

# 1

## Roots and Coefficients of a Quadratic Equation Summary

For a quadratic equation  $ax^2 + bx + c = 0$  with roots  $\alpha$  and  $\beta$ :

$$\text{Sum of roots} = \alpha + \beta = \frac{-b}{a}$$

and

$$\text{Product of roots} = \alpha\beta = \frac{c}{a}$$

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

Symmetrical functions of  $\alpha$  and  $\beta$  include:

$$\frac{1}{\alpha} \times \frac{1}{\beta} = \frac{1}{\alpha\beta}$$

and

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$$

$$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$$

therefore

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\alpha^3 \times \beta^3 = \alpha^3\beta^3 = (\alpha\beta)^3$$

and

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

### Example

Find the quadratic equation whose roots are  $\alpha + \frac{5}{\beta}$  and  $\beta + \frac{5}{\alpha}$  where  $\alpha$  and  $\beta$  are the roots of the quadratic equation  $2x^2 + 7x - 3 = 0$ .

From  $2x^2 + 7x - 3 = 0$ ,  $\alpha + \beta = -\frac{7}{2}$  and  $\alpha\beta = -\frac{3}{2}$ .

The sum of the new roots is:

$$\alpha + \frac{5}{\beta} + \beta + \frac{5}{\alpha} = \alpha + \beta + \frac{5}{\beta} + \frac{5}{\alpha}$$

$$= \alpha + \beta + \frac{5(\alpha + \beta)}{\alpha\beta}$$

$$= -\frac{7}{2} + \frac{35}{3}$$

$$= \frac{49}{6}$$

The new product of roots is:

$$\left(\alpha + \frac{5}{\beta}\right)\left(\beta + \frac{5}{\alpha}\right) = \alpha\beta + \frac{25}{\alpha\beta} + 10$$

$$= -\frac{49}{6}$$

So the new equation is  $6x^2 - 49x - 49 = 0$

### Equations with related roots:

If  $\alpha$  and  $\beta$  are the roots of the equation  $ax^2 + bx + c = 0$ , you can obtain an equation with roots  $2\alpha$  and  $2\beta$  by substituting in  $y=2x$ , thus  $x = \frac{1}{2}y$ .

### Example

Find a quadratic equation with roots  $2\alpha-1$  and  $2\beta-1$ , where  $\alpha$  and  $\beta$  are the roots of the equation  $4x^2 + 7x - 5 = 0$ .

$$4x^2 + 7x - 5 = 0$$

Use the variable  $y$  where  $y=2x-1$ , so that  $x = \frac{1}{2}(y + 1)$ .

The required equation is therefore:

$$4\left(\frac{1}{2}(y + 1)\right)^2 + 7\left(\frac{1}{2}(y + 1)\right) - 5 = 0$$

Which can be simplified to:

$$2y^2 + 11y - 1 = 0$$

# 2

## Series and Sums of Natural Numbers

The sum of the first  $n$  natural numbers is given by:

$$\sum_{1}^{n} r = \frac{1}{2}n(n + 1)$$

The sum of the squares of the first  $n$  natural numbers is given by:

$$\sum_{1}^{n} r^2 = \frac{1}{6}n(n + 1)(2n + 1)$$

The sum of the cubes of the first  $n$  natural numbers is given by:

$$\sum_{1}^{n} r^3 = \frac{1}{4}n^2(n + 1)^2$$

### Example

Find:

$$\sum_{70}^{100} (2r - 1)$$

Splitting the expression gives:

$$\sum_{70}^{100} (2r - 1) = 2 \sum_{70}^{100} r - \sum_{70}^{100} 5$$

Which gives:

$$\begin{aligned} \sum_{70}^{100} (2r - 1) &= 2 \left\{ \sum_{1}^{100} r - \sum_{1}^{69} r \right\} - \{(100 \times 5) - (69 \times 5)\} \\ &= 2(5050 - 2415) - 155 \\ &= 5115 \end{aligned}$$

**Example**

Find the sum of the squares of the odd numbers from 1 to 49.

The sum of the squares of the odd numbers from 1 to 49 is equal to:

Sum of all numbers from 1 to 49 – Sum of even numbers from 1 to 49

The sum of the even numbers is:

$$\begin{aligned} & 2^2+4^2+6^2+\dots+48^2 \\ & =2^2(1^2+2^2+3^2+4^2+\dots+24^2) \\ & =4\sum_{1}^{24}r^2 \end{aligned}$$

Therefore the sum of all odd numbers from 1 to 49 is:

$$\begin{aligned} & \sum_{1}^{49}r^2 - 4\sum_{1}^{24}r^2 \\ & =40425-19600=20825 \end{aligned}$$

**Example**

Find the value of

$$\sum_{r=n+1}^{2n}(4r^3 - 3)$$

$$\sum_{r=n+1}^{2n}(4r^3 - 3) = \sum_{r=1}^{2n}(4r^3 - 3) - \sum_{r=1}^n(4r^3 - 3)$$

Which gives:

$$\begin{aligned} \sum_{r=1}^{2n}(4r^3 - 3) & = 4\sum_{1}^{2n}r^3 - 3\sum_{1}^{2n}1 - \left(4\sum_{1}^nr^3 - 3\sum_{1}^n1\right) \\ & = 4n^2(2n+1)^2 - 6n - n^2(n+1)^2 + 3n \\ & = 15n^4 + 14n^3 + 3n^2 - 3n \end{aligned}$$

# 3

## Matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \pm \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} ap + bq \\ cp + dq \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}$$

The 2x2 identity matrix is called **I**, where:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

When a matrix is multiplied by the identity matrix **I**, it stays the same.

### Example

Find  $2\mathbf{A} - 5\mathbf{B}$ , where  $\mathbf{A} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$  and  $\mathbf{B} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ .

$$\begin{aligned} 2\mathbf{A} - 5\mathbf{B} &= 2 \begin{pmatrix} 8 \\ 2 \end{pmatrix} - 5 \begin{pmatrix} 1 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 16 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ -20 \end{pmatrix} \\ &= \begin{pmatrix} 11 \\ 24 \end{pmatrix} \end{aligned}$$

### Example

Find  $\mathbf{AB}$  and  $\mathbf{BA}$  where  $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$  and  $\mathbf{B} = \begin{pmatrix} 2 & -1 \\ 6 & 7 \end{pmatrix}$

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 6 & 7 \end{pmatrix} \\ &= \begin{pmatrix} 3 \times 2 + 4 \times 6 & 3 \times -1 + 4 \times 7 \\ 2 \times 2 + 5 \times 6 & 2 \times -1 + 5 \times 7 \end{pmatrix} \\ &= \begin{pmatrix} 30 & 25 \\ 34 & 33 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{BA} &= \begin{pmatrix} 2 & -1 \\ 6 & 7 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 2 \times 3 - 1 \times 2 & 2 \times 4 - 1 \times 5 \\ 6 \times 3 + 7 \times 2 & 6 \times 4 + 7 \times 5 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 32 & 59 \end{pmatrix} \end{aligned}$$

# 4

## Transformations

Every linear transformation can be represented by a 2 x 2 square matrix,  $\mathbf{M}$ , of the form  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ .

To find the matrix  $\mathbf{M}$ , representing a given transformation, you find the images of the two points (1, 0) and (0, 1).

### Example

Find the matrix,  $\mathbf{M}$ , representing an enlargement, scale factor 2, with the origin as the centre of enlargement.

By drawing a graph, the images of (1, 0) and (0, 1) are (2, 0) and (0, 2).

$$\text{So } \mathbf{M} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Using the image (1, 1) to check:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \text{ as expected.}$$

The general form for the matrix of rotation about the origin through angle  $\theta$  anticlockwise is  $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ .

The general form for the matrix of a reflection in the line  $y = (\tan\theta)x$  is:

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

### Common Transformations

$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$  is a stretch parallel to the y-axis with a scale factor  $a$ .

$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$  is a stretch parallel to the x-axis with scale factor  $a$ .

$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$  generally represents both a rotation and an enlargement.

$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$  generally represents both a reflection and an enlargement.

### Combining Two Transformations

If the image of  $(x, y)$  under the transformation  $T$  is transformed by a second transformation  $T_1$ , represented by the matrix  $M_1$ , the image of  $(x, y)$  under the two transformations is given by  $M_1M\begin{pmatrix} x \\ y \end{pmatrix}$ .

In general,  $MM_1 \neq M_1M$ .

#### Example

Find the matrix represented by the combined transformations:

- A stretch parallel to the x-axis of scale factor 2
- Followed by a stretch parallel to the y-axis of scale factor 5.

The two matrices are:

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}.$$

The combined transformation is given by  $M_1M$ .

$$\begin{aligned} M_1M &= \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}. \end{aligned}$$

# 5

## Graphs of Rational Functions

### Rational Functions with a Linear Numerator and Denominator:

This is when it is in the form

$$y = \frac{ax+b}{cx+d} \text{ where } a, b, c \text{ and } d \text{ are constants.}$$

Consider the function  $\frac{4x-8}{x+3}$ . To understand the function, it is useful to sketch its graph.

By observation, as  $x$  approaches the value  $-3$ , the numerator approaches  $-20$  but the denominator approaches  $0$ . As  $x$  keeps getting closer to  $-3$ , the  $y$  value will tend towards infinity without ever touching  $x=-3$ . Hence the line  $x=-3$  is an **asymptote**.

To find the next asymptote, divide both the numerator and denominator by  $x$ .

This gives  $y = \frac{4 - \frac{8}{x}}{1 + \frac{3}{x}}$ . As  $x$  gets very large, both  $-\frac{8}{x}$  and  $\frac{3}{x}$  approach the value zero.

Therefore as  $x$  tends towards infinity, both fractions above tend towards zero. As a result, the curve approaches another asymptote of  $y=4$ .

After finding the  $x$  and  $y$  intercepts by using  $x=0$  and  $y=0$ , you get:

- $x=-3$  as the vertical asymptote
- $y=4$  as the horizontal asymptote
- When  $x=0$ ,  $y=-\frac{8}{3}$
- When  $y=0$ ,  $x=2$



### Two Distinct Linear Factors in the Denominator

These are where the denominator is a factorised quadratic, or one which can be factorised. If the denominator has two real solutions, there will be two vertical asymptotes.

If the numerator is also a quadratic, the curve will usually cross the horizontal asymptote.

- To find the **horizontal** asymptote, divide through by  $x$ .
- To find the **vertical** asymptotes, equate the denominator to zero and then solve to find the asymptotes.
- To find where the curve crosses the  $x$  and  $y$  axes, substitute in  $x=0$  and  $y=0$ .
- To find where the curve crosses the horizontal asymptote, substitute in the value of the horizontal asymptote ( $y=a$ ).

If the numerator is a linear expression and the denominator has two linear factors, the horizontal asymptote is **always**  $y=0$ , because if you expand the denominator; then divide through by  $x^2$ ,  $y$  tends towards 0.

### Rational Functions with a Repeated Factor in the Denominator

If the numerator is a quadratic expression, and the denominator is a quadratic expression with equal factors, e.g.  $y = \frac{(x-3)(x+3)}{(x-2)^2}$   
There is only one vertical asymptote.

Considering the function above, as  $x$  approaches 2, the denominator approaches zero so the vertical asymptote is  $x=2$ .

- To find the **horizontal** asymptote, divide through by  $x$ .
- To find the **vertical** asymptotes, equate the denominator to zero and then solve to find the asymptotes.
- To find where the curve crosses the  $x$  and  $y$  axes, substitute in  $x=0$  and  $y=0$ .
- To find where the curve crosses the horizontal asymptote, substitute in the value of the horizontal asymptote ( $y=a$ ).

### Rational Functions with an Irreducible Quadratic in the Denominator

Not all curves in the form  $y = \frac{x^2 + ax + b}{x^2 + cx + d}$  have vertical asymptotes.

If the equation  $x^2 + cx + d = 0$  has no real solutions, then the curve will have no vertical asymptote.

### Stationary Points on the Graphs of Rational Functions

Several of the functions in this chapter have one or more stationary points. Now consider the curve:

$$y = \frac{x^2 + 2x - 3}{x^2 + 2x + 6}$$

By sketching this graph you will see it has a minimum point. To find the minimum point, consider the line  $y=k$  intersecting the curve. Therefore,

$$k = \frac{x^2 + 2x - 3}{x^2 + 2x + 6} \text{ This gives } (k - 1)x^2 + (2k - 2)x + 6k + 3 = 0 \quad (*)$$

Since there is a minimum point, the roots must be equal, Thus  $b^2 - 4ac = 0$ .

$$\text{Therefore } (2k - 2)^2 - 4(k - 1)(6k + 3) = 0$$

$$\rightarrow 5k^2 - k - 4 = 0$$

$$\rightarrow (5k^2 + 4)(k - 1) = 0$$

$$\rightarrow k = -\frac{4}{5} \text{ or } k = 1$$

As there is no point on the curve for which  $y=1$ , the minimum value is

$$y = -\frac{4}{5}$$

To find the value of  $x$  at the minimum point, sub the value of  $k$  ( $k=y$ ) into equation (\*). This gives  $x=-1$

### Inequalities

With inequalities, normal rules apply unless you are multiplying or dividing both sides by a negative number. If you do this, the inequality symbol must be reversed.

$$4x - 1 < 5 \quad \rightarrow \quad 4x < 6$$

$$3x + 2 > 11 \quad \rightarrow \quad 3x > 9$$

$$\text{however, } 2x > 4 \quad \rightarrow \quad x < -2$$

This means that you cannot solve an inequality such as  $\frac{ax + b}{cx + d} > 2$  because you don't know whether  $cx+d$  is positive or negative, so you're unable to multiply both sides out by this.

To solve an inequality like  $\frac{ax + b}{cx + d} > k$ , there are two methods.

- The first method is to multiply out both sides by  $(cx + d)^2$ , which cannot be negative.

- Or sketch  $\frac{ax + b}{cx + d} = y$ , solve  $\frac{ax + b}{cx + d} = k$  and then, by comparing these two results, write down the solution to the inequality.

# 6

## Conics

### The Parabola

The standard equation for a parabola is  $y^2 = 4ax$ .

#### Translation:

The translation  $\begin{pmatrix} a \\ b \end{pmatrix}$  on the parabola  $y^2 = 8x$  to the parabola with equation  $(y - b)^2 = 8(x - a)$

#### Reflection in the line $y=x$ :

Reflecting  $y^2 = 8x$  in the line  $y=x$  results in the new equation  $x^2 = 8y$ .

#### Stretch parallel to the x-axis:

A stretch, scale factor  $a$ , parallel to the x-axis transforms the parabola  $y^2 = 8x$  into the new equation  $y^2 = \frac{8x}{a}$

#### Stretch parallel to the y-axis:

A stretch, scale factor  $a$ , parallel to the y-axis transforms the parabola to the equation  $\left(\frac{y}{a}\right)^2 = 8x$

### The Ellipse

You can obtain an ellipse from a circle by applying a one-way stretch. Starting with a circle with the equation  $x^2 + y^2 = 1$ , then applying a stretch of scale factor  $a$  parallel to the x-axis followed by a stretch scale factor  $b$  parallel to the y-axis gives the standard equation of an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

#### Translation:

A translation of  $\begin{pmatrix} p \\ q \end{pmatrix}$  moves  $\frac{x^2}{25} + \frac{y^2}{4} = 1$  to  $\frac{(x - p)^2}{25} + \frac{(y - q)^2}{4} = 1$

#### Reflection in the line $y=x$ :

To reflect an ellipse in the line  $y=x$ , replace the x's with y's and vice versa.

#### Stretch parallel to the x-axis:

A stretch parallel to the x-axis scale factor  $p$ , goes to  $\frac{\left(\frac{x}{p}\right)^2}{25} + \frac{y^2}{4} = 1$

#### Stretch parallel to the y-axis:

A stretch, scale factor  $q$ , parallel to the y-axis goes to  $\frac{x^2}{25} + \frac{\left(\frac{y}{q}\right)^2}{4} = 1$

## The hyperbola

The standard equation for a hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

### Translation:

The translation of  $\begin{pmatrix} p \\ q \end{pmatrix}$  moves the hyperbola  $\frac{x^2}{25} - \frac{y^2}{25} = 1$  into a hyperbola with a new equation of:

$$\frac{(x-p)^2}{25} - \frac{(y-q)^2}{4} = 1$$

### Reflection in the line $y=x$ :

To reflect a hyperbola in the line  $y=x$ , swap around the  $x$  and  $y$ 's.

### Stretch parallel to the x-axis:

A stretch parallel to the x-axis scale factor  $p$ , goes to  $\frac{\left(\frac{x}{p}\right)^2}{25} - \frac{y^2}{4} = 1$

### Stretch parallel to the y-axis:

A stretch, scale factor  $q$ , parallel to the y-axis goes to  $\frac{x^2}{25} - \frac{\left(\frac{y}{q}\right)^2}{4} = 1$

## The rectangular hyperbola

The standard equation for a rectangular hyperbola is  $xy = c^2$

### Translation:

The translation of  $\begin{pmatrix} a \\ b \end{pmatrix}$  moves the rectangular hyperbola  $xy = 25$  into a rectangular hyperbola with a new equation of  $(x-a)(x-b) = 25$ .

### Reflection in the line $y=x$ :

To reflect a rectangular hyperbola in the line  $y=x$ , swap around the  $x$  and  $y$ 's, which will result in the same equation.

### Stretch parallel to the x-axis:

A stretch parallel to the x-axis scale factor  $p$ , goes to  $\left(\frac{x}{p}\right)y = 25$

### Stretch parallel to the y-axis:

A stretch, scale factor  $q$ , parallel to the y-axis goes to  $x\left(\frac{y}{q}\right) = 25$

# 7

## Complex Numbers

A complex number is a number in the form  $a + ib$ , where  $a$  and  $b$  are real numbers and  $i^2 = -1$ . In the complex number, the real number  $a$  is called the real part, and the real number  $b$  is called the imaginary part of the complex number.

- When  $a=0$ , the number is said to be wholly imaginary.
- When  $b=0$ , the number is real.
- If a complex number is equal to zero, both  $a$  and  $b$  are zero.

If two complex numbers are equal, their real parts are equal and their imaginary parts are equal.

If  $a + ib = c + id$ , then  $a = c$  and  $b = d$ .

If  $z$  is a complex number, its complex conjugate is denoted by  $z^*$ .

If  $z = x + iy$ , then  $z^* = x - iy$

The quadratic formula can be used to find the roots to an equation. If  $b^2 - 4ac$  is negative, you will have to introduce  $i$ .

From chapter one, we have also learnt that the sum of roots is equal to  $-\frac{b}{a}$  the sum of the equations from the quadratic formula.

The quadratic formula can be used to find the roots to an equation. If  $b^2 - 4ac$  is negative, you will have to introduce  $i$ .

From chapter one, we have also learnt that the sum of roots,  $-\frac{b}{a}$ , is equal to the sum of the equations from the quadratic formula.

Equations with  $z$  or  $z^*$  can be solved by substituting in  $z = x + iy$  or  $z^* = x - iy$ . Then you can equate the real and imaginary coefficients.

# 8

## Calculus

### Differentiating from First Principles

If you consider the curve  $y = x^2$ , the gradient at point A(2, 4) is given by the gradient of the tangent at A. To find the exact value of this gradient, you can differentiate from first principles.

If you consider the chord AP, where P is a point very close on the curve to A, the x-coordinate of P is  $2+h$ , where  $h$  is a very small number. As  $h$  gets smaller, the chord AP will get closer to the tangent at the point (2, 4).

The coordinates of P are  $(2+h, 4+4h+h^2)$ .

The chord AP will never be equal to the tangent, but the slopes will be more identical as  $h$  gets smaller.

Gradient of AP:

$$\begin{aligned} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{(2+h)^2 - (2)^2}{(2+h) - (2)} \\ &= \frac{4h + h^2}{h} \\ &= 4 + h \text{ (you can divide by } h \text{ as } h \neq 0. \end{aligned}$$

As  $h$  approaches 0, the gradient of AP tends towards 4. When  $h$  is zero, A and P are the same point, so it doesn't make sense to find the gradient.

Eg:

As  $h \rightarrow 0$ , the gradient of chord AP  $\rightarrow$  Gradient of tangent at A

Thus  $4+h \rightarrow 4$ , so the gradient of the curve  $y = x^2$  at point (2, 4) is 4.

### Example

Use first principles to find the gradient of the tangent to the curve  $y = 2x^2 + 3x - 18$  at the point B( $a, 2a^2 + 3a - 18$ )

The chord joining point B to the point P on the curve must have an x-coordinate of  $a+h$ .

Gradient of chord BP:

$$\begin{aligned} &= \frac{[2(a+h)^2 + 3(a+h) - 18] - [2a^2 + 3a - 18]}{(a+h) - a} \\ &= \frac{4ah + 2h^2 + 3h}{h} \\ &= 4a + 2h + 3 \end{aligned}$$

As  $h$  approaches 0, the gradient of the tangent at B is  $4a+3$ .

## Improper Integrals

There are two types of improper integrals:

- The integral has  $\infty$  or  $-\infty$  as one of its limits.
- The integrand is undefined either at one of its limits, or somewhere between them limits. The function may have a denominator which equals 0 at some point.

**-Limits of integration between  $\infty$  or  $-\infty$**

Replace  $\pm\infty$  with  $n$ , and once you have integrated, allow  $n$  to approach infinity. If the integral approaches a finite answer, then the integral can be found. If the integral doesn't approach a finite answer, the integral cannot be found.

### Example

Determine  $\int_1^{\infty} \frac{1}{x^2} dx$

Substitute  $n$  for  $\infty$ :

$$\int_1^n \frac{1}{x^2} dx = \int_1^n x^{-2} dx$$

$$= \left[ \frac{x^{-1}}{-1} \right]_1^n$$

$$= -\frac{1}{n} + 1$$

As  $n \rightarrow \infty$ , the value of the integral approaches 1. Therefore the value of the improper integral is 1.

### Example

Determine whether the integral  $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$  has a value. If so, find the value of the integral.

Substituting  $n$  for  $\infty$ , then integrating gives:

$$\left[ 2x^{\frac{1}{2}} \right]_1^n$$

$$= 2\sqrt{n} - 2$$

As  $n \rightarrow \infty$ , the value of this integral does not approach a finite number, and so the integral cannot be found.

### Integral Undefined at a Particular Value

If the integrand is undefined at one of the limits of integration, replace that limit of integration with  $p$ , and see what happens as  $p$  approaches the limit of integration. If the integral approaches a finite number, the integral can be found. If the integral doesn't approach a finite number, the integral cannot be found.

If the integrand is undefined at some point between the two limits of integration, then you need to split the integral into two parts, one to the left, and one to the right of the point where the integrand is undefined and use a similar approach as before.

#### Example

Determine  $\int_{-4}^2 \frac{1}{x^2} dx$

$$\int_{-4}^2 \frac{1}{x^2} dx = \int_{-4}^0 \frac{1}{x^2} dx + \int_0^2 \frac{1}{x^2} dx$$

When  $x=0$ ,  $0=p$ .

$$\rightarrow \left[ \frac{-1}{x} \right]_{-4}^p + \left[ \frac{-1}{x} \right]_p^2$$

$$\rightarrow \left[ \left( \frac{-1}{p} \right) - \left( \frac{1}{4} \right) \right] + \left[ \left( \frac{-1}{2} \right) - \left( \frac{-1}{p} \right) \right]$$

$$\rightarrow \frac{-2}{p} - \frac{3}{4}$$

As  $p$  tends towards 0, no finite answer is reached. Therefore, the integral cannot be found.



# 9

## Trigonometry

### Cosine:

The general solution of the equation  $\cos\theta = \cos\alpha$  for any angle  $\alpha$  is:

$$\theta = 360n^\circ \pm \alpha$$

or

$$\theta = 2n\pi \pm \alpha$$

### Sine:

The general solution of the equation  $\sin\theta = \sin\alpha$  for any angle  $\alpha$  is:

$$\theta = 180n^\circ + (-1)^n\alpha$$

or

$$\theta = n\pi + (-1)^n\alpha$$

### Tangent:

The general solution of the equation  $\tan\theta = \tan\alpha$  for any angle  $\alpha$  is:

$$\theta = 180n^\circ + \alpha$$

or

$$\theta = n\pi + \alpha$$

### Example

Find the general solution, in degrees, of the equation  $\cos 5\theta = \frac{\sqrt{3}}{2}$

$\frac{\sqrt{3}}{2}$  is the cosine of  $30^\circ$ .

So the general solution for  $5\theta$  is:

$$5\theta = 360n^\circ \pm 30^\circ$$

Therefore the general solution for  $\theta$  is:

$$\theta = 72n^\circ \pm 6^\circ$$

**Example**

Find the general solution, in radians to the equation  $\tan\left(\frac{\pi}{4} - 3x\right) = \sqrt{3}$

---

$$\text{As } \sqrt{3} = \tan\frac{\pi}{3},$$

$$\frac{\pi}{4} - 3x = n\pi + \frac{\pi}{3}$$

$$3x = -n\pi - \frac{\pi}{12}$$

$$x = -\frac{n}{3}\pi - \frac{\pi}{36}$$

To simplify the equation further, allow  $m=-n$ . Therefore,

$$x = \frac{m\pi}{3} - \frac{\pi}{36}$$

**Example**

Find the general solution, in radians, of the equation

$$\sin\left(2x + \frac{\pi}{4}\right) = \cos\left(2x + \frac{\pi}{4}\right)$$

---

Dividing both sides by the RHS gives:

$$\tan\left(2x + \frac{\pi}{4}\right) = 1$$

$$\text{As } \tan^{-1}(1) = \frac{\pi}{4}$$

$$2x + \frac{\pi}{4} = n\pi + \frac{\pi}{4}$$

$$x = \frac{n\pi}{2}$$

# 10

## Numerical Solution of Equations

### Interval Bisection

If you know there is a root of  $f(x)=0$  between  $x=a$  and  $x=b$ , you can try  $x = \frac{1}{2}(a + b)$ . Whether this is positive or negative determines which side of  $\frac{1}{2}(a + b)$  the root lies.

### Example

Show that the equation  $x^3 + 5x = 9$  has a root between 1 and 2. Use the method of interval bisection twice to obtain an interval of width 0.25 within which the root must lie.

$$f(x) = x^3 + 5x - 9$$

$$f(1) = 1 + 5 - 9 = -3$$

$$f(2) = 8 + 10 - 9 = 9$$

$f(1)$  is negative and  $f(2)$  is positive, therefore a root lies between 1 and 2.

$$f(1.5) = 1.875$$

$f(1)$  is negative and  $f(1.5)$  is positive, therefore a root lies between 1 and 1.5.

$$f(1.25) = -0.796875$$

As  $f(1.25)$  is negative and  $f(1.5)$  is positive, there must be a root between 1.25 and 1.5

### Linear Interpolation

In solving the equation  $f(x) = x^3 + 5x - 9$ , you can deduce from  $f(1) = -3$  and  $f(2) = 9$  that the root of  $f(x) = x^3 + 5x - 9 = 0$  is much more likely to be nearer to 1 than 2, since  $|f(2)| > |f(1)|$ .

By plotting (1, -3) and (2, 9) on a graph, and connecting them by a straight line, the estimated root is  $1+p$ . Using similar triangles, we can deduce:

$$\frac{p}{3} = \frac{1}{9+3}$$

Thus the estimated root is  $1+p$  which is  $1+0.25=1.25$

### Example

Show that the equation  $\cos \frac{\pi x}{2} = 4x - 1$  has a root between 0 and 1.

Use linear interpolation to find an approximate value of this root.

---

$$\text{Let } f(x) = \cos \frac{\pi x}{2} - 4x + 1$$

$$f(0) = 2$$

$$f(1) = -3$$

The change of sign indicates that  $f(x)=0$  is between  $x=0$  and  $x=1$ .

Using similar triangles, with the root at  $x=p$ , we can deduce that

$$\frac{p}{2} = \frac{1}{2+3}$$
$$p = 0.4$$

Therefore the required approximate value is 0.4

### The Newton- Raphson Method

If you are given an approximation to a root of  $f(x)=0$ , such as  $\alpha$ , then

$\alpha - \frac{f(\alpha)}{f'(\alpha)}$  is generally a better approximation.

In its iterative form, the **Newton- Raphson method** for solving  $f(x)=0$  gives:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

### Example

Use the Newton-Raphson method once, with an initial value of  $x=3$ , to find an approximation for a root of the equation  $14 + 8x + 5x^2 - x^4 = 0$ .  
Give your answer to three significant figures.

$$\text{Let } f(x) = 14 + 8x + 5x^2 - x^4$$

Differentiation gives:

$$f'(x) = 8 + 10x - 4x^3.$$

When  $x=3$ ,

$$f(x) = 2$$

$$f'(x) = -70$$

The required approximation is therefore:

$$3 - \frac{2}{-70} = 3.03 \text{ (3sf)}$$

### Step-by-Step Solution of Differential Equations

The Euler formula is:

$$y_{n+1} \approx y_n + hf(x_n)$$

This formula is used to find approximate solutions for a differential equation of the form  $\frac{dy}{dx} = f(x)$ , where 'f' is a given function. In the equation, h is the step length, and  $(x_n, y_n)$  are the points on a curve.

### Example

The variables  $x$  and  $y$  satisfy the differential equation  $\frac{dy}{dx} = \ln x$ , and  $y=5$  and  $x=2$ .

Use the Euler formula with step length 0.1 to find an approximation for the value of  $y$  when  $x=2.2$

$$x_n = 2 \text{ and } y_n = 5 \text{ and } h = 0.1$$

$$f(x_n) = \ln 2 \approx 0.693$$

$$\text{Thus, } y_{n+1} \approx 5 + (0.1)(0.693) = 5.0693$$

This is an approximation of  $y$  when  $x=2.1$

This time,  $x_n = 2.1$  and  $y_n = 5$

$$f(x_n) = \ln 2.1 \approx 0.742$$

$$\text{Thus, } y_{n+1} \approx 5.0693 + (0.1)(0.742) = 5.143$$

Therefore, when  $x=2.2$ , the approximate value of  $y=5.143$

# 11

## Linear Laws

If you have results for  $x$  and  $y$ , and you need to prove that these results satisfy a formula, such as  $y^3 = ax^2 + b$ , you will need to then calculate the values for  $y^3$  and  $x^2$ , and then plot the graph of  $y^3$  by  $x^2$ . The graph should be linear if it does satisfy the equation. Then, you can work out 'a' and 'b' by calculating the gradient and the y-intercept.

$$y = ax^n:$$

Two variables  $x$  and  $y$ , which are related by a law of the form  $y = ax^n$ , can be turned into a linear relation if you use logarithms.

$$\begin{aligned}\text{From } y &= ax^n \\ \log y &= \log(ax^n) \\ &= \log a + \log x^n \\ &= \log a + x \log n\end{aligned}$$

$$y = ab^x$$

Two variables  $x$  and  $y$ , which are related by a law of the form  $y = ab^x$ , can be turned into a linear relation if you use logarithms.

$$\begin{aligned}\log y &= \log(ab^x) \\ &= \log a + \log b^x \\ &= \log a + x \log b\end{aligned}$$