REVISION SHEET – FP1 (OCR)

ALGEBRA

The main ideas are:

- The relationships between roots and coefficients in polynomial equations
- Finding a polynomial equation with roots related to that of a given one

Before the exam you should know:

- Use the relations between the symmetric functions of the roots of polynomial equations and the coefficients.
- The method of substitution which is available for finding a polynomial equation with roots related to a given one.

E.g., if $3x^3+2x^2-7x+4=0$ has roots α,β,γ then $3(y-3)^3+2(y-3)^2-7(y-3)+4=0$ will have roots $\alpha+3,\beta+3,\gamma+3$.

Roots and coefficients in polynomial equations

Quadratic: If $ax^2+bx+c=0$ has roots α and β then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

Cubic: If $ax^3+bx^2+cx+d=0$ has roots α,β and γ then

$$\alpha+\beta+\gamma=-\frac{b}{a},\ \alpha\beta+\beta\gamma+\alpha\gamma=\frac{c}{a} \text{ and } \alpha\beta\gamma=-\frac{d}{a}.$$

Example

The cubic equation $2x^3+4x^2+3x+1=0$ has roots α,β and γ.

i) Write down the values of $\alpha+\beta+\gamma$, $\alpha\beta+\beta\gamma+\gamma\alpha$ and $\alpha\beta\gamma$.

ii) Find a cubic equation with integer coefficients with roots $2\alpha-1,2\beta-1,2\gamma-1$.

Solution

i) $\alpha+\beta+\gamma=-2$, $\alpha\beta+\beta\gamma+\gamma\alpha=\frac{3}{2}$ and $\alpha\beta\gamma=-\frac{1}{2}$.

ii) $(2\alpha-1)+(2\beta-1)+(2\gamma-1)=2(\alpha+\beta+\gamma)-3=-4-3=-7$

$(2\alpha-1)(2\beta-1)+(2\beta-1)(2\gamma-1)+(2\alpha-1)(2\gamma-1)=4(\alpha\beta+\beta\gamma+\alpha\gamma)-4(\alpha+\beta+\gamma)+3=6+8+3=17$

$(2\alpha-1)(2\beta-1)(2\gamma-1)=8\alpha\beta\gamma-4(\alpha\beta+\beta\gamma+\alpha\gamma)+2(\alpha+\beta+\gamma)-1=-4-6-4-1=-15$

Therefore a cubic with integer coefficient with roots $2\alpha-1,2\beta-1,2\gamma-1$ is $x^3+7x^2+17x+15=0$

Example (Substitution Method)

The cubic equation $2x^3+4x^2+3x+1=0$ has roots α,β and γ. Find a cubic equation with integer coefficients with roots $2\alpha+1,2\beta+1,2\gamma+1$.

Solution

Let $w=2z+1$ so that $z=\frac{w-1}{2}$. Since α,β and γ are the roots of $2x^3+4x^2+3x+1=0$,

$2\alpha+1,2\beta+1,2\gamma+1$ are the roots of $2\left(\frac{w-1}{2}\right)^3+4\left(\frac{w-1}{2}\right)^2+3\left(\frac{w-1}{2}\right)+1=0$.

Multiplying this out gives

$$(w-1)^3+4(w-1)^2+6(w-1)+4=0$$
$$w^3-3w^2+3w-1+4w^2-8w+4+6w-6+4=0$$
$$w^3+w^2+w+1=0$$

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COMPLEX NUMBERS

The main ideas are:

- Manipulating complex numbers
- Complex conjugates and roots of equations
- The Argand diagram

Before the exam you should know:

- How to manipulate complex numbers and be able to multiply two complex numbers quickly and in one step as this will save a lot of time in the exam.
- How to geometrically interpret $|z_1 - z_2|$ as the distance between the complex numbers z_1 and z_2 in the Argand diagram.
- The fact that $|z_1 + z_2| = |z_1 - (-z_2)|$ which equals the distance between z_1 and $-z_2$ in the Argand diagram.

Manipulating Complex Numbers.

Multiplying, dividing, adding and subtracting

- Multiplying, adding and subtracting are all fairly straightforward.

- Dividing is slightly more complicated. Whenever you see a complex number on the denominator of a fraction you can "get rid of it" by multiplying both top and bottom of the fraction by its complex conjugate.

  e.g. $\dfrac{3+2i}{1-i} = \left(\dfrac{3+2i}{1-i}\right)\left(\dfrac{1+i}{1+i}\right) = \dfrac{1+5i}{2}$

Complex Conjugates and Roots of Equations

The complex conjugate

The complex conjugate of $z = a + bi$ is $z* = a - bi$.

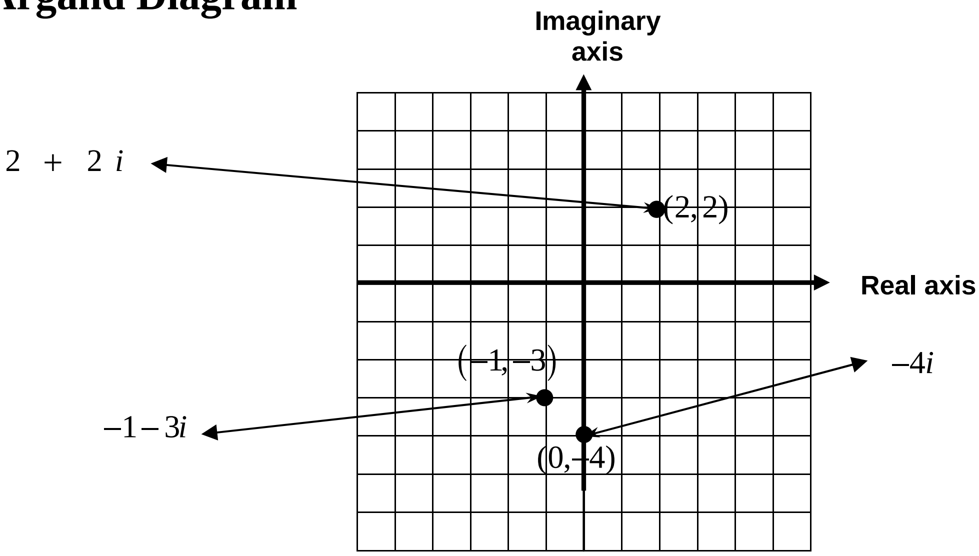
- Remember $zz*$ is a real number and it equals the square of the modulus of z.

- Complex roots of polynomial equations with real coefficients occur in conjugate pairs. This means that if you are told one complex root of a polynomial equation with real coefficients you are in fact being told two roots. This is key to answering some typical exam questions.

An example of an algebraic trick that it is very useful to know is:

$$\begin{aligned}(z-(3+2i))(z-(3-2i)) &= ((z-3)-2i)((z-3)+2i)\\ &= (z-3)^2 - 4i^2\\ &= z^2 - 6z + 13\end{aligned}$$

The Argand Diagram



- In the Argand diagram the point (x, y) corresponds to the complex number $x + yi$.

- You should be able to illustrate simple equations involving complex numbers by means of loci in an Argand diagram. For example, you should be aware that the set of complex numbers z with $|z-5+i|=6$ is a circle of radius 6 centered at $5-i$ (or (5, −1)) in the Argand plane. *(Note: Points on the diagram above do not correspond to this example)*

- You should also be able to illustrate inequalities involving complex numbers by means of loci in an Argand diagram, e.g. $|z-6+i|<b$.

- The argument of a complex number z, denoted $\arg(z)$ is the angle it makes with the positive real axis in the Argand diagram, measured anticlockwise and such that $-\pi<\arg(z)\le\pi$.

- When answering exam questions about points in the Argand diagram be prepared to used geometrical arguments based around equilateral triangles, similar triangles, isosceles triangles and parallel lines to calculate lengths and angles.

Other sets of points in the complex plane.

Where a and b are complex numbers, the set of complex numbers z such that

1. $\arg(z-a)=\theta$, is a half line starting from a in the direction θ.

2. $\arg(z-a)=\arg(z-b)$, is the line through a and b with the section between a and b (inclusive) removed.

3. $\arg(z-a)=\arg(z-b)+\pi$, is the line from a to b (not including a and b themselves).

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MATRICES

The main ideas are:

- Manipulating matrices
- Using matrices to represent transformations
- Matrices and simultaneous equations

Before the exam you should know:

- How to add, subtract and multiply matrices.
- How to calculate the determinant and inverse of 2×2 and 3×3 matrices.
- That matrix multiplication is associative, so $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ but not commutative, so $\mathbf{AB} \neq \mathbf{BA}$.
- The standard matrices for rotation, reflection and enlargement and understand how matrices can be combined to represent composite transformations.
- A linear transformation maps a straight line onto another straight line.
- The determinant of a transformation matrix gives the area factor of the transformation.
- How matrices can be used to represent simultaneous equations.

Manipulating matrices

Adding and subtracting matrices are straightforward.

Multiplying matrices is slightly more difficult. Matrices may only be multiplied if they are conformable, that is if the number of columns in the multiplying matrix (the left-hand one) is the same as the number of rows in the matrix being multiplied (the right-hand one). Matrix multiplication is **not** commutative. This means that for two matrices, **A** and **B**, it is **not** generally true that **AB** = **BA** (it is vital that you remember this).

If $\mathbf{A} = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 7 & 8 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 4 & -1 \\ 2 & 3 & -2 \\ 1 & 2 & 1 \end{pmatrix}$ then $\mathbf{AB} = \begin{pmatrix} 1\times2+3\times2+4\times1 & 1\times4+3\times3+4\times2 & 1\times-1+3\times-2+4\times1 \\ 2\times2+7\times2+8\times1 & 2\times4+7\times3+8\times2 & 2\times-1+7\times-2+8\times1 \end{pmatrix} = \begin{pmatrix} 12 & 21 & -3 \\ 26 & 45 & -8 \end{pmatrix}$

Notice that **BA** is not possible as, this way round, the matrices are not conformable.

The Determinant of 2×2 and 3×3 matrices

The determinant of a 2×2 matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is $ad - bc$. If the matrix is representing a 2 dimensional transformation, the determinant gives the **area** factor of the transformation.

If $\text{M} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ then $\det \text{M} = a_1\left(b_2c_3 - b_3c_2\right) - a_2\left(b_1c_3 - b_3c_1\right) + a_3\left(b_1c_2 - b_2c_1\right)$. This is "expanding by the first row", it is possible to expand by any row or column and you should know how to do this.

Matrices with zero determinant

Example Show that the matrix $\begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix}$ has determinant zero and that it maps all points onto the line $y = \frac{1}{2}x$.

Solution

The determinant of the matrix is $(2\times 4) - (1\times 8) = 0$. Taking a general point (x, y)

$\begin{pmatrix} 2 & 8 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+8y \\ x+4y \end{pmatrix}$. Since $(x+4y) = \frac{1}{2}(2x+8y)$ the point $(2x + 8y, x + 4y)$ is on $y = \frac{1}{2}x$.

The Inverse of 2×2 and 3×3 matrices

The inverse of matrix $\mathbf{A}$ is denoted $\mathbf{A}^{-1}$. It has the property that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. The 2×2 identity matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Only square matrices have inverses and an identity matrix. All identity matrices have 1's on the left to right downwards diagonal and 0's everywhere else. The inverse of a 2×2 matrix is given by swapping the entries on the left to right downwards diagonal, changing the sign of the entries on the other diagonal and dividing the whole thing by the determinant of the original matrix so,

if $\mathbf{A} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$.

For a 3×3 matrix, if $\mathrm{M} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ then $\mathrm{M}^{-1} = \frac{1}{\det \mathrm{M}}\begin{pmatrix} A_1 & -B_1 & C_1 \\ -A_2 & B_2 & -C_2 \\ A_3 & -B_3 & C_3 \end{pmatrix}$,

where A_1 for example is the determinant of the 2×2 matrix which is left after removing the column and row containing a_1 from M, B3 is the determinant of the 2×2 matrix which is left after removing the column and row containing b_3 from M *etc.*

Using matrices to represent transformations

These are some of the standard transformation matrices, and are worth remembering. Remember that the first column of a matrix is where (1, 0) moves to and the second column is where (0, 1) moves to. This can be useful in exams because using this idea it is possible to derive the matrix of a transformation described in words.

- Rotation through angle θ, anticlockwise about the origin: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.
- Enlargement, scale factor *k*, centre the origin: $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.
- Stretch, factor *a* horizontally, factor *b* vertically: $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

A composite transformation is made up of two or more standard transformations, for example a rotation through angle π, followed by a reflection in $y = x$. The matrix representing a composite transformation is obtained by multiplying the component transformation matrices together. The order is important. The matrix for the composite of the transformation with matrix $\mathbf{M}$, followed by the transformation with matrix $\mathbf{N}$ is $\mathbf{NM}$. The order is right to left.

Matrices and simultaneous equations

Matrices can be used to represent systems of simultaneous equations. Matrix techniques can be used to solve them.

E.g. $\left.\begin{matrix} x+2y=7 \\ 3x+7y=24 \end{matrix}\right\}$ is equivalent to: $\mathrm{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 24 \end{pmatrix}$, where $\mathrm{M} = \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}$.

Using the inverse matrix: $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1}\begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 7 \\ 24 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow x = 1$ and $y = 3$.

This can be extended to systems of 3 equations in 3 unknowns.

It should be noted that if det M ≠ 0, M is non-singular, M^{-1} exists and the equations have a unique solution. If det M = 0, M is singular, M^{-1} does not exist and either:

(i) the equations are inconsistent and have no solutions
(ii) the equations are consistent and have infinitely many solutions.

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SERIES AND INDUCTION

The main ideas are:

- Summing Series using standard formulae
- Method of Differences
- Proof by Induction

Before the exam you should know:

- The standard formula:

$$\sum_{r=1}^{n} r, \sum_{r=1}^{n} r^2, \sum_{r=1}^{n} r^3$$

- And be able to spot that a series like

$$(1\times 2)+(2\times 3)+...+n(n+1)$$

can be written in sigma notation as:

$$\sum_{r=1}^{n} r(r+1)$$

- How to do proof by induction

Summing Series

Using standard formulae

Fluency is required in manipulating and simplify standard formulae sums like:

$$\sum_{r=1}^{n} r\left(r^2+1\right) = \sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r = \frac{n^2\left(n+1\right)^2}{4} + \frac{n\left(n+1\right)}{2}$$

$$= \frac{1}{4}n\left(n+1\right)\left[n\left(n+1\right)+2\right]$$

$$= \frac{1}{4}n\left(n+1\right)\left(n^2+n+2\right).$$

The Method of Differences (Telescoping)

Since $\frac{r+4}{r(r+1)(r+2)} = \frac{2}{r} - \frac{3}{r+1} + \frac{1}{r+2}$ (frequently in exam questions you are told to show that this is true first) it is possible to demonstrate that:

$$\sum_{r=1}^{n} \frac{r+4}{r(r+1)(r+2)} = \left(2-\frac{3}{2}+\frac{1}{3}\right)+\left(\frac{2}{2}-\frac{3}{3}+\frac{1}{4}\right)+\left(\frac{2}{3}-\frac{3}{4}+\frac{1}{5}\right)+...$$

$$+\left(\frac{2}{n-2}-\frac{3}{n-1}+\frac{1}{n}\right)+\left(\frac{2}{n-1}-\frac{3}{n}+\frac{1}{n+1}\right)+\left(\frac{2}{n}-\frac{3}{n+1}+\frac{1}{n+2}\right)$$

In this kind of expression many terms cancel with each other. For example, the (+)$\frac{1}{3}$ in the first bracket cancels with the (–)$\frac{3}{3}$ in the second bracket and the (+)$\frac{2}{3}$ in the third bracket. (subsequent fractions that are cancelling are doing so with terms in the "…" part of the sum.)

This leaves $\sum_{r=1}^{n} \frac{r+4}{r(r+1)(r+2)} = \frac{3}{2} - \frac{2}{n+1} + \frac{1}{n+2}$.

(Note: You should be able to recognise when a series is convergent and find its sum to infinity. If you are not familiar with this please see notes from your C2 module.)

Proof by Induction

1. Using proof by induction to prove a formula for the summation of a series,

$$\text{E.g., Prove that} \sum_{r=1}^{n}(2r-1)=n^2 .$$

2. Other miscellaneous questions. These are usually very easy, in fact easier than the questions which fall into the categories above, so long as you don't panic, keep a clear head and apply what you know.

E.g., show that if $M=\begin{pmatrix}5 & 8\\ -2 & -3\end{pmatrix}$ then $M^n=\begin{pmatrix}1+4n & 8n\\ -2n & 1-4n\end{pmatrix}$ for all natural numbers n.

Example

Prove by induction that, for all positive integers n, $\sum_{r=1}^{n}3r+1=\frac{1}{2}n(3n+5)$.

Solution

When n = 1 the left hand side equals $(3\times1)+1=4$. The right hand side is $\frac{1}{2}\times1\times((3\times1)+5)=4$. So the statement is true when n = 1.

Assume the statement is true when n = k. In other words $\sum_{r=1}^{k}3r+1=\frac{1}{2}k(3k+5)$.

We must show the statement would then be true when n = k + 1, i.e. that $\sum_{r=1}^{k+1}3r+1=\frac{1}{2}(k+1)(3k+8)$.

Now,

$$\begin{aligned}\sum_{r=1}^{k+1}(3r+1)&=\sum_{r=1}^{k}(3r+1)+(3(k+1)+1)\\&=\frac{1}{2}k(3k+5)+(3k+4)\\&=\frac{1}{2}\left[3k^2+5k+6k+8\right]\\&=\frac{1}{2}\left[3k^2+11k+8\right]\\&=\frac{1}{2}(k+1)(3k+8)\end{aligned}$$

So the statement is true when n = 1 and if it's true when n = k, then it's also true when n = k +1.

Hence, by induction the statement is true for all positive integers, n.