Solutionbank FP1
Edexcel AS and A Level Modular Mathematics

Examination style paper
Exercise A, Question 1

Question:

Use the standard results for $\sum_{r=1}^{n} r$ and for $\sum_{r=1}^{n} r^2$ to show that, for all positive integers n, $\sum_{r=1}^{n}(r+1)(3r+2) = n(an^2+bn+c)$, where the values of a, b and c should be stated.

Solution:

$$\sum_{r=1}^{n}(r+1)(3r+2) = \sum_{r=1}^{n}(3r^2+5r+2)$$
Multiply out brackets first

$$= 3\sum_{r=1}^{n} r^2 + 5\sum_{r=1}^{n} r + 2\sum_{r=1}^{n} 1$$
Split into three separate parts to isolate $\sum r^2$, $\sum r$ and $\sum 1$

$$= 3\frac{n}{6}(n+1)(2n+1) + 5\frac{n}{2}(n+1) + 2n$$
Use standard formulae for $\sum r^2$, $\sum r$ and remember that $\sum_{r=1}^{n} 1 = n$.

$$= \frac{n}{2}[(n+1)(2n+1) + 5(n+1) + 4]$$
Take out factor $\frac{n}{2}$

$$= \frac{n}{2}\left[2n^2 + 3n + 1 + 5n + 5 + 4\right]$$
Multiply out the terms in the bracket.

$$= \frac{n}{2}\left[2n^2 + 8n + 10\right]$$
Simplify the bracket.

$$= n\left[n^2 + 4n + 5\right]$$
Take out factor of 2 from bracket which will then be 'cancelled' by the $\frac{1}{2}$ term to give the answer.

So $a = 1$, $b = 4$ and $c = 5$.

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Examination style paper
Exercise A, Question 2

Question:

$f(x) = x^3 + 3x - 6$

The equation $f(x) = 0$ has a root α in the interval [1, 1.5].

a Taking 1.25 as a first approximation to α, apply the Newton–Raphson procedure once to $f(x)$ to obtain a second approximation to α. Give your answer to three significant figures.

b Show that the answer which you obtained is an accurate estimate to three significant figures.

Solution:

a

$$f(x) = x^3 + 3x - 6$$
$$f'(x) = 3x^2 + 3$$

Differentiate $f(x)$ to give $f'(x)$

Using the Newton-Raphson procedure with $x_1 = 1.25$

$$x_2 = 1.25 - \frac{f(1.25)}{f'(1.25)}$$

State the Newton-Raphson procedure.

$$= 1.25 - \frac{[1.25^3 + 3 \times 1.25 - 6]}{[3 \times 1.25^2 + 3]}$$
$$= 1.25 - \frac{[-0.296875]}{7.6875}$$
$$= 1.25 + .0386\ldots$$

Substitute 1.25.

$$= 1.29 (\text{to } 3 \text{ sf})$$

Give your answer to the required accuracy.

b

$$f(1.285) = -0.023\ldots < 0$$
$$f(1.295) = 0.0567\ldots > 0$$

Check the sign of $f(x)$ for the lower and upper bounds of values which round to 1.29 (to 3 sf).

As there is a change of sign and $f(x)$ is continuous the root α satisfies

State 'sign change' and draw a conclusion.

$$1.285 < \alpha < 1.295$$

$\therefore\ \alpha = 1.29$(correct to 3 sf).

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Examination style paper
Exercise A, Question 3

Question:

$\mathbf{R} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ and $\mathbf{S} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$

a Describe fully the geometric transformation represented by each of **R** and **S**.

b Calculate **RS**.

The unit square, U, is transformed by the transformation represented by **S** followed by the transformation represented by **R**.

c Find the area of the image of U after both transformations have taken place.

Solution:

a

R represents a rotation of 135° anti-clockwise about 0.

R takes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$ so is rotation.

S represents an enlargement scale factor $\sqrt{2}$ centre 0

S is of the form $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ so is enlargement with scale factor k.

b

$$\mathbf{RS} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

Use the process of matrix multiplication eg $(ab)\begin{pmatrix} c \\ d \end{pmatrix} = ac + bd$.

c

Determinant of **RS** = 2

$\therefore$ Area scale factor of U is 2.

$\therefore$ Image of U has area 2.

Recall that the determinant of matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$ and that this represents an area scale factor.

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Examination style paper
Exercise A, Question 4

Question:

$f(z) = z^4 + 3z^2 - 6z + 10$

Given that $1 + i$ is a complex root of $f(z) = 0$,

a state a second complex root of this equation.

b Use these two roots to find a quadratic factor of $f(z)$, with real coefficients.

Another quadratic factor of $f(z)$ is $z^2 + 2z + 5$.

c Find the remaining two roots of $f(z) = 0$.

Solution:

a

$1 - i$ is a second root.

This is the conjugate of $1 + i$, and complex roots of polynomial equations with real coefficients occur in conjugate pairs.

b

$[z - (1 + i)][z - (1 - i)]$ is a quadratic factor.

$\therefore\ z^2 - 2z + 2$ is the factor.

Multiply the two linear factors to give a quadratic factor.

c

If $z^2 + 2z + 5 = 0$

$$z = \frac{-2 \pm \sqrt{4 - 20}}{2}$$
$$= -1 \pm \frac{1}{2}\sqrt{16}\,i$$
$$= -1 \pm 2i$$

Use the quadratic formula $z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Remaining roots are $-1 + 2i$ and $-1 - 2i$.

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Exercise A, Question 5

Question:

The rectangular hyperbola H has equation $xy = c^2$. The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ lie on the hyperbola H.

a Show that the gradient of the chord PQ is $-\frac{1}{pq}$.

The point $R, \left(3c, \frac{c}{3}\right)$ also lies on H and PR is perpendicular to QR.

b Show that this implies that the gradient of the chord PQ *is* 9.

Solution:

a

The gradient of the chord PQ is $\dfrac{\frac{c}{p} - \frac{c}{q}}{cp - cq}$

Use gradient $= \dfrac{y_2 - y_1}{x_2 - x_1}$.

$= c\dfrac{(q-p)}{pq} \div c(p-q)$

Use a common denominator to combine the fractions.

$= c\dfrac{(q-p)}{pq} \times \dfrac{1}{c(p-q)}$

$= -\dfrac{(p-q)}{pq(p-q)}$

Express $(q-p)$ as $-(p-q)$

$= \dfrac{-1}{pq}$

Divide numerator and denominator by the factor $(p-q)$.

b

PR has gradient $\dfrac{-1}{3p}$

Use the result established in part (a) to deduce these gradients.

QR has gradient $\dfrac{-1}{3q}$

These lines are perpendicular

$\therefore \dfrac{-1}{3p} \times \dfrac{-1}{3q} = -1$

Use the condition for perpendicular lines $mm' = -1$.

$\therefore \dfrac{1}{9pq} = -1$

$\therefore \dfrac{1}{pq} = -9$

$\therefore$ Gradient of $PQ = \dfrac{-1}{pq} = 9$.

Find the value of $\dfrac{-1}{pq}$.

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Examination style paper
Exercise A, Question 6

Question:

$\mathbf{M} = \begin{pmatrix} x & 2x-7 \\ -1 & x+4 \end{pmatrix}$

a Find the inverse of matrix **M**, in terms of x, given that **M** is non-singular.

b Show that **M** is a singular matrix for two values of x and state these values.

Solution:

a The determinant of **M** is

$$x(x+4)-(-1)(2x-7)$$
$$= x^2+4x+2x-7$$
$$= x^2+6x-7$$

The inverse of **M** is

$$\frac{1}{x^2+6x-7}\begin{pmatrix} x+4 & 7-2x \\ 1 & x \end{pmatrix}$$

Use the result that the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

b **M** is singular when

$$x^2+6x-7=0$$
ie: $(x+7)(x-1)=0$
$\therefore\ x=-7$ or 1.

Put the value of the determinant of **M** equal to zero.

Then solve the quadratic equation.

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Examination style paper
Exercise A, Question 7

Question:

The complex numbers z and w are given by $z = \frac{7-\text{i}}{1-\text{i}}$, and $w = \text{i}z$.

a Express z and w in the form $a + \text{i}b$, where a and b are real numbers.

b Find the argument of w in radians to two decimal places.

c Show z and w on an Argand diagram

d Find $|z - w|$.

Solution:

a

$$z = \frac{7-\text{i}}{1-\text{i}} = \frac{(7-\text{i})(1+\text{i})}{(1-\text{i})(1+\text{i})} = \frac{8+6\text{i}}{2} = 4+3\text{i}$$

Multiply numerator and denominator by the conjugate of $1-\text{i}$.

Remember $\text{i}^2 = -1$

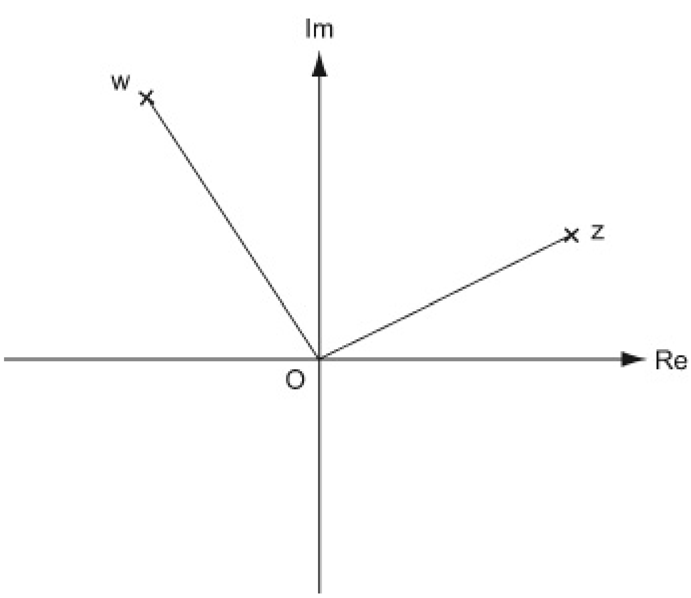
$$w = 1z = \text{i}(4+3\text{i}) = -3+4\text{i}$$

b

$$\arg w = \pi - \left(\tan^{-1} 4/3\right) = 2.21$$

As w is in the second quadrant in the Argand diagram.

c



d

$$\begin{aligned} z-w &= 7-\mathrm{i} \\ |z-w| &= \sqrt{7^2+(-1)^2} \\ &= \sqrt{50} \\ &= 5\sqrt{2}. \end{aligned}$$

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Examination style paper
Exercise A, Question 8

Question:

The parabola C has equation $y^2 = 16x$.

a Find the equation of the normal to C at the point P, (1, 4).

The normal at P meets the directrix to the parabola at the point Q.

b Find the coordinates of Q.

c Give the coordinates of the point R on the parabola, which is equidistant from Q and from the focus of C.

Solution:

a

$$y^2 = 16x \Rightarrow y = 4x^{\frac{1}{2}}$$
$$\frac{dy}{dx} = 4 \times \frac{1}{2}x^{\frac{-1}{2}}$$
$$= 2x^{\frac{-1}{2}}$$

At (1, 4) gradient is 2

$\therefore$ Gradient of normal is $\frac{-1}{2}$

The equation of the normal is $y - 4 = \frac{-1}{2}(x - 1)$

ie: $y = \frac{-1}{2}x + 4\frac{1}{2}$

Find the gradient of the curve at (1, 4).

Use $mm' = -1$ as the normal is perpendicular to the curve.

Use $y - y_1 = m(x - x_1)$

b

The directrix has equation $x = -4$.

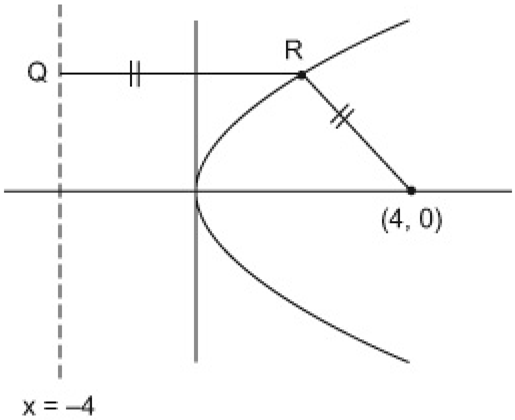
Substitute $x = -4$ into normal equation

$\therefore$ $y = 6\frac{1}{2}$

So Q is the point $\left(-4, 6\frac{1}{2}\right)$.

The directrix of the parabola $y^2 = 4ax$ has equation $x = -a$.

c



$$\text{At } R \ y = 6\frac{1}{2}$$

The point R must have the same y co-ordinate as the point Q.

$$\therefore \left(6\frac{1}{2}\right)^2 = 16x$$

$$\therefore x = \frac{6\frac{1}{2} \times 6\frac{1}{2}}{16} = \frac{169}{64}$$

So R is the point $\left(\frac{169}{64}, \frac{13}{2}\right)$

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Examination style paper
Exercise A, Question 9

Question:

a Use the method of mathematical induction to prove that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} r + \left(\frac{1}{2}\right)^{r-1} = \frac{1}{2}(n^2 + n + 4) - \left(\frac{1}{2}\right)^{n-1}.$$

b $f(n) = 3^{n+2} + (-1)^n 2^n, n \in \mathbb{Z}^+$.

By considering $2f(n+1) - f(n)$ and using the method of mathematical induction prove that, for $n \in \mathbb{Z}^+$, $3^{n+2} + (-1)^n 2^n$ is divisible by 5.

Solution:

a Let $n = 1$

$$LHS = 1 + \left(\frac{1}{2}\right)^0 = 1 + 1 = 2$$

$$RHS = \frac{1}{2}\left(1^2 + 1 + 4\right) - \left(\frac{1}{2}\right)^0$$
$$= \frac{1}{2} \times 6 - 1 = 2$$

Show that the result is true when $n = 1$.

$\therefore$ $LHS = RHS$ so result is true for $n = 1$

Assume that the result is true for $n = k$

ie: $$\sum_{r=1}^{k}\left[r + \left(\frac{1}{2}\right)^{r-1}\right] = \frac{1}{2}\left(k^2 + k + 4\right) - \left(\frac{1}{2}\right)^{k-1}$$

Add $(k+1) + \left(\frac{1}{2}\right)^k$ to each side.

Show that assuming the result is true for $n = k$ implies that it is also true for $n = k + 1$

$$\therefore \sum_{r=1}^{k+1} r + \left(\frac{1}{2}\right)^{r-1} = \frac{1}{2}\left(k^2 + k + 4\right) + (k+1) - \left(\frac{1}{2}\right)^{k-1} + \left(\frac{1}{2}\right)^k$$

$$= \frac{1}{2}\left(k^2 + k + 4 + 2k + 2\right) + \left(\frac{1}{2}\right)^{k-1}\left(-1 + \frac{1}{2}\right)$$

Collect the similar terms together.

$$= \frac{1}{2}\left(k^2 + 3k + 6\right) - \frac{1}{2}\left(\frac{1}{2}\right)^{k-1}$$

$$= \frac{1}{2}\left((k+1)^2 + (k+1) + 4\right) - \left(\frac{1}{2}\right)^k$$

ie : $$\sum_{r=1}^{n} r + \left(\frac{1}{2}\right)^{r-1} = \frac{1}{2}\left(n^2 + n + 4\right) - \left(\frac{1}{2}\right)^{n-1}$$

where $n = k + 1$

ie: Result is implied for $n = k + 1$.

$\therefore$ By induction, as result is true for $n = 1$ then it is implied for $n = 2$, $n = 3$, etc… ie: for all positive integer values for n.

Conclude that this implies by induction that the result is true for all positive integers.

b

$f(n) = 3^{n+2} + (-1)^n 2^n \; n \in Z^+$

Let $n = 1$

$$\begin{aligned} f(1) &= 3^3 + (-1)^1 2^1 \\ &= 27 - 2 \\ &= 25 \end{aligned}$$

Show that the result is true when $n = 1$.

This is divisible by 5.

Let $f(k)$ be divisible by 5

Assume that $f(k)$ is divisible by 5

ie: $3^{k+2} + (-1)^k 2^k = 5A$ *

Consider

$$2f(k+1) - f(k) = 2.3^{k+3} + 2(-1)^{k+1}2^{k+1} - 3^{k+2} - (-1)^k 2^k$$

Follow the hint given in the question

$$\begin{aligned} &= 3^{k+2}[2.3 - 1] + 2^k(-1)^k[-4 - 1] \\ &= 3^{k+2} \times 5 - 5.(-1)^k 2^k \\ &= 5\left(3^{k+2} - (-1)^k 2^k\right). \end{aligned}$$

Collect similar terms together and look for common factor of 5.

$\therefore$ $2f(k+1) - f(k)$ is divisible by 5.

$= 5B$

$$\begin{aligned} \therefore\ 2f(k+1) &= 5B + f(k) \\ &= 5(B + a) \end{aligned}$$

As $f(k)$ and $2f(k+1) - f(k)$ are each divisible by 5, deduce that $f(k+1)$ is also divisible by 5.

ie: $2f(k+1)$ is divisible by $5 \Rightarrow f(k+1)$ is divisible by 5.

So by induction as f(1) is divisible by 5 then so is f(2) and so is f(3) …. and by induction $f(n)$ is divisible by 5 for all positive integers n.

Use induction to complete your proof.