' + 8y'^2 \Rightarrow (1)y''' = 4(0)(3) + 8(1)^2 \Rightarrow y''' = 8$

$\therefore \; y = \frac{1}{2} + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 \ldots$

8. In the Argand diagram the point P represents the complex number z.

Given that $\arg\left(\dfrac{z-2\mathrm{i}}{z+2}\right)=\dfrac{\pi}{2}$,

(a) sketch the locus of P,

(4)

(b) deduce the value of $|z+1-\mathrm{i}|$.

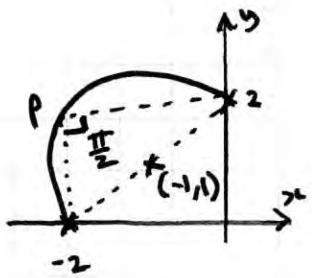
(2)

The transformation T from the z-plane to the w-plane is defined by

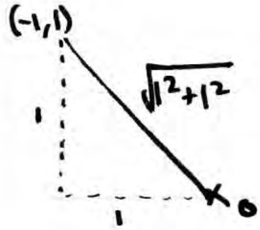
$$w=\frac{2(1+\mathrm{i})}{z+2}, \qquad z\neq -2$$

(c) Show that the locus of P in the z-plane is mapped to part of a straight line in the w-plane, and show this in an Argand diagram.

(6)(Total 12 marks)



(b)



$|z+1-i| = \sqrt{2}$

c)

$$z+2 = \frac{2(1+i)}{w} \Rightarrow z = \frac{2(1+i)}{w} - 2 \Rightarrow z = \frac{2(1+i)-2w}{w}$$

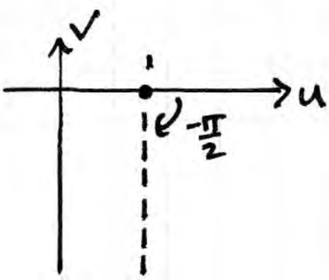
$$z-2i = \frac{2(1+i)-2w}{w} - 2i \Rightarrow z-2i = \frac{2(1+i)-2w(1+i)}{w}$$

$$\therefore \frac{z-2i}{z+2} = \frac{\dfrac{2(1+i)-w\,2(1+i)}{w}}{\dfrac{2(1+i)}{w}} = \frac{2(1+i)-w\,2(1+i)}{2(1+i)}$$

$$\therefore \frac{z-2i}{z+2} = 1-w \Rightarrow \arg|1-w| = \frac{\pi}{2}$$

$$\Rightarrow \arg[(-1)(w-1)] = \frac{\pi}{2} \Rightarrow \arg(-1) + \arg(w-1) = \frac{\pi}{2}$$

$$\Rightarrow \pi + \arg(w-1) = \frac{\pi}{2} \quad \therefore \arg(w-1) = -\frac{\pi}{2}$$



$\therefore$ locus of P in z-plane maps to half line $\arg(w-1) = -\frac{\pi}{2}$.