REVISION SHEET – FP1 (MEI)

ALGEBRA

The main ideas are:

- Identities
- The relationships between roots and coefficients in polynomial equations
- Finding a polynomial equation with roots related to that of a given one

Before the exam you should know:

- If an expression is an identity then it is true for all values of the variable it contains.

  If A and B are constants and $A(x - 1) + B(x + 1) = 4x + 2$ is an identity, then by substituting $x = 1$, $2B = 6$ and by substituting $x = -1$, $-2A = -2$. So $B = 3$ and $A = 1$.

- The method of substitution for finding a polynomial equation with roots related to a given one.

  E.g., if $3x^3 + 2x^2 - 7x + 4 = 0$ has roots α, β, γ then $3(y-3)^3 + 2(y-3)^2 - 7(y-3) + 4 = 0$ will have roots $\alpha + 3, \beta + 3, \gamma + 3$.

Identities

In mathematics, an identity is a statement which is true for all values of the variables it contains.

For example $3(x+2) \equiv 3x+6$ is an identity because no matter what value of x you substitute in, the left hand side is always the same as the right hand side.

$3(x+2) = 3$ is not an identity, because the left hand side only equals the right hand side when $x = -2$.

If you know that a statement is an identity then you can substitute any values for the variables in it and know that the resultant expression is true. The following question is typical.

Example

You are given that $A(x-1)(x-2) + Bx(x-1) + Cx(x-2) \equiv x^2 + x + 1$ is an identity. Find the values of A, B and C.

Solution

Since the statement is an identity, it is true for any value of x.

Substituting $x = 1$ gives: $A(1-1)\times(1-2) + B\times1\times(1-1) + C\times1\times(1-2) \equiv 1^2 + 1 + 1$

$\Rightarrow -C = 3 \Rightarrow C = -3$.

Substituting $x = 2$ gives: $A(2-1)\times(2-2) + B\times2\times(2-1) + C\times2\times(2-2) \equiv 2^2 + 2 + 1$

$$\Rightarrow 2B = 7 \Rightarrow B = \frac{7}{2}$$

Substituting $x = 0$ gives: $A(0-1)\times(0-2) + B\times0\times(2-1) + C\times0\times(2-2) \equiv 0^2 + 0 + 1$

$$\Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$$

Roots and coefficients in polynomial equations

Quadratic: If $ax^2+bx+c=0$ has roots α and β then

$$\alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}.$$

Cubic: If $ax^3+bx^2+cx+d=0$ has roots α,β and γ then

$$\alpha+\beta+\gamma=-\frac{b}{a},\ \alpha\beta+\beta\gamma+\alpha\gamma=\frac{c}{a} \text{ and } \alpha\beta\gamma=-\frac{d}{a}.$$

Example

The cubic equation $2x^3+4x^2+3x+1=0$ has roots α,β and γ.

i) Write down the values of $\alpha+\beta+\gamma$, $\alpha\beta+\beta\gamma+\gamma\alpha$ and $\alpha\beta\gamma$.

ii) Find a cubic equation with integer coefficients with roots $2\alpha-1, 2\beta-1, 2\gamma-1$.

Solution

i) $\alpha+\beta+\gamma=-2$, $\alpha\beta+\beta\gamma+\gamma\alpha=\dfrac{3}{2}$ and $\alpha\beta\gamma=-\dfrac{1}{2}$.

ii) $(2\alpha-1)+(2\beta-1)+(2\gamma-1)=2(\alpha+\beta+\gamma)-3=-4-3=-7$

$(2\alpha-1)(2\beta-1)+(2\beta-1)(2\gamma-1)+(2\alpha-1)(2\gamma-1)=4(\alpha\beta+\beta\gamma+\alpha\gamma)-4(\alpha+\beta+\gamma)+3=6+8+3=17$

$(2\alpha-1)(2\beta-1)(2\gamma-1)=8\alpha\beta\gamma-4(\alpha\beta+\beta\gamma+\alpha\gamma)+2(\alpha+\beta+\gamma)-1=-4-6-4-1=-15$

Therefore a cubic with integer coefficient with roots $2\alpha-1, 2\beta-1, 2\gamma-1$ is $x^3+7x^2+17x+15=0$

Example (Substitution Method)

The cubic equation $2x^3+4x^2+3x+1=0$ has roots α,β and γ. Find a cubic equation with integer coefficients with roots $2\alpha+1, 2\beta+1, 2\gamma+1$.

Solution

Let $w=2z+1$ so that $z=\dfrac{w-1}{2}$.

Since α,β and γ are the roots of $2x^3+4x^2+3x+1=0$, $2\alpha+1, 2\beta+1, 2\gamma+1$ are the roots of

$$2\left(\frac{w-1}{2}\right)^3+4\left(\frac{w-1}{2}\right)^2+3\left(\frac{w-1}{2}\right)+1=0.$$

Multiplying this out and multiplying both sides by 4 gives

$$(w-1)^3+4(w-1)^2+6(w-1)+4=0$$
$$w^3-3w^2+3w-1+4w^2-8w+4+6w-6+4=0$$
$$w^3+w^2+w+1=0$$

REVISION SHEET – FP1 (MEI)

COMPLEX NUMBERS

The main ideas are:

- Manipulating complex numbers
- Complex conjugates and roots of equations
- The Argand diagram
- Polar Form

Before the exam you should know:

- How to multiply two complex numbers quickly and in one step as this will save a lot of time in the exam.
- How to geometrically interpret $|z_1 - z_2|$ as the distance between the complex numbers z_1 and z_2 in the Argand diagram.
- The fact that $|z_1 + z_2| = |z_1 - (-z_2)|$ which equals the distance between z_1 and $-z_2$ in the Argand diagram.
- The exact values of the sine and cosine angles which are multiplies of $\frac{\pi}{6}$ and $\frac{\pi}{4}$, e.g. $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

Manipulating Complex Numbers.

Multiplying, dividing, adding and subtracting

- Multiplying, adding and subtracting are all fairly straightforward.
- Dividing is slightly more complicated. Whenever you see a complex number on the denominator of a fraction you can "get rid of it" by multiplying both top and bottom of the fraction by its complex conjugate.

e.g. $\frac{3+2\text{j}}{1-\text{j}} = \left(\frac{3+2\text{j}}{1-\text{j}}\right)\left(\frac{1+\text{j}}{1+\text{j}}\right) = \frac{1+5\text{j}}{2}$

Complex Conjugates and Roots of Equations

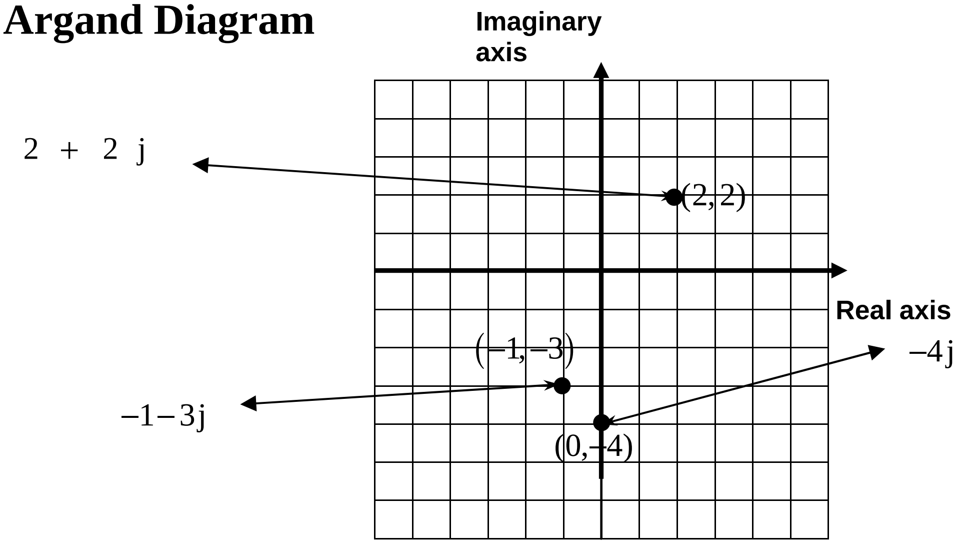
The complex conjugate of $z = a + bj$ is $z^* = a - bj$.

- Remember zz^* is a real number and it equals the square of the modulus of z.
- Complex roots of polynomial equations with real coefficients occur in conjugate pairs. This means that if you are told one complex root of a polynomial equation with real coefficients you are in fact being told two roots. This is key to answering some typical exam questions.

An example of an algebraic trick that it is very useful to know is:

$$\begin{aligned}(z-(3+2\text{j}))(z-(3-2\text{j})) &= ((z-3)-2\text{j})((z-3)+2\text{j}) \\ &= (z-3)^2 - 4\text{j}^2 \\ &= z^2 - 6z + 13\end{aligned}$$

The Argand Diagram



- In the Argand diagram the point (x, y) corresponds to the complex number $x + y$j.
- You should be aware that the set of complex numbers z with for example $|z-5+j|=6$ is a circle of radius 6 centred at $5-\mathrm{j}$ (or (5, –1)) in the Argand plane. *(Note: Points on the diagram above do not correspond to this example)*
- The argument of a complex number z, denoted $\arg(z)$ is the angle it makes with the positive real axis in the Argand diagram, measured anticlockwise and such that $-\pi < \arg(z) \le \pi$.
- When answering exam questions about points in the Argand diagram be prepared to used geometrical arguments based around equilateral triangles, similar triangles, isosceles triangles and parallel lines to calculate lengths and angles.

Other sets of points in the complex plane.

Where a and b are complex numbers, the set of complex numbers z such that

1. $\arg(z-a)=\theta$, is a half line starting from a in the direction θ.
2. $\arg(z-a)=\arg(z-b)$, is the line through a and b with the section between a and b (inclusive) removed.
3. $\arg(z-a)=\arg(z-b)+\pi$, is the line from a to b (not including a and b themselves).

Polar Form

If $z=x+y\mathrm{j}$ has $|z|=r$ and $\arg(z)=\theta$ then $z=r(\cos\theta+\mathrm{j}\sin\theta)$. This is called the *polar* or *modulus-argument* form.

Example Write $z=3-3\mathrm{j}$ in polar form.

Solution

$|z|=\sqrt{3^2+(-3)^2}=3\sqrt{2}$ and $\arg(z)=\frac{3\pi}{4}$. Therefore in polar form z is $z=3\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right)+\mathrm{j}\sin\left(\frac{3\pi}{4}\right)\right)$

Example If $z=6\left(\cos\left(\frac{3\pi}{7}\right)+\mathrm{j}\sin\left(\frac{3\pi}{7}\right)\right)$, what are $|z|$ and $\arg(z)$?

Solution Since z is given in polar form we can just read off that $|z|=6$ and $\arg(z)=\frac{3\pi}{7}$

REVISION SHEET – FP1 (MEI)

GRAPHS AND INEQUALITIES

The main ideas are:

- Sketching Graphs of Rational Functions
- Solving Inequalities

Before the exam you should know:

- There are three main cases of horizontal asymptotes.
  - One is a curve which is a linear polynomial divided by a linear polynomial, for example $y = \frac{4x+1}{3x-2}$. This has a horizontal asymptote at $y = \frac{\text{coefficient of } x \text{ on the top}}{\text{coefficient of } x \text{ on the bottom}}$. Here this would be $y = \frac{4}{3}$.
  - The second is a curve given by a quadratic polynomial divided by a quadratic polynomial, for example, $y = \frac{5x^2+x+5}{2x^2-2x+1}$ as $x \to \pm\infty$, This has a horizontal asymptote at $y = \frac{\text{coefficient of } x^2 \text{ on the top}}{\text{coefficient of } x^2 \text{ on the bottom}}$. Here this would be $y = \frac{5}{2}$.
  - Thirdly, when the curve is given by a linear polynomial divided by a quadratic polynomial, it will generally have the *x*-axis (y = 0) as a horizontal asymptote.
- When you solve an inequality, try substituting a few of the values for which you are claiming it is true back into the original inequality as a check.

Graph Sketching.

Rational functions.

To sketch the graph of $y = \frac{N(x)}{D(x)}$:

- Find the intercepts – that is where the graph cuts the axes.
- Find any asymptotes – the vertical asymptotes occur at values of *x* which make the denominator zero.
- Examine the behaviour of the graph near to the vertical asymptotes; a good way to do this is to find out what the value of *y* is for values of *x* very close to the vertical asymptote
- Examine the behaviour around any non-vertical asymptotes, i.e. as *x* tends to $\pm\infty$.

Example Sketch the curve $y = \frac{x^2-2}{4-x^2}$

Solution (Sketch)

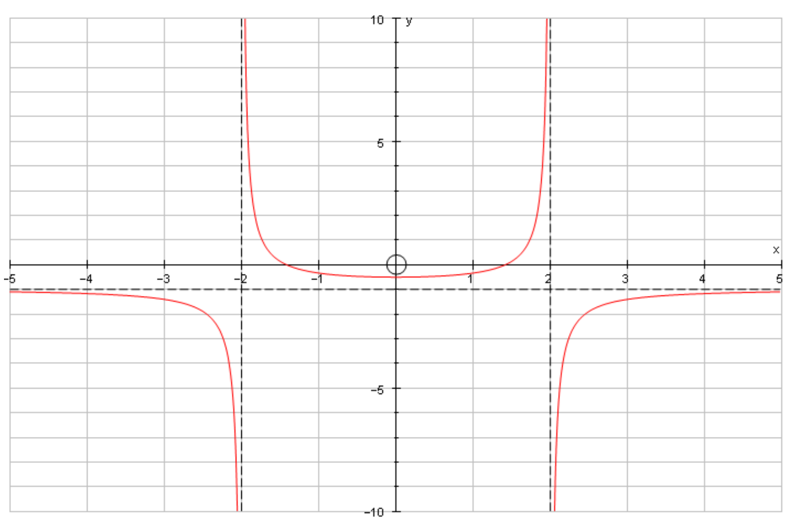
The curve can be written as $y = \frac{x^2-2}{(x+2)(x-2)}$.

If $x = 0$ then $y = -0.5$. So the *y* intercept is (0, -0.5)
Setting $y = 0$ gives, $x = -2$ and $x = 2$. So the *x* intercepts are $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$.

The denominator is zero when $x = -2$ and when $x = 2$ so these are the vertical asymptotes

Also $y = \frac{x^2-2}{4-x^2} = \frac{-1(-x^2+4)+2}{4-x^2} = -1 + \frac{2}{4-x^2}$

so $y = -1$ is a horizontal asymptote.



Solving Inequalities

Broadly speaking inequalities can be solved in one of three ways, or sometimes in a combination of more than one of these ways.

Method 1

Draw a "sketch" of the inequality. For example, if you are asked to solve an inequality of the form $g(x) \le f(x)$ then sketch both f and g, and identify points where the graph of f is lower than the graph of g. These points will lie between points x_1 for which $g(x_1) = f(x_1)$ and so these usually need to be calculated.

Method 2

Use algebra to find an equivalent inequality which is easier to solve. When dealing with inequalities remember there are certain rules which need to be obeyed when performing algebraic manipulations. The main one is "DON'T MULTIPLY BY A NUMBER UNLESS YOU KNOW IT'S SIGN, IF IT'S NEGATIVE YOU MUST REVERSE THE INEQUALITY SIGN, IF IT'S POSITIVE THEN LEAVE THE INEQUALITY SIGN AS IT IS." For example, don't multiply by $(x – 2)$ because that's positive when $x > 2$ and negative when $x < 2$. On the other hand $(x – 2)^2$ is always positive so you can safely multiply by this (with no need to reverse the inequality sign).

Method 3

Sometimes is easier to rearrange an inequality of the from $g(x) \le f(x)$ to $g(x) - f(x) \le 0$ (you don't have to worry about reversing the inequality for such a rearrangement). Identify points where $g(x) – f(x) = 0$ or where $g(x) – f(x)$ has a vertical asymptote. Finally test whether the inequality is true in the various regions between these points.

Example Solve the inequality $3x - 2 \le \dfrac{x+2}{x-1}$

Solution (using Method 3)

$$3x - 2 \le \frac{x+2}{x-1} \Leftrightarrow 3x - 2 - \frac{x+2}{x-1} \le 0$$

$$\Leftrightarrow \frac{(3x-2)(x-1)-(x+2)}{x-1} \le 0$$

$$\Leftrightarrow \frac{3x^2 - 6x}{x-1} \le 0$$

$$\Leftrightarrow \frac{3x(x-2)}{x-1} \le 0$$

Looking at the expression, $3x = 0$ if $x = 0$, $x – 2 = 0$ if $x = 2$ and $x – 1 = 0$ if $x = 1$.
This means that the truth of the inequality should be tested in each of the following regions

$x < 0$ | 0 | $0 < x < 1$ | 1 | $1 < x < 2$ | 2 | $x > 2$

It can be seen that the inequality is TRUE if $x < 0$, false if $0 < x < 1$, TRUE if $1 < x < 2$ and FALSE if $x > 2$. The solution is therefore $x \le 0$, $1 < x \le 2$. Can you see why $x = 0$ and $x = 2$ are included as values for which the inequality is true, but $x = 1$ is not?

REVISION SHEET – FP1 (MEI)

MATRICES

The main ideas are:

- Manipulating matrices
- Using matrices to represent transformations
- Matrices and simultaneous equations
- Invariant Points

Before the exam you should know:

- How to add, subtract and multiply matrices.
- How to calculate the determinant of a 2×2 matrix.
- How to find the inverse of a 2×2 matrix.
- That matrix multiplication is associative, so $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ but not commutative, so $\mathbf{AB} \neq \mathbf{BA}$.
- The standard matrices for rotation, reflection and enlargement and understand how matrices can be combined to represent composite transformations.
- A linear transformation maps a straight line onto another straight line.
- The origin is invariant under a linear transformation.
- The determinant of a transformation matrix gives the area factor of the transformation.
- How matrices can be used to represent simultaneous equations.
- The invariant points of a transformation are not moved by the transformation.
- And are able to identify the invariant points of transformations represented by matrices.

Manipulating matrices.

- Adding and subtracting matrices are straightforward.
- Multiplying matrices is slightly more difficult. Matrices may only be multiplied if they are conformable, that is if the number of columns in the multiplying matrix (the left-hand one) is the same as the number of rows in the matrix being multiplied (the right-hand one). Matrix multiplication is **not** commutative. This means that for two matrices, **A** and **B**, it is **not** generally true that **AB** = **BA** (it is vital that you remember this).

If $\mathbf{A} = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 7 & 8 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 4 & -1 \\ 2 & 3 & -2 \\ 1 & 2 & 1 \end{pmatrix}$ then

$$\mathbf{AB} = \begin{pmatrix} 1\times 2+3\times 2+4\times 1 & 1\times 4+3\times 3+4\times 2 & 1\times -1+3\times -2+4\times 1 \\ 2\times 2+7\times 2+8\times 1 & 2\times 4+7\times 3+8\times 2 & 2\times -1+7\times -2+8\times 1 \end{pmatrix} = \begin{pmatrix} 12 & 21 & -3 \\ 26 & 45 & -8 \end{pmatrix}$$

Notice that **BA** is not possible as, this way round, the matrices are not conformable.

- The determinant of a 2×2 matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is $ad - bc$. If the matrix is representing a 2 dimensional transformation, the determinant gives the **area** factor of the transformation.

- The inverse of matrix **A** is denoted $\mathbf{A}^{-1}$. It has the property that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, where **I** is the identity matrix. Only square matrices have inverses. Identity matrices are always square. The 2×2 identity matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. All identity matrices have 1's on the left to right downwards diagonal and 0's everywhere else. The inverse of a 2×2 matrix is given by swapping the entries on the left to right downwards diagonal, changing the sign of the entries on the other diagonal and dividing the whole thing by the determinant of the original matrix so, if $\mathbf{A} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, $\mathbf{A}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ (note, you can only do this when the determinant isn't zero!).

Using matrices to represent transformations

These are some of the standard transformation matrices, and are worth remembering. Remember that the first column of a matrix is where (1, 0) moves to and the second column is where (0, 1) moves to. This can be useful in exams because using this idea it is possible to derive the matrix of a transformation described in words.

- Rotation through angle θ, anticlockwise about the origin: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.
- Enlargement, scale factor *k*, centre the origin: $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.
- Stretch, factor *a* horizontally, factor *b* vertically: $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

A composite transformation is made up of two or more standard transformations, for example a rotation through angle π, followed by a reflection in $y = x$. The matrix representing a composite transformation is obtained by multiplying the component transformation matrices together. The order is important. The matrix for the composite of the transformation with matrix **M**, followed by the transformation with matrix **N** is **NM**. The order is right to left.

Matrices and simultaneous equations

Matrices can be used to represent systems of simultaneous equations. Matrix techniques can be used to solve them.

E.g. $\left.\begin{array}{r} x+2y=7 \\ 3x+7y=24 \end{array}\right\}$ is equivalent to $\begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 24 \end{pmatrix}$. Using the inverse matrix we have

$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1}\begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 7 \\ 24 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow x = 1$ and $y = 3$. This can be extended to systems of 3 equations in 3 unknowns.

Invariant Points

The invariant points of a transformation are not affected by the transformation. (0, 0) is an invariant point of any matrix transformation, but there could be others too:

Example Find the invariant points of the matrix $\begin{pmatrix} 2 & 3 \\ 2 & 7 \end{pmatrix}$.

Solution

$\begin{pmatrix} 2 & 3 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{array}{c} 2x+3y=x \\ 2x+7y=y \end{array} \Rightarrow \begin{array}{c} x+3y=0 \\ 2x+6y=0 \end{array} \Rightarrow \begin{array}{c} y=-\frac{1}{3}x \\ y=-\frac{1}{3}x \end{array}$ So any point on $y = -\frac{1}{3}x$ is an invariant point

Matrices with zero determinant

Example Show that the matrix $\begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix}$ has determinant zero and that it maps all points onto the line $y = \frac{1}{2}x$.

Solution

The determinant of the matrix is $(2\times 4)-(1\times 8)=0$. Taking a general point (x, y)

$\begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+8y \\ x+4y \end{pmatrix}$. Since $(x+4y) = \frac{1}{2}(2x+8y)$ the point $(2x + 8y, x + 4y)$ is on $y = \frac{1}{2}x$.

REVISION SHEET – FP1 (MEI)

SERIES AND INDUCTION

The main ideas are:

- Summing Series using standard formulae
- Telescoping (or method of differences)
- Proof by Induction

Before the exam you should know:

- The standard formula:

$$\sum_{r=1}^{n} r \ , \sum_{r=1}^{n} r^2 \ , \sum_{r=1}^{n} r^3$$

- And be able to spot that a series like

$$(1\times 2)+(2\times 3)+...+n(n+1)$$

can be written in sigma notation as:

$$\sum_{r=1}^{n} r(r+1)$$

- How to do proof by induction

Summing Series

Using standard formulae

Fluency is required in manipulating and simplify standard formulae sums like:

$$\sum_{r=1}^{n} r\left(r^2+1\right)=\sum_{r=1}^{n} r^3+\sum_{r=1}^{n} r=\frac{n^2\left(n+1\right)^2}{4}+\frac{n\left(n+1\right)}{2}$$

$$=\frac{1}{4}n\left(n+1\right)\left[n\left(n+1\right)+2\right]$$

$$=\frac{1}{4}n\left(n+1\right)\left(n^2+n+2\right).$$

The Method of Differences (Telescoping)

Since $\frac{r+4}{r(r+1)(r+2)}=\frac{2}{r}-\frac{3}{r+1}+\frac{1}{r+2}$ (frequently in exam questions you are told to show that this is true first) it is possible to demonstrate that:

$$\sum_{r=1}^{n}\frac{r+4}{r(r+1)(r+2)}=\left(2-\frac{3}{2}+\frac{1}{3}\right)+\left(\frac{2}{2}-\frac{3}{3}+\frac{1}{4}\right)+\left(\frac{2}{3}-\frac{3}{4}+\frac{1}{5}\right)+...$$

$$+\left(\frac{2}{n-2}-\frac{3}{n-1}+\frac{1}{n}\right)+\left(\frac{2}{n-1}-\frac{3}{n}+\frac{1}{n+1}\right)+\left(\frac{2}{n}-\frac{3}{n+1}+\frac{1}{n+2}\right)$$

In this kind of expression many terms cancel with each other. For example, the (+)$\frac{1}{3}$ in the first bracket cancels with the (–)$\frac{3}{3}$ in the second bracket and the (+)$\frac{2}{3}$ in the third bracket. (subsequent fractions that are cancelling are doing so with terms in the "…" part of the sum.)

This leaves $\sum_{r=1}^{n}\frac{r+4}{r(r+1)(r+2)}=\frac{3}{2}-\frac{2}{n+1}+\frac{1}{n+2}$.

Proof by Induction

1. Using proof by induction to prove a formula for the summation of a series,

E.g., Prove that $\sum_{r=1}^{n}(2r-1)=n^2$.

2. Other miscellaneous questions. These are usually very easy, in fact easier than the questions which fall into the categories above, so long as you don't panic, keep a clear head and apply what you know.

E.g., show that if $M=\begin{pmatrix}5 & 8\\ -2 & -3\end{pmatrix}$ then $M^n=\begin{pmatrix}1+4n & 8n\\ -2n & 1-4n\end{pmatrix}$ for all natural numbers *n*.

Example

Prove by induction that, for all positive integers *n*, $\sum_{r=1}^{n}3r+1=\frac{1}{2}n(3n+5)$.

Solution

When *n* = 1 the left hand side equals $(3\times1)+1=4$. The right hand side is $\frac{1}{2}\times1\times((3\times1)+5)=4$. So the statement is true when *n* = 1.

Assume the statement is true when *n* = *k*. In other words $\sum_{r=1}^{k}3r+1=\frac{1}{2}k(3k+5)$.

We must show the statement would then be true when *n* = *k* + 1, i.e. that $\sum_{r=1}^{k+1}3r+1=\frac{1}{2}(k+1)(3k+8)$.

Now,

$$\begin{aligned}\sum_{r=1}^{k+1}(3r+1)&=\sum_{r=1}^{k}(3r+1)+(3(k+1)+1)\\&=\frac{1}{2}k(3k+5)+(3k+4)\\&=\frac{1}{2}\left[3k^2+5k+6k+8\right]\\&=\frac{1}{2}\left[3k^2+11k+8\right]\\&=\frac{1}{2}(k+1)(3k+8)\end{aligned}$$

So the statement is true when *n* = 1 and if it's true when *n* = k, then it's also true when *n* = *k* +1.

Hence, by induction the statement is true for all positive integers, *n*.