General Certificate of Education
January 2008
Advanced Level Examination

MATHEMATICS **MFP2**
Unit Further Pure 2

Thursday 31 January 2008 9.00 am to 10.30 am

For this paper you must have:
- an 8-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP2.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 (a) Express $4 + 4\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(3 marks)*

(b) Solve the equation

$$z^5 = 4 + 4\mathrm{i}$$

giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. *(5 marks)*

2 (a) Show that

$$(2r + 1)^3 - (2r - 1)^3 = 24r^2 + 2$$ *(3 marks)*

(b) Hence, using the method of differences, show that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n + 1)(2n + 1)$$ *(6 marks)*

3 A circle C and a half-line L have equations

$$|z - 2\sqrt{3} - \mathrm{i}| = 4$$

and $$\arg(z + \mathrm{i}) = \frac{\pi}{6}$$

respectively.

(a) Show that:

(i) the circle C passes through the point where $z = -\mathrm{i}$; *(2 marks)*

(ii) the half-line L passes through the centre of C. *(3 marks)*

(b) On one Argand diagram, sketch C and L. *(4 marks)*

(c) Shade on your sketch the set of points satisfying both

$$|z - 2\sqrt{3} - \mathrm{i}| \leqslant 4$$

and $$0 \leqslant \arg(z + \mathrm{i}) \leqslant \frac{\pi}{6}$$ *(2 marks)*

4 The cubic equation

$$z^3 + \mathrm{i}z^2 + 3z - (1 + \mathrm{i}) = 0$$

has roots α, β and γ.

(a) Write down the value of:

(i) $\alpha + \beta + \gamma$; *(1 mark)*

(ii) $\alpha\beta + \beta\gamma + \gamma\alpha$; *(1 mark)*

(iii) $\alpha\beta\gamma$. *(1 mark)*

(b) Find the value of:

(i) $\alpha^2 + \beta^2 + \gamma^2$; *(3 marks)*

(ii) $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$; *(4 marks)*

(iii) $\alpha^2\beta^2\gamma^2$. *(2 marks)*

(c) Hence write down a cubic equation whose roots are α^2, β^2 and γ^2. *(2 marks)*

5 Prove by induction that for all integers $n \geqslant 1$

$$\sum_{r=1}^{n} (r^2 + 1)(r!) = n(n + 1)!$$ *(7 marks)*

Turn over for the next question

6 (a) (i) By applying De Moivre's theorem to $(\cos\theta + \mathrm{i}\sin\theta)^3$, show that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$ *(3 marks)*

(ii) Find a similar expression for $\sin 3\theta$. *(1 mark)*

(iii) Deduce that

$$\tan 3\theta = \frac{\tan^3\theta - 3\tan\theta}{3\tan^2\theta - 1}$$ *(3 marks)*

(b) (i) Hence show that $\tan\frac{\pi}{12}$ is a root of the cubic equation

$$x^3 - 3x^2 - 3x + 1 = 0$$ *(3 marks)*

(ii) Find two other values of θ, where $0 < \theta < \pi$, for which $\tan\theta$ is a root of this cubic equation. *(2 marks)*

(c) Hence show that

$$\tan\frac{\pi}{12} + \tan\frac{5\pi}{12} = 4$$ *(2 marks)*

7 (a) Given that $y = \ln\tanh\frac{x}{2}$, where $x > 0$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \operatorname{cosech} x$$ *(6 marks)*

(b) A curve has equation $y = \ln\tanh\frac{x}{2}$, where $x > 0$. The length of the arc of the curve between the points where $x = 1$ and $x = 2$ is denoted by s.

(i) Show that

$$s = \int_1^2 \coth x \,\mathrm{d}x$$ *(2 marks)*

(ii) Hence show that $s = \ln(2\cosh 1)$. *(4 marks)*

END OF QUESTIONS