MEI

Mathematics in Education and Industry

MEI Structured Mathematics

Module Summary Sheets

FP1, Further Concepts for Advanced Mathematics

Topic 1: Matrices

Topic 2: Complex Numbers

Topic 3: Curve Sketching

Topic 4: Algebra

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Summary FP1 Topic 1: ***Matrices***

References:
Chapter 1
Pages 2-3

Matrices are rectangular arrays of numbers which can be used to convey information.
A matrix with n rows and m columns is known as an $n \times m$ matrix.
A matrix with equal numbers of rows and columns is **square**.
The entries in a matrix are called **elements**.
If all elements are 0 then matrix is known as the zero matrix.
The Identity Matrix for any sized square matrix is one where all elements in the leading diagonal are 1 and all other elements are 0.

i.e., for a 2×2 matrix $\mathbf{I}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

Exercise 1A
Q. 2, 5

If a matrix is multiplied by a number then every element is multiplied by that number.
Matrices are conformable if they are the same size.

Conformable matrices may be added or subtracted.
Matrix addition is commutative
i.e. $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$

Matrix addition is associative.
i.e. $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$

E.g. $\mathbf{A}=\begin{pmatrix}1&2\\3&4\end{pmatrix}, \mathbf{B}=\begin{pmatrix}5&6\\7&8\end{pmatrix}, \mathbf{C}=\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}$

(i) Find, if possible, $\mathbf{A}+\mathbf{B}$, $\mathbf{A}+\mathbf{C}$
(ii) Show that $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$.

(i) $\mathbf{A}+\mathbf{B}=\begin{pmatrix}1+5&2+6\\3+7&4+8\end{pmatrix}=\begin{pmatrix}6&8\\10&12\end{pmatrix}$

$\mathbf{A}+\mathbf{C}$ is not possible as $\mathbf{A}$ and $\mathbf{C}$ are not conformable.

(ii) $\mathbf{B}+\mathbf{A}=\begin{pmatrix}5+1&6+2\\7+3&8+4\end{pmatrix}=\begin{pmatrix}6&8\\10&12\end{pmatrix}=\mathbf{A}+\mathbf{B}$

E.g. $\mathbf{A}=\begin{pmatrix}2&3\\1&4\end{pmatrix}$, $\mathbf{A}+\mathbf{A}=2\mathbf{A}=2\begin{pmatrix}2&3\\1&4\end{pmatrix}=\begin{pmatrix}4&6\\2&8\end{pmatrix}$

E.g. $2\mathbf{I}=\begin{pmatrix}2&0\\0&2\end{pmatrix}$

E.g. If $\mathbf{A}$ and $\mathbf{B}$ are as given above, and $\mathbf{D}=\begin{pmatrix}2&-2\\-1&0\end{pmatrix}$

then show that $(\mathbf{A}+\mathbf{B})+\mathbf{D}=\mathbf{A}+(\mathbf{B}+\mathbf{D})$

$(\mathbf{A}+\mathbf{B})+\mathbf{D}=\begin{pmatrix}6&8\\10&12\end{pmatrix}+\begin{pmatrix}2&-2\\-1&0\end{pmatrix}=\begin{pmatrix}8&6\\9&12\end{pmatrix}$

$=\mathbf{A}+(\mathbf{B}+\mathbf{D})=\begin{pmatrix}1&2\\3&4\end{pmatrix}+\begin{pmatrix}7&4\\6&8\end{pmatrix}=\begin{pmatrix}8&6\\9&12\end{pmatrix}$

References:
Chapter 1
Pages 6-12

Transformations may be represented by matrices.

If $\mathbf{X}$, $\begin{pmatrix}x\\y\end{pmatrix}$ represents the position vector of the point (x, y)

and $\mathbf{X'}$, $\begin{pmatrix}x'\\y'\end{pmatrix}$ represents the position vector of the point (x', y')

and if $\mathbf{M}=\begin{pmatrix}a&c\\b&d\end{pmatrix}$ then $\mathbf{X}$ is transformed to $\mathbf{X'}$

by $\mathbf{M}$ if $\mathbf{X'}=\mathbf{MX}$.

$\begin{pmatrix}1\\0\end{pmatrix}$ is transformed to $\begin{pmatrix}a\\b\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}$ is transformed to $\begin{pmatrix}c\\d\end{pmatrix}$.

$\begin{pmatrix}0&1\\1&0\end{pmatrix}$ is the matrix for a reflection in the line $y=x$.

$\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$ is the matrix for a rotation of θ anticlockwise about the origin.

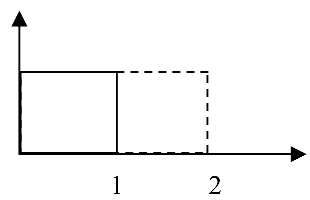
$\begin{pmatrix}2&0\\0&2\end{pmatrix}$ is the matrix for an enlargement, centre the origin and scale factor 2.

Exercise 1B
Q. 3(i), 4(i)

E.g. Describe the transformation given by $\mathbf{M}=\begin{pmatrix}2&0\\0&1\end{pmatrix}$.

$\begin{pmatrix}1\\0\end{pmatrix}\to\begin{pmatrix}2\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}\to\begin{pmatrix}0\\1\end{pmatrix}$

$\Rightarrow$ **M** represents a stretch parallel to the x axis.



E.g. Describe the transformation given by

$\mathbf{M}=\begin{pmatrix}0.8&-0.6\\0.6&0.8\end{pmatrix}$

$\begin{pmatrix}1\\0\end{pmatrix}\Rightarrow\begin{pmatrix}0.8\\0.6\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}\Rightarrow\begin{pmatrix}-0.6\\0.8\end{pmatrix}$

i.e. rotation of 53.1^0 anticlockwise about the origin.

Notation
A transformation is described by a bold italic letter; e.g. ***T***.
The matrix that represents this transformation is denoted by a bold upright capital letter; e.g. **T**.

FP1; Further Concepts for Advanced Mathematics —page 2
Competence statements FP1, m1, m2, m4

References:
Chapter 1
Pages 14-20

Exercise 1C
Q. 1(i), (ix), 5

Multiplication of matrices

2 matrices, **A** and **B** may be multiplied provided the number of columns in **A** is equal to the number of rows in **B**

i.e. If **A** is $m \times n$ and **B** is $n \times p$ then **AB** is a matrix of size $m \times p$
i.e. Size(**A**) = 2×3, Size(**B**) = 3×5
Then size(**AB**) = 2×5.

Note that **BA** cannot be calculated as the number of columns of **B** does not equal the number of rows in **A**.

If **A** and **B** are square then **AB** and **BA** can be calculated but they are not, in general, equal. So matrix multiplication is **not** commutative.
i.e. **AB** ≠ **BA**
Matrix multiplication is associative.
i.e. **A**(**BC**) = (**AB**)**C**

$$\mathbf{A}=\begin{pmatrix}1&2\\3&4\\5&6\end{pmatrix},\quad \mathbf{B}=\begin{pmatrix}7&8&9&10\\11&12&13&14\end{pmatrix}$$

$$3\times2 \qquad 2\times4 \qquad \Rightarrow AB = 3\times4$$

$$\mathbf{AB}=\begin{pmatrix}1\times7+2\times11 & 1\times8+2\times12 & 1\times9+2\times13 & 1\times10+2\times14\\ 3\times7+4\times11 & 3\times8+4\times12 & 3\times9+4\times13 & 3\times10+4\times14\\ 5\times7+6\times11 & 5\times8+6\times12 & 5\times9+6\times13 & 5\times10+6\times14\end{pmatrix}$$

$$\mathbf{AB}=\begin{pmatrix}29&32&35&38\\65&72&79&86\\101&112&123&134\end{pmatrix}$$

$$\mathbf{C}=\begin{pmatrix}1&2\\3&4\end{pmatrix},\ \mathbf{D}=\begin{pmatrix}-1&1\\3&-2\end{pmatrix},$$

$$\mathbf{CD}=\begin{pmatrix}5&-3\\9&-5\end{pmatrix},\ \mathbf{DC}=\begin{pmatrix}2&2\\-3&-2\end{pmatrix}$$

E.g. If **A** and **B** are as given above, and $\mathbf{D}=\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$

then show that $(\mathbf{AB})\mathbf{D}=\mathbf{A}(\mathbf{BD})$

$$(\mathbf{AB})\mathbf{D}=\begin{pmatrix}29&32&35&38\\65&72&79&86\\101&112&123&134\end{pmatrix}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}=\begin{pmatrix}-6\\-14\\-22\end{pmatrix}=\mathbf{A}(\mathbf{BD})$$

Notation

The point P has position vector **p.** The image of P under the transformation ***T*** can be denoted ***T***(P) or P'. The image, ***T***(P), has position vector **p'** = ***T***(**p**).
T(**p**) is found by evaluating the matrix product **Tp**.

References:
Chapter 1
Pages 23-25

Exercise 1D
Q. 3, 7

Composition of transformations

The transformation $\mathbf{MN}$ represents the transformation $\boldsymbol{N}$ followed by $\boldsymbol{M}$.

E.g. If $\boldsymbol{N}$ is a reflection in the y axis,

then $\mathbf{N}=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$.

If $\boldsymbol{M}$ is a reflection in the x axis,

then $\mathbf{M}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$.

$$\mathbf{MN}=\begin{pmatrix}-1&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}.$$

This represents a rotation through 180^0 about the origin.

E.g. $\mathbf{M}=\begin{pmatrix}1&0\\2&-1\end{pmatrix}$, $\mathbf{N}=\begin{pmatrix}2&-1\\0&1\end{pmatrix}$

(i) Find **MN**.
(ii) Find the image of the point (1,2) under the transformation N followed by M.

(i) $\mathbf{MN}=\begin{pmatrix}1&0\\2&1\end{pmatrix}\begin{pmatrix}2&-1\\0&1\end{pmatrix}=\begin{pmatrix}2&-1\\4&-1\end{pmatrix}.$

(ii) $\mathbf{p}=\begin{pmatrix}1\\2\end{pmatrix}$, $\mathbf{MNp}=\begin{pmatrix}2&-1\\4&-1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}0\\2\end{pmatrix}$

References:
Chapter 1
Pages 28-30

Determinants

If $\mathbf{A}=\begin{pmatrix}a&b\\c&d\end{pmatrix}$, the number $ad-bc$ is called the **Determinant** of **A**.

The value is written $|\mathbf{A}|$ or Det(**A**).

E.g. $\mathbf{A}=\begin{pmatrix}2&3\\4&-1\end{pmatrix}\Rightarrow|\mathbf{A}|=2\times(-1)-4\times3=-14$

$\mathbf{B}=\begin{pmatrix}3&1\\6&2\end{pmatrix}\Rightarrow|\mathbf{B}|=3\times2-6\times1=0$

Note:

$\mathbf{AB}=\begin{pmatrix}24&8\\6&2\end{pmatrix}\Rightarrow$ Det(**AB**) = 0 = Det(**A**)Det(**B**)

References: Chapter 1 Pages 28-30

Inverse Matrices
The inverse of a square matrix **A** is written $\mathbf{A}^{-1}$ and is such that
$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Note that the inverse of the 2×2 matrix **A** can be written as follows:
Interchange the two elements of the leading diagonal
Change the sign of each element in the trailing diagonal
Divide each term by the determinant of **A.**

Exercise 1E Q. 1(v),(vii)

Only square matrices have inverses.
An inverse of a matrix only exists if $|\mathbf{A}| \neq 0$.
If $|\mathbf{A}| = 0$ then the matrix is said to be **singular.**

E.g. Given that $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$
find $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$

$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} \Rightarrow |\mathbf{A}| = -14$

$\Rightarrow \mathbf{A}^{-1} = \frac{1}{-14}\begin{pmatrix} -1 & -3 \\ -4 & 2 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 1 & 3 \\ 4 & -2 \end{pmatrix}$

$\mathbf{B} = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} \Rightarrow |\mathbf{B}| = 0 \Rightarrow \mathbf{B}^{-1}$ does not exist.

Exercise: Take any **A** and **B** and show that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

References: Chapter 1 Pages 31-34

Use of determinants
The determinant of the matrix **A** is the (signed) area scale factor of the transformation represented by A.

i.e. the matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ transforms the unit square into a parallelogram with area $ad - bc$.
A singular matrix tranforms all points into a line through the origin.

Exercise 1F Q. 2, 7

E.g. A transformation is a shear parallel to the y axis such that each point is moved by 3 times its distance from the x axis.
Give the matrix that represents this transformation and show that area is preserved. (i.e. the area of the parallelogram that results from transforming a square has the same area.

$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \Rightarrow |\mathbf{M}| = 1$

References: Chapter 1 Pages 36-39

Simultaneous Equations
The simultaneous equations

$$ax + by = c$$
$$dx + ey = f$$

can be written $\begin{pmatrix} a & b \\ d & e \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ f \end{pmatrix}$

As a matrix equation this can be written **AX = B**
Multiply both sides by the inverse of A $\Rightarrow \mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$
i.e. $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.
A solution can be found if $|\mathbf{A}| \neq 0$

Exercise 1G Q. 3(i), 5

E.g. Solve $2x + 5y = 17,\ 4x - y = 1$

$\mathbf{AX} = \mathbf{B}$ where $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 4 & -1 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$,

$\mathbf{B} = \begin{pmatrix} 17 \\ 1 \end{pmatrix}$

$|\mathbf{A}| = -2 - 20 = -22 \Rightarrow \mathbf{A}^{-1} = \frac{1}{-22}\begin{pmatrix} -1 & -5 \\ -4 & 2 \end{pmatrix}$

$\mathbf{AX} = \mathbf{B} \Rightarrow \mathbf{X} = \mathbf{A}^{-1}\mathbf{B} = \frac{1}{-22}\begin{pmatrix} -1 & -5 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 17 \\ 1 \end{pmatrix}$

$= \frac{1}{-22}\begin{pmatrix} -22 \\ -66 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

i.e. $x = 1,\ y = 3$

References: Chapter 1 Pages 41-43

Invariant points and lines
If a matrix represents a reflection in a line, then any point on the line remains on the line.
If all points on a line AB are mapped onto points in AB (not necessarily the same point) then the line AB is known as an **invariant line**.

If a point is transformed onto itself then the point is known as an **invariant point**.

Exercise 1H Q. 1(i), (iii), 4(i)

$\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

(i) Show that $(1, -3)$ is an invariant point.
(ii) Find the equation of the invariant line.

(i) $\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 4-3 \\ 3-6 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

(ii) For any point (x, y), $\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow 4x + y = x \Rightarrow y = -3x$

Note that $(1, -3)$ lies on this line.
Alternatively, we have already found that $(1, -3)$ is an invariant point; so also is $(0, 0)$ and so the invariant line is the line joining these two points.

FP1; Further Concepts for Advanced Mathematics —page 4
Competence statements FP1 m6, m7, m8, m9, m10, m11, m12, m13

References:
Chapter 2
Pages 46-49

Complex Numbers

The complex number, **j**, is such that $\mathbf{j}^2 = -1$. Thus $\mathbf{j} = \sqrt{-1}$.

A complex number takes the general form $a + b\mathbf{j}$.

Complex numbers can be added and subtracted in the usual algebraic way:
E.g. $(a + b\mathbf{j}) + (c + d\mathbf{j}) = (a + c) + (b + d)\mathbf{j}$

Complex numbers can be multiplied in the usual way,
using $\mathbf{j}^2 = -1$.
E.g. $(a + b\mathbf{j})(c + d\mathbf{j}) = ac + ad\mathbf{j} + bc\mathbf{j} + bd\mathbf{j}^2$
$= (ac - bd) + (ad + bc)\mathbf{j}$.

Exercise 2A
Q. 1(ii), (vi), 2(ii), (vi)

A quadratic can now be described as always having two roots, though they may be complex. (Note that if one is complex then so is the other.)

E.g. $z^2 + 4z + 13 = 0 \Rightarrow z = \dfrac{-4 \pm \sqrt{16-52}}{2} = \dfrac{-4 \pm \sqrt{-36}}{2}$
$= \dfrac{-4 \pm 6\mathbf{j}}{\mathbf{2}} = -2 \pm 3\mathbf{j}$

Note that two complex numbers can be added to give a real number
E.g. (3 + 2j) + (3 – 2j) = 6

Two complex numbers may also be multiplied to give a real number.
E.g. (3 + 2j) × (3 – 2j) = 9 + 4 = 13

E.g. $(3+2\mathbf{j})+(4-3\mathbf{j}) = (3+4)+(2-3)\mathbf{j}$
$= 7 - \mathbf{j}$

E.g. $(3+2\mathbf{j})-(4-3\mathbf{j}) = (3-4)+(2+3)\mathbf{j}$
$= -1 + 5\mathbf{j}$

E.g. $(3+2\mathbf{j})\times(4-3\mathbf{j})$
$= 3\times4 + 3\times(-3\mathbf{j}) + 2\mathbf{j}\times4 + 2\mathbf{j}\times(-3\mathbf{j})$
$= 12 + 8\mathbf{j} - 9\mathbf{j} - 6\mathbf{j}^2$
$= 12 - \mathbf{j} + 6 = 18 - \mathbf{j}$

E.g. Solve the equation $z^2 + 4z + 7 = 0$

$$z = \frac{-4 \pm \sqrt{4^2 - 4\times7}}{2} = \frac{-4 \pm \sqrt{-12}}{2}$$
$$= -2 \pm \sqrt{3}\mathbf{j}$$

Solution by the method of completing the square:

$z^2 + 4z + 7 = 0 \Rightarrow z^2 + 4z = -7$

$\Rightarrow z^2 + 4z + 4 = 4 - 7 = -3 \Rightarrow (z+2)^2 = -3$

$\Rightarrow z + 2 = \pm\sqrt{-3} \Rightarrow z = -2 \pm \sqrt{3}\mathbf{j}$

References:
Chapter 2
Page 49

Complex conjugates

If $z = x + y\mathbf{j}$, then $z^* = x - y\mathbf{j}$ is called the complex conjugate of z.

Exercise 2A
Q. 5

Complex roots of a quadratic equation occur in conjugate pairs.
E.g. If $2 + 3\mathbf{j}$ is a root then the other one is $2 - 3\mathbf{j}$

E.g. A root of a quadratic equation is $-5-3\mathbf{j}$
(i) State the other root,
(ii) Find the equation.

(i) One root is $-5-3\mathbf{j}$ so the other is $-5+3\mathbf{j}$
(ii) The equation is $\left(z-(-5-3\mathbf{j})\right)\left(z-(-5+3\mathbf{j})\right) = 0$
$\Rightarrow z^2 - z(-5+3\mathbf{j}) - z(-5-3\mathbf{j}) + (-5-3\mathbf{j})(-5+3\mathbf{j}) = 0$
$\Rightarrow z^2 + 10z + (25+9) = 0$
$\Rightarrow z^2 + 10z + 34 = 0$

References:
Chapter 2
Pages 49-52

Division

Remember that $(x + y\mathbf{j})(x - y\mathbf{j}) = x^2 + y^2$

Then $\dfrac{1}{x + y\mathbf{j}} = \dfrac{x - y\mathbf{j}}{(x + y\mathbf{j})(x - y\mathbf{j})} == \dfrac{x - y\mathbf{j}}{x^2 + y^2}$

Exercise 2B
Q. 1(ii), 2(ii) 4(iii),9

Therefore a fraction with a complex number in the denominator may be “rationalised” by multiplying top and bottom of the fraction by the complex conjugate of the denominator.

N.B. The process is the same as for surds:

Then $\dfrac{1}{4+\sqrt{3}} = \dfrac{4-\sqrt{3}}{(4+\sqrt{3})(4-\sqrt{3})} == \dfrac{4-\sqrt{3}}{16-3} = \dfrac{1}{13}(4-\sqrt{3})$

E.g. Rationalise $\dfrac{2+3\mathbf{j}}{3+4\mathbf{j}}$

$\dfrac{2+3\mathbf{j}}{3+4\mathbf{j}} = \dfrac{(2+3\mathbf{j})(3-4\mathbf{j})}{(3+4\mathbf{j})(3-4\mathbf{j})}$
$= \dfrac{6 - 8\mathbf{j} + 9\mathbf{j} + 12}{9+16}$
$= \dfrac{1}{25}(18 + \mathbf{j})$

E.g. Solve the equation $2x + \mathbf{j} = 2 - \mathbf{j}x$
$2x + \mathbf{j} = 2 - \mathbf{j}x \Rightarrow 2x + \mathbf{j}x = 2 - \mathbf{j}$
$\Rightarrow (2+\mathbf{j})x = (2-\mathbf{j}) \Rightarrow x = \dfrac{2-\mathbf{j}}{2+\mathbf{j}}$
$= \dfrac{(2-\mathbf{j})(2-\mathbf{j})}{(2+\mathbf{j})(2-\mathbf{j})} = \dfrac{4 - 2\mathbf{j} - 2\mathbf{j} - 1}{4+1}$
$= \dfrac{1}{5}(3 - 4\mathbf{j})$

Summary FP1 Topic 2: ***Complex Numbers***

References: Chapter 2 Pages 55-57

Exercise 2C Q 1(iii), 2(ii),(v)

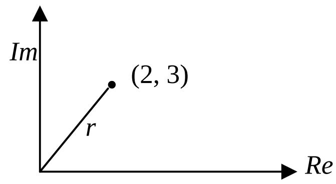
The Argand diagram
The complex number $x + y\mathbf{j}$ can be represented geometrically on a coordinate system by the point (x, y).
Real numbers (i.e. $(x, 0)$) form the x axis which is therefore called the real axis.
Imaginary numbers (i.e. $(0, y)$) form the y axis which is therefore called the imaginary axis.
Such a representation is called an **Argand diagram.**

Complex conjugates are reflections in the imaginary (y) axis.

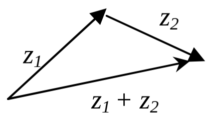
$x + y\mathbf{j}$ can be described by the vector $\begin{pmatrix} x \\ y \end{pmatrix}$. The sum of complex numbers can therefore be seen as the sum of vectors.

The modulus is the length of the complex number, $\sqrt{x^2 + y^2}$

E.g. Show the number $r = 2 + 3\mathbf{j}$ on an argand diagram.



$z_1 = 2 + 2\mathbf{j}$, $z_2 = 2 - \mathbf{j}$



References: Chapter 2 Pages 58-59

Exercise 2D Q. 1(v), 2

Loci

The distance between z_1 and z_2 is $|z_1 - z_2|$.
$|z - z_1| = k$ is satisfied by all points, z, which are k units from z_1.
i.e. the locus of z is a circle, centre z_1 and radius k

E.g. The locus of all points satisfying

$$|z - 2 - 3\mathbf{j}| = 5$$

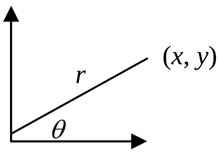
is a circle, centre $2 + 3\mathbf{j}$ radius 5

References: Chapter 2 Pages 61-65

Exercise 2E Q 1(ii), 2(ix), 3(iii)

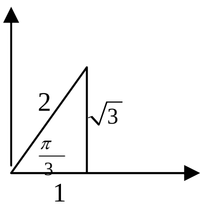
Modulus and Argument form for complex numbers.
A complex number can be described in Cartesian form.
i.e. $z = x + y\mathbf{j}$ describes the position vector from the origin to the point (x, y).

It can also be described by means of the length of the number together with the angle it makes with the positive real axis.



The distance, r, is the modulus of z.
The angle is measured anticlockwise from the real axis and is measured in radians. Without restricting the range of θ this is not unique, so it is usual to define θ as the principal angle in the range $-\pi < \theta \le \pi$.

E.g. the point $(1, \sqrt{3})$ can be written in modulus-argument form as $\left(2, \frac{\pi}{3}\right)$.



References: Chapter 2 Pages 66-67

Loci
$\arg z$ = const. is a straight line.

Exercise 2F Q. 1(ii), 3

References: Chapter 2 Pages 69-71

Exercise 2G Q. 3, 9

Equations
On page 6 it was seen that if a quadratic equation has no real roots then the two complex roots occur in complex conjugate pairs.
i.e. if one root is $a + b\mathbf{j}$ then the other is $a - b\mathbf{j}$.
This applies more generally to a polynomial equation of degree n.

If a polynomial equation has real coefficients then complex roots occur in complex conjugate pairs.

i.e. a cubic equation has either three real roots or one real root and two complex roots which are a conjugate pair.

E.g. Solve the equation $z^3 + 2z - 3 = 0$.
By inspection f(1) = 0 so one root is $z = 1$

$\Rightarrow (z-1)(z^2 + z + 3) = 0$

The roots of $z^2 + z + 3 = 0$ are $z = \dfrac{-1 \pm \sqrt{1-12}}{2}$

$\Rightarrow z = \dfrac{1}{2}\left(-1 \pm \sqrt{11}\mathbf{j}\right)$ and $z = 1$

FP1; Further Concepts for Advanced Mathematics —page 6
Competence statements FP1, j5, j6, j7, j8, j9

Summary FP1 Topic 3: **Curve Sketching**

References:
Chapter 3
Pages 74-76

Discontinuities and Asymptotes
If a function contains a fraction with an expression for x in the denominator such that for a given value of x the denominator is zero, then there is a discontinuity at that value of x.

E.g. $y = \dfrac{x}{x-3} + 5$ has a discontinuity at $x = 3$.
The curve approaches, but does not cross, the line $x = 3$.
This line is called an **asymptote**.

The **power** of a fraction is the difference between the powers in the numerator and denominator.

E.g. $y = \dfrac{x}{x-3} + 5$ has power $1 - 1 = 0$.

$y = \dfrac{x^2}{x-3} + 5$ has power $2 - 1 = 1$.

$y = \dfrac{x}{x^2-3} + 5$ has power $1 - 2 = -1$.

If the fraction has a negative power then the curve will approach $y = 0$
If the fraction has power 0 then it will approach the line $y = a$.
If the power is 1 then the curve will approach the line $y = mx + c$.

References:
Chapter 3
Pages 77-85

Exercise 3A
Q. 1, 7, 15

Sketching Curves
See also the topic in C1 and C2.

1. Find where the curve cuts the axes.
2. Find the vertical asymptotes and examine the behaviour of the graph either side of them.
3. Examine the behaviour as x tends to infinity.
4. Complete the sketch.

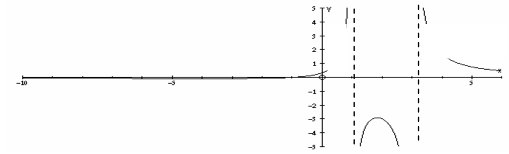
References:
Chapter 3
Pages 87-92

Exercise 3B
Q. 1, 6(iii)

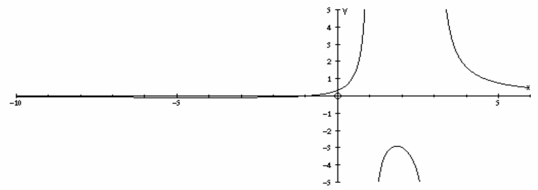
Inequalities
There are two ways to solve an inequality f(x) >0
(i) Draw the graph of y = f(x), identifying the region satisfied by the inequality
(ii) Manipulate the inequality algebraically.
Inequalities may be manipulated like equations ***except***:
Any number multiplying or dividing both sides must be positive
Care is needed with subtraction.

E.g. Sketch the curve $y = \dfrac{x+1}{(x-1)(x-3)}$

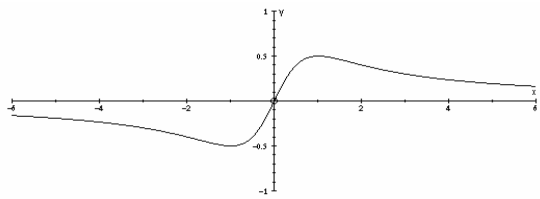
(i) $x = 0, y = 0.33;\ \ y = 0, x = -1$
(ii) Discontinuities at $x = 1,\ x = 3$



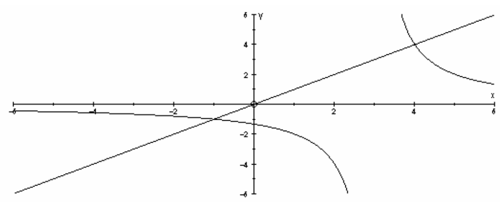
(iii) $x > 3 \Rightarrow y > 0;\ 1 < x < 3 \Rightarrow y < 0,$
$-1 < x < 1 \Rightarrow y > 0;\ \ x < -1 \Rightarrow y < 0$
(iv) As $x \to -\infty,\ y \to 0$ from below
As $x \to \infty,\ y \to 0$ from above



E.g. Sketch the graph of $y = \dfrac{x}{x^2+1}$.
(i) When $x = 0, y = 0$.
(ii) There are no discontinuities.
(iii) As $x \to \infty, y \to 0$ from above
As $x \to -\infty, y \to 0$ from below



E.g. Solve the inequality $x < \dfrac{4}{x-3}$ by algebraic means and also by drawing graphs.
Providing $x > 3,\ x(x-3) < 4 \Rightarrow x^2 - 3x - 4 < 0$
$\Rightarrow (x-4)(x+1) < 0 \Rightarrow -1 < x < 4$
i.e. $3 < x < 4$
If $x < 3,\ x(x-3) > 4 \Rightarrow x^2 - 3x - 4 > 0$
$\Rightarrow (x-4)(x+1) > 0 \Rightarrow x > 4$ and $x < -1$
i.e. $x < -1$
i.e. $3 < x < 4$ and $x < -1$
From the two graphs the inequalities derived above can be deduced.



FP1; Further Concepts for Advanced Mathematics —page 7
Competence statements FP1, c1, c2, c3, a4

Summary FP1 Topic 4: **Algebra**

References:
Chapter 4
Pages 97-99

Exercise 4A
Q. 4, 5

Identities
Two different ways of writing the same expression are identically equal. The expressions will have the same value for all values of x.

E.g. $2(x+3) \equiv 2x+6$

$x^2-4x+3 \equiv (x-3)(x-1)$

In an identity the coefficient of equal powers on each side will be the same.

E.g. If $x^2+2x-5 \equiv (x-1)(x-a)+b$, find a and b.

R.H.S $\equiv x^2-(a+1)x+(a+b)$

Compare with coefficients on L.H.S

$\Rightarrow a+1=-2 \Rightarrow a=-3$,

$a+b=-5 \Rightarrow b=-2$

i.e. $x^2+2x-5 \equiv (x-1)(x+3)-2$

References:
Chapter 4
Pages 100-104

Exercise 4B
Q. 3, 4

References:
Chapter 4
Pages 105-107

Exercise 4C
Q 1, 8, 9

Roots of polynomials

1. A quadratic equation has two roots. They may be real and distinct, real and coincident, or complex (in which case they are a complex conjugate pair).

If the two roots of $ax^2+bx+c=0$ are α and β

$\alpha+\beta=-\frac{b}{a}$, $\alpha\beta=\frac{c}{a}$

2. A cubic has 3 roots, at least one of which is real. The other two roots may be real and coincident or complex (in which case they are a complex conjugate pair.)

If the three roots of $ax^3+bx^2+cx+d=0$ are α, β and γ

$\alpha+\beta+\gamma=\sum\alpha=-\frac{b}{a}$,

$\alpha\beta+\beta\gamma+\gamma\alpha=\sum\alpha\beta=\frac{c}{a}$,

$\alpha\beta\gamma=-\frac{d}{a}=\sum\alpha\beta\gamma$

You should be familiar with the process of finding an equation whose roots are related to a given equation.

E.g. If the roots of an equation are α,β and γ and the roots of a second equation are $\alpha',\beta'$and γ' where there is a symmetric relationship (e.g. $\alpha'=\alpha+1$, etc) then you need to be able to find $\alpha'+\beta'+\gamma'$, $\alpha'\beta'+\beta'\gamma'+\gamma'\alpha'$ and $\alpha'\beta'\gamma'$ in terms of α,β and γ.

References:
Chapter 4
Pages 111-112

Exercise 4D
Q. 1, 4

3. A quartic equation has 4 roots. Complex roots occur in conjugate pairs, so there are either 0, 2 or 4 real roots (some of which may be coincident).

If the four roots of $ax^4+bx^3+cx^2+dx+e=0$ are α, β, γ and δ

$\alpha+\beta+\gamma+\delta=\sum\alpha=-\frac{b}{a}$,

$\alpha\beta+\beta\gamma+\gamma\delta+\alpha\gamma+\alpha\delta+\beta\delta=\sum\alpha\beta=\frac{c}{a}$,

$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=\sum\alpha\beta\gamma=-\frac{d}{a}$,

$\alpha\beta\gamma\delta=\frac{e}{a}=\sum\alpha\beta\gamma\delta$

E.g. The roots of $x^2+2x+5=0$ are α and β.

Find the equation with roots α^2 and β^2.

$\Rightarrow \alpha+\beta=-2,\ \alpha\beta=5$,

$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=4-10=-6$

$\alpha^2\beta^2=25$

i.e. $x^2+6x+25=0$

E.g. If the roots of the equation

$$x^3+2x+4=0$$

are α,β and γ then find the equation whose roots are $\alpha+1$, $\beta+1$ and $\gamma+1$.

$\alpha+\beta+\gamma=0$,

$\alpha\beta+\beta\gamma+\gamma\alpha=2,\ \ \alpha\beta\gamma=-4$

$\Rightarrow \alpha+1+\beta+1+\gamma+1=3$

$(\alpha+1)(\beta+1)+(\beta+1)(\gamma+1)$
$\qquad +(\gamma+1)(\alpha+1)$

$=(\alpha\beta+\beta\gamma+\gamma\alpha)+2(\alpha+\beta+\gamma)+3$

$=2+3=5$

$(\alpha+1)(\beta+1)(\gamma+1)$

$=\alpha\beta\gamma+(\alpha\beta+\beta\gamma+\gamma\alpha)+(\alpha+\beta+\gamma)+1$

$=-4+2+1=-1$

$\Rightarrow x^3-3x^2+5x+1=0$

Notice that this may have also been achieved by the transformation $w=x+1$.

$x^3+2x+4=0 \Rightarrow (w-1)^3+2(w-1)+4=0$

$\Rightarrow w^3-3w^2+3w+2w-1-2+4=0$

$\Rightarrow w^3-3w^2+5w+1=0$

E.g. If the roots of the equation $x^2+3x+8=0$ are α and then find the equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

Substitute $x=\frac{1}{z}$. The equation in z will have roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

$\Rightarrow 1+3z+8z^2=0$

References:
Chapter 5
Pages 115-118

Exercise 5A
Q. 1, 8, 9

Induction
Given a "conjecture", it is proved by induction as follows:
1. Show that it is true for a particular value of n, typically, $n = 0$ or 1.
2. Assume it to be true for $n = k$.
3. Based on this assumption prove that it is also true for $n = k + 1$.

A conjecture can be disproved by a single counter-example.

The conjecture might be a formula for the sum of a finite series.

E.g. Prove that $1\times3+2\times4+...+n\times(n+2)=\frac{1}{6}n(n+1)(2n+7)$

It is true for $n=1$, for $1\times3=\frac{1}{6}.1.2.9$

Assume true for $n=k$.

i.e. $S_k=\frac{1}{6}k(k+1)(2k+7)$

Then $S_{k+1}=\frac{1}{6}k(k+1)(2k+7)+(k+1)(k+3)$

$$=\frac{1}{6}(k+1)(k(2k+7)+6(k+3))$$
$$=\frac{1}{6}(k+1)(2k^2+13k+18)$$
$$=\frac{1}{6}(k+1)(k+2)(2k+9)$$

which is of the same form as the formula given with k replaced by $k+1$.

So if true for $n=k$ then also true for $n=k+1$

But since it is true for $n=1$, it is true for all positive integers, n.

References:
Chapter 5
Page 120

Sequences
If a sequence is defined *inductively* – i.e. if each term is defined by the term before it – then a formula may be derived, or if given may be proved by induction.

E.g. A sequence is defined by $u_{n+1} = 2u_n + 1$ for positive integers, n with $u_1 = 1$. Prove that $u_n = 2^n - 1$

It is true for $n = 1$, since $u_1 = 2^1 - 1$.

Assume true for $n = k$.
i.e. $u_k = 2^k - 1$.
Then $u_{k+1} = 2u_k + 1 = 2(2^k - 1) + 1 = 2^{k+1} - 2 + 1$
$= 2^{k+1} - 2 + 1$

So if true for $n = k$ then it is also true for $n = k + 1$
But since it is true for $n = 1$, the formula is true for all positive integers, n.

References:
Chapter 5
Page 120

Exercise 5B
Q. 1, 5

Summation of Series to infinity.
A series is said to **converge** if , as n approaches ∞, the sum approaches a finite number.
You met the condition for a geometric series to converge in C2.

$$S_n=a+ar+ar^2+....+ar^{n-1}=\frac{a(1-r^n)}{1-r}$$

$S_n\to S=\frac{a}{1-r}$ providing $|r|<1$.

Other series also converge.

E.g. Show that $\frac{1}{r^2-1}=\frac{1}{2}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)$.

Hence find $S_n=\sum_2^n\frac{1}{r^2-1}$ and deduce the sum to infinity.

$$\frac{1}{2}\left(\frac{1}{r-1}-\frac{1}{r+1}\right)=\frac{1}{2}\left(\frac{r+1-(r-1)}{(r+1)(r-1)}\right)=\frac{1}{r^2-1}$$

$$S_n=\sum_2^n\frac{1}{r^2-1}=\frac{1}{2}\sum_2^n\left(\frac{1}{r-1}-\frac{1}{r+1}\right)$$
$$=\frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+......+\frac{1}{n-2}-\frac{1}{n}+\frac{1}{n-1}-\frac{1}{n+1}\right)$$
$$=\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}\right)=\frac{1}{2}\left(\frac{3}{2}-\frac{2n+1}{n(n+1)}\right)$$

$$\Rightarrow S_\infty=\frac{3}{4}$$

References:
Chapter 5
Pages 122-124

Exercise 5C
Q. 1, 5

The method of differences
If the terms of one series can be expressed as the difference of consecutive terms of another series then the series can be summed by the cancellation of middle terms.

E.g. If $t_r=s_r-s_{r-1}$

then $\sum_1^n t_r=(s_1-s_0)+(s_2-s_1)+...+(s_n-s_{n-1})$

$$=s_n-s_0$$

Standard results may sometimes be used.

E.g. Find $\sum_{r=1}^n(2r+1)$.

$\sum_{r=1}^n(2r+1)=2\sum_{r=1}^n r+n$ Since $\sum_{r=1}^n r=\frac{1}{2}n(n+1)$

$\sum_{r=1}^n(2r+1)=n(n+1)+n=n(n+2)$

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Competence statements FP1, p3, p4, a1, a2, a3