Further Pure 1 Summary Notes

1. Roots of Quadratic Equations

For a quadratic equation **$ax^2 + bx + c = 0$** with roots α and β

Sum of the roots

$$\alpha + \beta = -\frac{b}{a}$$

Product of roots

$$\alpha\beta = \frac{c}{a}$$

- If the coefficients a,b and c are real then either α and β **are real** or α and β are **complex conjugates**

Once the value α + β and αβ have been found, new quadratic equations can be formed with roots :

Roots		
α^2 and β^2	Sum of roots $(\alpha + \beta)^2 - 2\alpha\beta$	Product of roots $(\alpha\beta)^2$
α^3 and β^3	Sum of roots $(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$	Product of roots $(\alpha\beta)^3$
$\frac{1}{\alpha}$ and $\frac{1}{\beta}$	Sum of roots $\frac{\alpha + \beta}{\alpha\beta}$	Product of roots $\frac{1}{\alpha\beta}$

The new equation becomes

$$x^2 - (\text{sum of new roots})x + (\text{product of new roots}) = 0$$

The questions often ask for integer coefficients!

Don't forget the "= 0"

Example
The roots of the quadratic equation $3x^2 + 4x - 1 = 0$ are α and β.
Determine a quadratic equation with integer coefficients which has roots $\alpha^3\beta$ and $\alpha\beta^3$

Step 1 : $\alpha + \beta = -\frac{4}{3}$ $\alpha\beta = -\frac{1}{3}$

Step 2 : *Sum of new roots*

$$\alpha^3\beta + \alpha\beta^3 = \alpha\beta(\alpha^2 + \beta^2)$$

$$= -\frac{1}{3} \times ((\alpha + \beta)^2 - 2\alpha\beta)$$

$$= -\frac{1}{3} \times \left(\frac{16}{9} + \frac{2}{3}\right)$$

$$= -\frac{22}{27}$$

Step 3 : Product of roots

$$\alpha^3\beta \times \alpha\beta^3 = \alpha^4 b^4 = (\alpha\beta)^4 = \frac{1}{81}$$

Step 4 : Form the new equation

$$x^2 + \frac{22}{27}x + \frac{1}{81} = 0$$

$$81x^2 + 66x + 1 = 0$$

2. Summation of Series

These are given in the formula booklet

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

REMEMBER :

$$\sum_{r=1}^{n} 1 = n$$

$$\sum 5r^2 = 5\sum r^2$$

Always multiply brackets before attempting to evaluate summations of series

- Look carefully at the limits for the summation

  $$\sum_{r=7}^{20} = \sum_{r=1}^{20} - \sum_{r=1}^{6} \qquad \sum_{r=n+1}^{2n} = \sum_{r=1}^{2n} - \sum_{r=1}^{n}$$

- Summation of **ODD / EVEN** numbers

  Example : Find the sum of the odd square numbers from 1 to 49

  Sum of odd square numbers
  = Sum of all square numbers – Sum of even square numbers

Sum of even square numbers $= 2^2+4^2 + \ldots\ldots\ldots\ldots 48^2$

$= 2^2 (1^2 + 2^2 + 3^2 + \ldots\ldots\ldots 24^2)$

$= 4\sum_{r=1}^{24} r^2$

Sum of odd numbers between 1 and 49 is $\sum_{1}^{49} r^2 - 4\sum_{r=1}^{24} r^2$

$$= \left(\frac{1}{6} \times 49 \times 50 \times 99\right) - 4\left(\frac{1}{6} \times 24 \times 25 \times 49\right)$$

$$= 40125 - 19600$$

$$= 20825$$

3. Matrices

- ORDER

$$\begin{bmatrix} a \\ b \end{bmatrix} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

2 x 1   2 x 2

- Addition and Subtraction – **must have the same order**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \pm \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{bmatrix}$$

- Multiplication

$$3\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3a \\ 3b \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 10 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 \times 1 + 4 \times 3 & 3 \times 10 + 4 \times 0 \\ 5 \times 1 + 2 \times 3 & 5 \times 10 + 2 \times 0 \end{bmatrix} = \begin{bmatrix} 15 & 30 \\ 11 & 50 \end{bmatrix}$$

NB : Order matters Do not assume that **AB = BA**
Do not assume that **$A^2 - B^2 = (A-B)(A+B)$**

- Identity Matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **$AI = IA = A$**

4. Transformations

- Make sure you know the exact trig ratios

Angle θ	sin θ	cos θ	tan θ
0°	0	1	0
30°	½	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	½	$\sqrt{3}$
90°	1	0	Undefined

- To calculate the coordinates of a point after a transformation
  Multiply the Transformation Matrix by the coordinate

Find the position of point (2,1) after a stretch of Scale factor 5 parallel to the x-axis

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \end{bmatrix} \quad (10,1)$$

- **To Identify a transformation from its matrix**
  Consider the points (1,0) and (0,1)

$$\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

(1,0) ⇨ (4,0)   (0,1) ⇨ (0,2)

Stretch Scale factor 4 parallel to the x-axis and scale factor 2 parallel to the y-axis

<u>Standard Transformations</u>

REFLECTIONS

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ in the y-axis $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ in the x-axis

In the formula booklet

Reflection in y = x $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Reflection in the line y= (tan θ)x $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

If all elements have the same magnitude then look at 2θ = 45° (reflection in y = (tan22.5)x) as one of the transformations

ENLARGEMENT

$\begin{bmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{bmatrix}$

Scale factor k
Centre (0,0)

STRETCH

$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ Scale factor **a** parallel to the x-axis
Scale factor **b** parallel to the y-axis

ROTATION

In the formula booklet →

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ Rotation through θ **anti-clockwise** about origin (0,0)

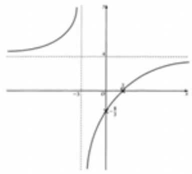
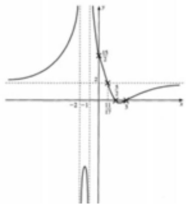
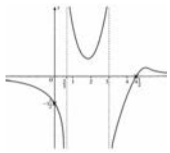
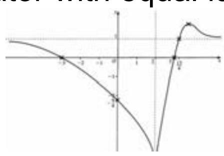
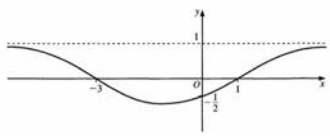
$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ Rotation through θ **Clockwise** about origin (0,0)

If all elements have the same magnitude then a rotation through 45 is likely to be one of the transformations (usually the second)

ORDER MATTERS !!!! *– make sure you multiply the matrices in the correct order*
A figure is transformed by $\boldsymbol{M_1}$ *followed by* $\boldsymbol{M_2}$

Multiply $\boldsymbol{M_2 M_1}$

5. Graphs of Rational Functions

$y = \dfrac{4x - 8}{x + 3}$	Linear numerator and linear denominator 	1 horizontal asymptote 1 vertical asymptote
$y = \dfrac{(x - 3)(2x - 5)}{(x + 1)(x + 2)}$	**2 distinct linear factors in the denominator** – quadratic numerator 	2 vertical asymptotes 1 horizontal asymptote **The curve will usually cross the horizontal asymptote**
$y = \dfrac{2x - 9}{3x^2 - 11x + 6}$	2 distinct linear factors in the denominator – linear numerator 	2 vertical asymptotes 1 horizontal asymptote horizontal asymptote is $y = 0$
$y = \dfrac{(x - 3)(x + 3)}{(x - 2)^2}$	Quadratic numerator – quadratic denominator with equal factors 	1 vertical asymptote 1 horizontal asymptote
$y = \dfrac{x^2 + 2x - 3}{x^2 + 2x + 6}$	Quadratic numerator with no real roots for denominator (irreducible) 	The curve does not have a vertical asymptote

Vertical Asymptotes – Solve "denominator = 0" to find $x = a$, $x = b$ etc

Horizontal Asymptotes – multiply out any brackets – look for highest power of x in the denominator – and divide all terms by this – as x goes to infinity majority of terms will disappear to leave either $y = 0$ or $y = a$

To find stationary points

$$k = \frac{x^2 + 2x - 3}{x^2 + 2x + 6}$$ rearrange to form a quadratic $ax^2 + bx + c = 0$ *

$b^2 - 4ac < 0$

the line(s) $y = k$ do not intersect the curve

$b^2 - 4ac = 0$

stationary point(s) occur when $y = k$ subs into * to find x coordinate

$b^2 - 4ac > 0$

the line(s) $y = k$ intersect the curve subs into * to find x coordinate

INEQUALITIES

- The questions are **unlikely** to lead to simple or single solutions such as x > 5 so **Sketch the graph** (often done already in a previous part of the question)

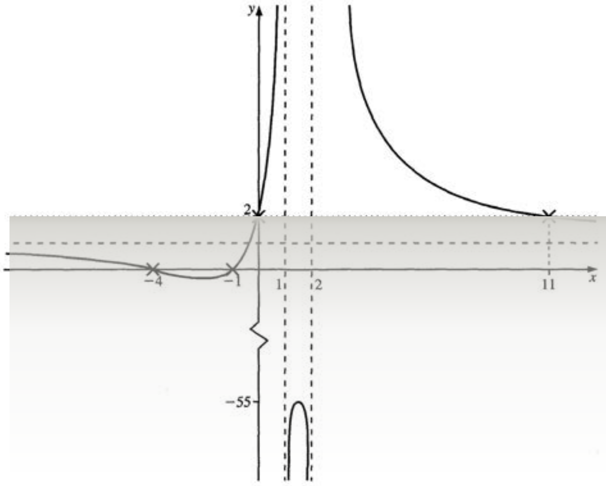
Solve the inequality

$$\frac{(x + 1)(x + 4)}{(x - 1)(x - 2)} < 2$$

The shaded area is where y < 2

So the solution is

x < 0 , 1 < x < 2 , x > 11

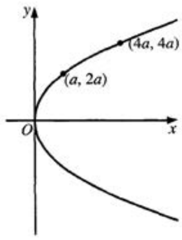


6. Conics and transformations

- You must learn the standard equations and the key features of each graph type
- Mark on relevant coordinates on any sketch graph

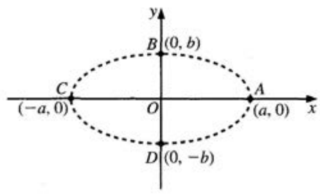
Parabola

$$y^2 = 4ax$$



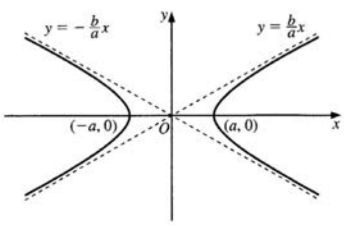
Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$



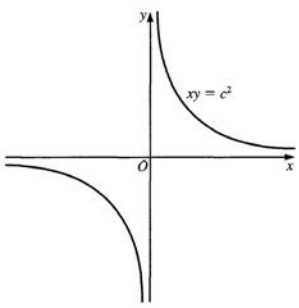
Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$$



Rectangular Hyperbola

$$xy = c^2$$



- You may need to **complete the square**

$$x^2 - 4x + y^2 - 6y = -12$$

$$(x - 2)^2 - 4 + (y - 3)^2 - 9 = -12$$

$$(x - 2)^2 + (y - 3)^2 = 1$$

Standard equations are given in the formula booklet but NOT graphs

Transformations

Translation $\begin{bmatrix} a \\ b \end{bmatrix}$ Replace x with (x – a)
Replace y with (y - b)

Circle radius 1 centre (2,.3)
$(x - 2)^2 + (y - 3)^2 = 1$

Reflection in the line y = x
Replace x with y and vice versa

Stretch Parallel to the x-axis scale factor a Replace x with $\frac{x}{a}$

Stretch Parallel to the y-axis scale factor b Replace y with $\frac{y}{b}$

Describe a geometrical transformation that maps the curve $y^2=8x$ onto the curve $y^2=8x-16$

x has been replaced by (x-2) to give $y^2= 8(x-2)$ Translation $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

7. Complex Numbers

$$z = a + ib$$

real (a) imaginary (ib)

$$\mathbf{i} = \sqrt{-1} \qquad \mathbf{i}^2 = -1$$

- **Addition and Subtraction**

  (2 + 3i) + (5 –2i) = 7 + i (add/subtract real part then imaginary part)

- **Multiplication** - multiply out the same way you would (x-2)(x+4)

  $$\begin{aligned}(2 - 3i)(6 + 2i) &= 12 + 4i - 18i - 6i^2 \\ &= 12 - 14i + 6 \\ &= 18 - 14i\end{aligned}$$

- **Complex Conjugate z***
  If **z = a + ib** then its complex conjugate is **z* = a – ib**

  - always collect the 'real' and 'imaginary' parts before looking for the conjugate

- **Solving Equations** - if two complex numbers are equal, their real parts are equal and their imaginary parts are equal.

  Find z when 5z –2z = 3 – 14i*

  Let z = x + iy and so z= x – iy*

  $$5(x + iy) - 2(x - iy) = 3 - 14i$$

  $$3x + 7iy = 3 - 14i$$

  Equating real : 3x = 3 so x = 1
  Equating imaginary : 7y = -14 so y= -2

  z = 1 –2i

8. Calculus

Differentiating from first principles

- Gradient of curve or tangent at x is $f'(x) = \dfrac{dy}{dx} = \lim_{h \to 0}\left(\dfrac{f(x+h) - f(x)}{h}\right)$

- You may need to use the binomial expansion

Differentiate from first principles to find the gradient of the curve $y = x^4$ at the point (2,16)

$f(x) = 2^4$ $\quad\quad f(2 + h) = (2 + h)^4$

$= 2^4 + 4(2^3h) + 6(2^2h^2) + 4(2h^3) + h^4$

$= 16 + 32h + 24h^2 + 8h^3 + h^4$

$$\frac{f(2+h) - f(2)}{h} = \frac{16 + 32h + 24h^2 + 8h^3 + h^4 - 16}{h}$$

$$= 32 + 24h + 8h^2 + h^3$$

As h approaches zero Gradient = 32

- You may need to give the **equation** of the tangent/normal to the curve – easy to do once you know the gradient and have the coordinates of the point

Improper Integrals

Improper if

- **one or both of the limits is infinity**

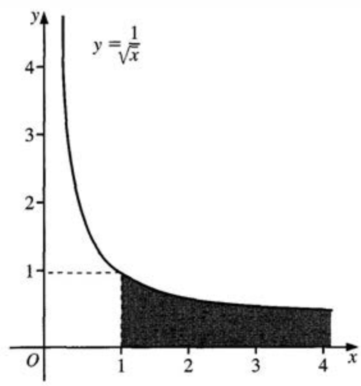
Determine whether the integral $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ has a value. If so, find the value of the integral.

substitute n for ∞:

$$\int_1^n \frac{1}{\sqrt{x}}\,dx = \int_1^n x^{-\frac{1}{2}}\,dx$$
$$= [2x^{\frac{1}{2}}]_1^n$$
$$= 2\sqrt{n} - 2$$

As $n \to \infty$, the value of this integral does not approach a finite number, and so the integral cannot be found.

Very important to include these statements



- the integrand is **undefined at one of the limits or somewhere in between the limits**

Determine $\int_0^1 \frac{1}{\sqrt{x}}\,dx$.

This is an improper integral since the integral is undefined when $x = 0$.

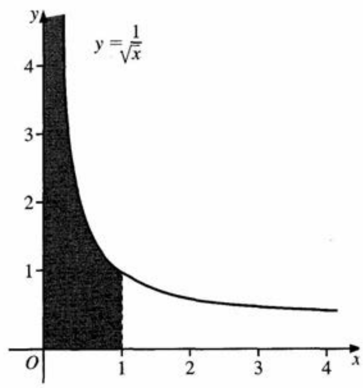
Replace the lower limit of integration by p.

$$\int_p^1 \frac{1}{\sqrt{x}}\,dx = \int_p^1 x^{-\frac{1}{2}}\,dx$$
$$= [2x^{\frac{1}{2}}]_p^1$$
$$= 2 - 2\sqrt{p}$$

As p approaches zero, the value of $2 - 2\sqrt{p}$ approaches 2.

Therefore the improper integral $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ can be found, and its value is 2.

Very important to include these statements



9. Trigonometry

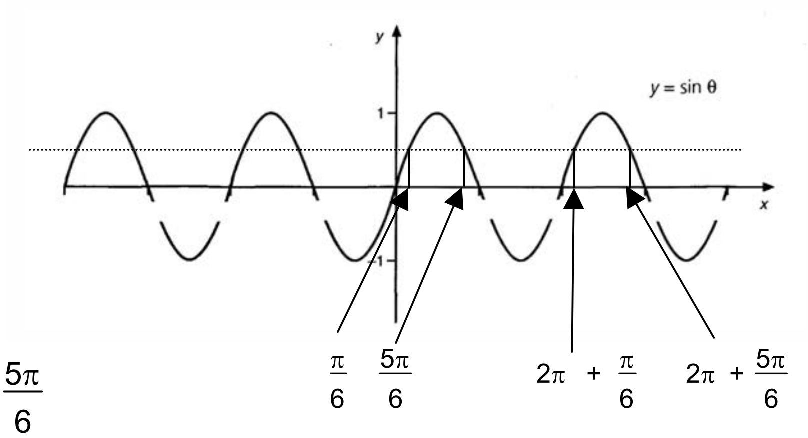
- **GENERAL SOLUTION –** don't just give one answer – there should be an 'n' somewhere!!
- **SKETCH** the graph of the basic Trig function before you start
- Check the question for **Degrees** or **Radians**
- **MARK** the first solution (from your calculator/knowledge) on your graph – mark a few more to see the pattern
- Find the general solution **before rearranging** to get x or θ on it's own.

Example
Find the general solution, in radians, of the equation $2\cos^2 x = 3\sin x$

$2(1 - \sin^2 x) = 3\sin x$ (Using $\cos^2 x + \sin^2 x = 1$)
$2\sin^2 x + 3\sin x - 2 = 0$
$(\sin x + 2)(2\sin x - 1) = 0$

no solutions for sin x = -2

sinx = ½



General Solutions

$$x = 2\pi n + \frac{\pi}{6}, \quad x = 2\pi n + \frac{5\pi}{6}$$

- You may need to use the fact that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ to solve equations of the form sin (2x – 0.1) = cos (2x – 0.1)

10. Numerical solution of equations –

- Rearrange into the form f(x) = 0
- To show the root lies within a given interval – evaluate f(x) for the upper and lower interval bounds

One should be positive and one negative
change of sign indicates a root within the interval

- **Interval Bisection**
  - Determine the **nature** of f(Lower) and f(upper) – sketch the graph of the interval
  - Investigate f(midpoint)- positive or negative ?
  - Continue investigating 'new' midpoints until you have an interval to the degree of accuracy required

- **Linear Interpolation**
  - Determine the Value of f(Lower) and f(upper) – sketch the graph of the interval
  - Join the Lower and Upper points together with a **straight line**
    - Mark "p" the approximate root
    - Use similar triangles to calculate p (equal ratios)

When working with Trig functions you probably need radians – check carefully!

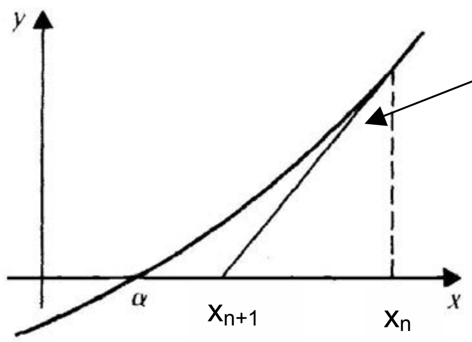
- **Newton- Raphson Method**
  - given in formula book as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Given in formula book

value of new approximation (x_{n+1}) — value of previous approximation (x_n)

- you may be required to draw a diagram to illustrate your method



tangent to the curve at x_n

Gradient of the tangent = $f'(x_n)$

$$f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}}$$

NB : When the initial approximation is not close to f(x) the method may fail!

DIFFERENTIAL EQUATIONS

- looking to find y when dy/dx is given

- **EULER's FORMULA** $y_{n+1} = y_n + hf(x_n)$

Given in formula book

$$\frac{dy}{dx} = f(x)$$ **h = step size**

- allows us to find an approximate value for y close to a given point

Example

$\frac{dy}{dx} = e^{\cos x}$, given that when y = 3 when x = 1, use the Euler Formula with step size 0.2 to find an approximation for y when x = 1.4

$x_1 = 1$ $y_1 = 3$ h =0.2 $f(x) = e^{\cos\theta}$

$y_2 = 3 + 0.2(e^{\cos 1})$
$= 3.343$ (approximate value of y when x = 1.2)

$y_3 = 3.323 + 0.2(e^{\cos 1.2})$
$= \underline{3.631}$ (approximate value of y when x = 1.4)

11. Linear Laws

- using straight line graphs to determine equations involving two variables
- remember the equation of a straight line is

y = mx + c where m is the gradient
c is the point of interception with the y-axis

- **Logarithms** needed when $y = ax^n$ or $y = ab^x$

Remember : Log ab = Log a + Log b

$\text{Log } a^x = x \text{ Log } a$

- equations must be rearranged/substitutions made to a linear form

- $y^3 = ax^2 + b$ **plot** y^3 against x^2

- $y^3 = ax^5 + bx^2$ $(\div x^2)$

  $\frac{y^3}{x^2} = ax^3 + b$ **plot** $\frac{y^3}{x^2}$ against x^3

- $y = ax^n$ (taking logs)

  $\log y = \log a + n \log x$ **plot** log y against log x

- $y = ab^x$ (taking logs)

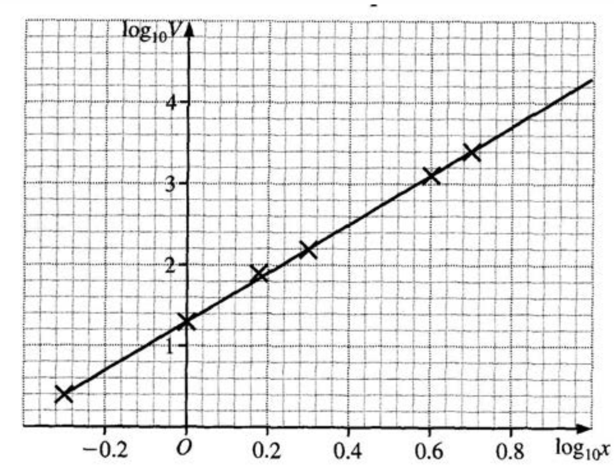
  $\log y = \log a + x \log b$ **plot** log y against x

if working in logs remember the inverse of log x is 10^x

EXAMPLE
It is thought that V and x are connected by the equation $V = ax^b$

The equation is reduced to linear from by taking logs

$\text{Log } V = \text{Log } a + b \log x$



Using data given Log V is plotted against Log x

The **gradient** b is $\frac{1.50}{0.5} = 3$

The **intercept** on the log V axis is 1.3

So Log a = 1.3
$a = 10^{1.3}$
$= 19.95\ldots$

The relationship between V and x is therefore

$V = 20x^3$