Review exercise 2

1 a **A** has 3 columns and **B** has 2 rows.

The number of columns in **A** is not the same as the number of rows in **B**.

Therefore, the product **AB** does not exist.

b **B** has 2 columns and **A** has 2 rows.

The number of columns in **B** is the same as the number of rows in **A**.

Therefore, the product **BA** exists:

$$\mathbf{BA} = \begin{pmatrix} q & 0 \\ 3 & -1 \end{pmatrix} \times \begin{pmatrix} 3 & 2 & p \\ 0 & 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3q & 2q & pq \\ 9 & 4 & 3p+1 \end{pmatrix}$$

c **BA** has 3 columns and **C** has 3 rows.

The number of columns in **BA** is the same as the number of rows in **A**.

Therefore, the product **BAC** exists:

$$\mathbf{BAC} = \begin{pmatrix} 3q & 2q & pq \\ 9 & 4 & 3p+1 \end{pmatrix} \times \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 12q - 6q + pq \\ 36 - 12 + 3p + 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6q + pq \\ 3p + 25 \end{pmatrix}$$

d **C** has 1 column and **B** has 2 rows.

The number of columns in **C** is not the same as the number of rows in **B**.

Therefore, the product **CBA** does not exist.

2 $$\mathbf{M}^2 = \begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix}$$

$\mathbf{M}^2$ is quite complicated to work out and it is sensible to calculate this before working out $\mathbf{M}^2 + a\mathbf{M} + b\mathbf{I}$

$$= \begin{pmatrix} 0\times0+3\times(-1) & 0\times3+3\times2 \\ (-1)\times0+2\times(-1) & (-1)\times3+2\times2 \end{pmatrix}$$

$$= \begin{pmatrix} 0-3 & 0+6 \\ 0-2 & -3+4 \end{pmatrix} = \begin{pmatrix} -3 & 6 \\ -2 & 1 \end{pmatrix}$$

Then consider $\mathbf{M}^2 + a\mathbf{M} + b\mathbf{I} = \mathbf{O}$

$$\begin{pmatrix} -3 & 6 \\ -2 & 1 \end{pmatrix} + a\begin{pmatrix} 0 & 3 \\ -1 & 2 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 6 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 3a \\ -a & 2a \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -3+b & 6+3a \\ -2-a & 1+2a+b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Equating the top left elements

$$-3+b=0 \Rightarrow b=3$$

Equating the top right elements

$$6+3a=0 \Rightarrow a=-2$$

$$a=-2,\ b=3$$

There are four elements which could be equated but you only need to equate two of them to find a and b. You could use the others to check your working. For example; if $a = -2$, $b = 3$ then $1 + 2a + b = 1 - 4 + 3$ which does equal 0.

3 $\mathbf{A}^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix}$

$\mathbf{A}^2 - (a+d)\mathbf{A}$

$$= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} - \begin{pmatrix} (a+d)a & (a+d)b \\ (a+d)c & (a+d)d \end{pmatrix}$$

$$= \begin{pmatrix} a^2+bc-a^2-ad & ab+bd-ab-bd \\ ac+cd-ac-cd & bc+d^2-ad-d^2 \end{pmatrix}$$

$$= \begin{pmatrix} bc-ad & 0 \\ 0 & bc-ad \end{pmatrix} = \lambda\mathbf{I} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so

$\lambda\mathbf{I} = \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

You can write down the results of simple calculations like this without showing all of the working.

Equating the top left (or bottom right elements)

$$\lambda = bc - ad$$

Note that $\lambda = -\det(\mathbf{A})$.

4 a $\det(\mathbf{A}) = 2\times(-1) - 3\times p = -2-3p$

If A is singular, $\det(\mathbf{A}) = 0$.

$$-2-3p=0 \Rightarrow 3p=-2 \Rightarrow p=-\frac{2}{3}$$

You need to know that, if $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(\mathbf{A}) = ad - bc$.

b As in part (a), $\det(\mathbf{A}) = -2-3p$

$$-2-3p=4 \Rightarrow -3p=6 \Rightarrow p=-2$$

c $\mathbf{A}^2 = \begin{pmatrix} 2 & 3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ -2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 4-6 & 6-3 \\ -4+2 & -6+1 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ -2 & -5 \end{pmatrix}$$

$$\mathbf{A}^2 - \mathbf{A} = \begin{pmatrix} -2 & 3 \\ -2 & -5 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = -4\mathbf{I}$$

This is the required result with $k = -4$.

5 a For a matrix $\mathbf{M}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathbf{M}^{-1}=\frac{1}{\det(\mathbf{M})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Here, $\det(\mathbf{A})=4\times 2-(-1)\times(-6)=8-6=2$

$$\mathbf{A}^{-1}=\frac{1}{2}\begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix}=\begin{pmatrix} 1 & \frac{1}{2} \\ 3 & 2 \end{pmatrix}$$

b $(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}$

$$=\begin{pmatrix} 2 & 0 \\ 3 & p \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ 3 & 2 \end{pmatrix}$$

$$=\begin{pmatrix} 2 & 1 \\ 3p+3 & 2p+\frac{3}{2} \end{pmatrix}$$

You need to remember this property of the inverse of matrices. The order of **A** and **B** is reversed in this formula.

c $(\mathbf{AB})(\mathbf{AB})^{-1}=\mathbf{I}$

$$\begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3p+3 & 2p+\frac{3}{2} \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The product of any matrix and its inverse is **I**. This applies to a product matrix, **AB** in this case, as well as to a matrix such as **A**.

Equating the upper left elements

$$-1\times 2+2(3p+3)=1$$
$$-2+6p+6=1$$
$$6p=-3$$
$$p=-\frac{1}{2}$$

Finding all four of the elements of the product matrix of the left hand side of this equation would be lengthy. To find p, you only need one equation, so you only need to consider one element. Here the upper left hand element has been used but you could choose any of the four elements.

6 a $\det(\mathbf{A})=4p\times q-(-q)\times(-3p)$

$$=4pq-3pq=pq$$

$$\mathbf{A}^{-1}=\frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}$$

In your final answer, you could multiply each term in the matrix by $\frac{1}{pq}$, which would give

$$\mathbf{A}^{-1}=\begin{pmatrix} \frac{1}{p} & \frac{1}{p} \\ \frac{3}{q} & \frac{4}{q} \end{pmatrix}$$

b $\mathbf{AX}=\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$

Multiply both sides on the left by $\mathbf{A}^{-1}$

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$$

$$\mathbf{X} = \frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$$

$$= \frac{1}{pq}\begin{pmatrix} 2pq - pq & 3q^2 + q^2 \\ 6p^2 - 4p^2 & 9pq + 4pq \end{pmatrix}$$

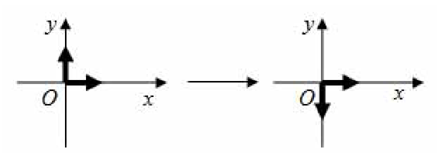
$$= \frac{1}{pq}\begin{pmatrix} pq & 4q^2 \\ 2p^2 & 13pq \end{pmatrix}$$

It is important to multiply by $\mathbf{A}^{-1}$ on the correct side of the expression. As shown here, multiplying on the left of $\mathbf{AX}$, you get $\mathbf{A}^{-1}\mathbf{AX} = (\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{IX} = \mathbf{X}$, which is what you are asked to find.

If instead you multiplied both sides on the right by $\mathbf{A}^{-1}$ you would get $\mathbf{AXA}^{-1}$, which does not simplify, and no further progress can be made.

Working out $\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}\frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}$ instead of $\frac{1}{pq}\begin{pmatrix} q & q \\ 3p & 4p \end{pmatrix}\begin{pmatrix} 2p & 3q \\ -p & q \end{pmatrix}$ is a common error.

7 a Reflection in the x-axis transforms



$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

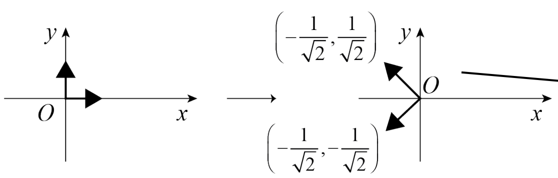
(1, 0) lies on the x-axis and so is not changed by reflection in the x-axis.

So

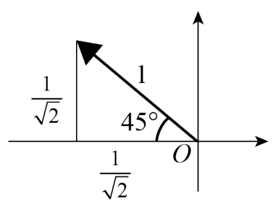
$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Unless the question states otherwise, it is acceptable to write down a simple matrix like this without showing your working.

Rotation of +135° about (0,0) transforms



The geometry of the vector to which $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is transformed is



$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}-\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$$

So

$$\mathbf{B} = \begin{pmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$$

Arrows have been added so that you can see where the columns in **B** come from.

$$\mathbf{C} = \mathbf{AB} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

7 b $\mathbf{C}^2 = \begin{pmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$

As an example of the calculations;

$$-\frac{1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} = +\frac{1\times 1}{\sqrt{2}\times\sqrt{2}} = \frac{1}{2}$$

$$= \begin{pmatrix}\left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)+\left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) & \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)+\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\\ \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right) & \left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{2}+\frac{1}{2} & \frac{1}{2}-\frac{1}{2}\\ \frac{1}{2}-\frac{1}{2} & \frac{1}{2}+\frac{1}{2}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \mathbf{I}, \text{as required.}$$

Alternatively, to make the calculation simpler you could initially write

$\mathbf{C} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & -1\\ -1 & 1\end{pmatrix}$ and square this expression.

8 a $\mathbf{M}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$

$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}a\\c\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}$

Equating the elements:

$a=3, c=2$

Also,

$\mathbf{M}\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$

Using $a=3, c=2$ from above,

$\begin{pmatrix}3 & b\\2 & d\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}6+b\\4+d\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$

Equating the upper elements

$6+b=2 \Rightarrow b=-4$

Equating the lower elements

$4+d=1 \Rightarrow d=-3$

$a=3, b=-4, c=2, d=-3$

In questions about transformations, you need to write the coordinates of points as column vectors. For example, the coordinate (1, 0) is written as the column vector $\begin{pmatrix}1\\0\end{pmatrix}$.

b $\mathbf{M}=\begin{pmatrix}3 & -4\\2 & -3\end{pmatrix}$

$\mathbf{M}^2=\begin{pmatrix}3 & -4\\2 & -3\end{pmatrix}\begin{pmatrix}3 & -4\\2 & -3\end{pmatrix}=\begin{pmatrix}9-8 & -12+12\\6-6 & -8+9\end{pmatrix}$

$=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=\mathbf{I}$, as required

c As $\mathbf{M}^2=\mathbf{I}, \mathbf{M}^{-1}\mathbf{M}^2=\mathbf{M}^{-1}\mathbf{I}$

$\mathbf{M}=\mathbf{M}^{-1}$

The matrix **M** is its own inverse. This follows from the result in part **b**. In more detail:

$$\mathbf{M}^2=\mathbf{I}$$
$$\mathbf{MM}=\mathbf{I}$$
$$\mathbf{M}^{-1}(\mathbf{MM})=\mathbf{M}^{-1}\mathbf{I}$$
$$(\mathbf{M}^{-1}\mathbf{M})\mathbf{M}=\mathbf{M}^{-1}$$
$$\mathbf{IM}=\mathbf{M}^{-1}$$
$$\mathbf{M}=\mathbf{M}^{-1}$$

$\mathbf{M}\begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}8\\-3\end{pmatrix}$

$\mathbf{M}^{-1}\mathbf{M}\begin{pmatrix}p\\q\end{pmatrix}=\mathbf{M}^{-1}\begin{pmatrix}8\\-3\end{pmatrix}$

$\mathbf{I}\begin{pmatrix}p\\q\end{pmatrix}=\mathbf{M}\begin{pmatrix}8\\-3\end{pmatrix}$

In this question, as **M** is its own inverse, you can replace $\mathbf{M}^{-1}$ by **M**.

$\begin{pmatrix}p\\q\end{pmatrix}=\begin{pmatrix}3 & -4\\2 & -3\end{pmatrix}\begin{pmatrix}8\\-3\end{pmatrix}=\begin{pmatrix}24+12\\16+9\end{pmatrix}=\begin{pmatrix}36\\25\end{pmatrix}$

Hence $p=36, q=25$

9 a $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2y - x \\ 3y \end{pmatrix} = \begin{pmatrix} -1x + 2y \\ 0x + 3y \end{pmatrix}$

$= \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

So $\mathbf{C} = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}$

b $\det(\mathbf{C}) = -1 \times 3 - 2 \times 0 = -3$

$\mathbf{C}^{-1} = \frac{1}{-3}\begin{pmatrix} 3 & -2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}$

You are given the results of transforming the points by T and are asked to find the original points. You are "working backwards" to the original points and so you will need the inverse matrix.

Let the coordinates of A, B and C be $(x_A, y_A), (x_B, y_B)$ and (x_C, y_C) respectively.

$\mathbf{C}\begin{pmatrix} x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix} = \begin{pmatrix} 0 & 10 & 10 \\ 3 & 15 & 12 \end{pmatrix}$

$\mathbf{C}^{-1}\mathbf{C}\begin{pmatrix} x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix} = \mathbf{C}^{-1}\begin{pmatrix} 0 & 10 & 10 \\ 3 & 15 & 12 \end{pmatrix}$

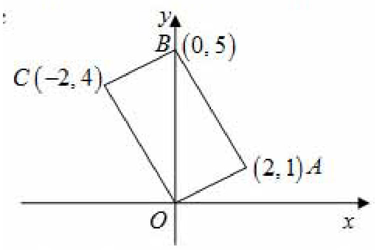
$\begin{pmatrix} x_A & x_B & x_C \\ y_A & y_B & y_C \end{pmatrix} = \begin{pmatrix} -1 & \frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 0 & 10 & 10 \\ 3 & 15 & 12 \end{pmatrix}$

$= \begin{pmatrix} 2 & -10+10 & -10+8 \\ 1 & 5 & 4 \end{pmatrix}$

$= \begin{pmatrix} 2 & 0 & -2 \\ 1 & 5 & 4 \end{pmatrix}$

Hence $A:(2,1),\ B:(0,5),\ C:(-2,4)$

9 c



Considering the gradients of the sides

$$m_{OA} = \frac{1}{2};\ m_{CB} = \frac{5-4}{0-(-2)} = \frac{1}{2}$$

So OA is parallel to CB.

$$m_{OC} = \frac{4-0}{-2-0} = -2;\ m_{AB} = \frac{5-1}{0-2} = \frac{4}{-2} = -2$$

So *OC* is parallel to *AB*.
The opposite sides of *OABC* are parallel to each other and so *OABC* is a parallelogram.
To show that *OABC* is specifically a rectangle (not just any parallelogram), we must show that one interior angle is a right-angle:
Consider the angle *AOC*

$$m_{OA} \times m_{OC} = \frac{1}{2} \times -2 = -1.$$

So *OA* is perpendicular to *OC,* and hence *AOC* is a right angle.
So the parallelogram *OABC* contains a right angle, and hence *OABC* is a rectangle.

> Using the properties of quadrilaterals you learnt for GCSE, there are many alternative ways of showing that *OABC* is a rectangle. This is just one of many possibilities, using the result you learnt in the C1 module that the gradient of the line joining (x_1, y_1) to (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$.

10 a $\det(\mathbf{A}) = k \times 2k - (k-1)\times(-3)$

$$= 2k^2 + 3k - 3$$

b The triangle has been enlarged by a factor of

$$\frac{198}{18} = 11$$

So $\det(\mathbf{A}) = 11$

$$2k^2 + 3k - 3 = 11$$

$$2k^2 + 3k - 14 = 0$$

$$(2k+7)(k-2) = 0$$

$$k = -\frac{7}{2}, 2$$

> The determinant is the area scale factor in transformations. This is equivalent to $\frac{\text{area of image}}{\text{area of object}} = \det(\mathbf{A})$. So the scale factor is part **b** must equal the determinant in part **a**.

11 a

$$\det\mathbf{M} = \begin{vmatrix} \frac{3}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{vmatrix}$$

$$= \frac{3}{\sqrt{2}} \times \frac{3}{\sqrt{2}} - \left(-\frac{3}{\sqrt{2}}\right) \times \frac{3}{\sqrt{2}}$$

$$= \frac{9}{2} + \frac{9}{2} = 9$$

Area scale factor $= 9$

Scale factor $= \sqrt{9} = 3$

For a linear transformation represented by matrix **M**, det **M** represents the scale factor for the change in area.
Hence $\sqrt{\det \mathbf{M}}$ represents the linear scale factor for the enlargement.

b $$\begin{pmatrix} \frac{3}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3\cos\theta & -3\sin\theta \\ 3\sin\theta & 3\cos\theta \end{pmatrix}$$

$$3\cos\theta = \frac{3}{\sqrt{2}}$$

$$\cos\theta = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\theta = 45^\circ$ anti-clockwise about $(0,0)$

Checking using the elements gives **M**.

11 c Let the coordinates of P be (x, y)

Let the coordinates of P be (x, y)

Applying the matrix **M** to $\begin{pmatrix} x \\ y \end{pmatrix}$ gives $\begin{pmatrix} p \\ q \end{pmatrix}$

$$\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}$$

Left multiply by the inverse $\mathbf{M}^{-1}$

$$\mathbf{M}^{-1}\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} p \\ q \end{pmatrix}$$

$$\mathbf{I}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{9}\begin{pmatrix} \frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ -\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3p}{9\sqrt{2}} + \frac{3q}{9\sqrt{2}} \\ -\frac{3p}{9\sqrt{2}} + \frac{3q}{9\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{p+q}{3\sqrt{2}} \\ \frac{-p+q}{3\sqrt{2}} \end{pmatrix}$$

The coordinates of P are $\left(\frac{p+q}{3\sqrt{2}}, \frac{-p+q}{3\sqrt{2}}\right)$

12 $\sum_{r=1}^{n}(2r-1)^2 = \sum_{r=1}^{n}(4r^2-4r+1)$

$= 4\sum_{r=1}^{n}r^2 - 4\sum_{r=1}^{n}r + n$

$= \frac{4}{6}n(n+1)(2n+1) - 2n(n+1) + n$

$= \frac{2}{3}n(n+1)(2n+1) - \frac{6}{3}n(n+1) + \frac{3}{3}n$

$= \frac{1}{3}n\left[2(n+1)(2n+1) - 6(n+1) + 3\right]$

$= \frac{1}{3}n\left[4n^2 + 6n + 2 - 6n - 6 + 3\right]$

$= \frac{1}{3}n(4n^2-1)$ as required

13 $\sum_{r=1}^{n}r(r^2-3) = \sum_{r=1}^{n}r^3 - 3\sum_{r=1}^{n}r$

$= \frac{1}{4}n^2(n+1)^2 - \frac{3}{2}n(n+1)$

$= \frac{1}{4}n^2(n+1)^2 - \frac{6}{4}n(n+1)$

$= \frac{1}{4}n(n+1)\left[n(n+1) - 6\right]$

$= \frac{1}{4}n(n+1)(n^2+n-6)$

$= \frac{1}{4}n(n+1)(n-2)(n+3)$ as required

14 a $\sum_{r=1}^{n} r(2r-1) = 2\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r$

$$= \frac{2}{6}n(n+1)(2n+1) - \frac{1}{2}n(n+1)$$
$$= \frac{2}{6}n(n+1)(2n+1) - \frac{3}{6}n(n+1)$$
$$= \frac{1}{6}n(n+1)[2(2n+1)-3]$$
$$= \frac{1}{6}n(n+1)(4n-1)$$
$$= \frac{n(n+1)(4n-1)}{6} \text{ as required}$$

b $\sum_{r=11}^{30} r(2r-1) = \sum_{r=1}^{30} r(2r-1) - \sum_{r=1}^{10} r(2r-1)$

$$= \frac{1}{6}(30)(30+1)(4(30)-1) - \frac{1}{6}(10)(10+1)(4(10)-1)$$
$$= 18\,445 - 715$$
$$= 17\,730$$

15 a $\sum_{r=1}^{n} (6r^2+4r-5) = 6\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - 5n$

$$= n(n+1)(2n+1) + 2n(n+1) - 5n$$
$$= n\left[(2n^2+3n+1) + 2(n+1) - 5\right]$$
$$= n\left[2n^2+3n+1+2n+2-5\right]$$
$$= n(2n^2+5n-2) \text{ as required}$$

b $\sum_{r=10}^{25} (6r^2+4r-5) = \sum_{r=1}^{25} (6r^2+4r-5) - \sum_{r=1}^{9} (6r^2+4r-5)$

$$= (25)\left(2(25)^2+5(25)-2\right) - (9)\left(2(9)^2+5(9)-2\right)$$
$$= 34\,325 - 1845$$
$$= 32\,480$$

16 a $\sum_{r=1}^{n} r(r+1) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$

$$= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$$
$$= \frac{1}{6}n(n+1)(2n+1) + \frac{3}{6}n(n+1)$$
$$= \frac{1}{6}n(n+1)[(2n+1)+3]$$
$$= \frac{1}{6}n(n+1)(2n+4)$$
$$= \frac{1}{3}n(n+1)(n+2) \text{ as required}$$

b $\sum_{r=n}^{3n} r(r+1) = \sum_{r=1}^{3n} r(r+1) - \sum_{r=1}^{n-1} r(r+1)$

$$= \frac{1}{3}(3n)(3n+1)(3n+2) - \frac{1}{3}(n-1)(n)(n+1)$$
$$= \frac{1}{3}n[3(3n+1)(3n+2) - (n-1)(n+1)]$$
$$= \frac{1}{3}n[(27n^2+27n+6) - (n^2-1)]$$
$$= \frac{1}{3}n(26n^2+27n+7)$$
$$= \frac{1}{3}n(2n+1)(13n+7)$$

Therefore $p = 13$ and $q = 7$

17 a $\sum_{r=1}^{n} r^2(r-1) = \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r^2$

$$= \frac{1}{4}n^2(n+1)^2 - \frac{1}{6}n(n+1)(2n+1)$$

$$= \frac{3}{12}n^2(n+1)^2 - \frac{2}{12}n(n+1)(2n+1)$$

$$= \frac{1}{12}n(n+1)\left[3n(n+1) - 2(2n+1)\right]$$

$$= \frac{1}{12}n(n+1)(3n^2 + 3n - 4n - 2)$$

$$= \frac{1}{12}n(n+1)(3n^2 - n - 2)$$

So $p = 3$, $q = -1$ and $r = -2$

b $\sum_{r=50}^{100} r^2(r-1) = \sum_{r=1}^{100} r^2(r-1) - \sum_{r=1}^{49} r^2(r-1)$

$$= \frac{(100)}{12}(101)\left(3(100)^2 - (100) - 2\right) - \frac{(49)}{12}(50)\left(3(49)^2 - (49) - 2\right)$$

$$= 25\,164\,150 - 1\,460\,200$$

$$= 23\,703\,950$$

18 $\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$

Let $n = 1$.

The left-hand side becomes

$$\sum_{r=1}^{1} r(r+3) = 1(1+3) = 4$$

$\sum_{r=1}^{1} r(r+3)$ consists of just one term. That is $r(r+3)$ with 1 substituted for r.

The right-hand side becomes

$$\frac{1}{3}\times 1(1+1)(1+5) = \frac{1}{3}\times 2\times 6 = 4$$

The left-hand side and the right-hand side are equal and so the summation is true for $n = 1$.

Assume the summation is true for $n = k$.

That is $\sum_{r=1}^{k} r(r+3) = \frac{1}{3}k(k+1)(k+5) \ldots \ldots *$

This is often called the **induction hypothesis**.

$$\sum_{r=1}^{k+1} r(r+3) = \sum_{r=1}^{k} r(r+3) + (k+1)(k+4)$$

The sum from 1 to $k+1$ is the sum from 1 to k plus one extra term.

In this case, the extra term is found by replacing each r in $r(r+3)$ by $k+1$.

$$= \frac{1}{3}k(k+1)(k+5) + (k+1)(k+4), \text{using}*$$

$$= \frac{1}{3}k(k+1)(k+5) + \frac{3}{3}(k+1)(k+4)$$

$$= \frac{1}{3}(k+1)[k(k+5) + 3(k+4)]$$

$$= \frac{1}{3}(k+1)[k^2 + 5k + 3k + 12]$$

$$= \frac{1}{3}(k+1)[k^2 + 8k + 12]$$

$$= \frac{1}{3}(k+1)(k+2)(k+6)$$

$$= \frac{1}{3}(k+1)((k+1)+1)((k+1)+5)$$

Multiplying out the brackets would give you an awkward cubic expression which would be difficult to factorise. You should try to simplify the working by looking for any common factors and taking them outside a bracket. Here $k+1$ is a common factor.

This expression is $\frac{1}{3}n(n+1)(n+5)$ with each n replaced by $k+1$.

This is the result obtained by substituting $n = k+1$ into the right-hand side of the summation and so the summation is true for $n = k+1$.

The summation is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k+1$.

By mathematical induction the summation is true for all positive integers *n*.

19 $\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$

Let $n=1$.
The left-hand side becomes

$$\sum_{r=1}^{1}(2r-1)^2 = (2-1)^2 = 1^2 = 1$$

$\sum_{r=1}^{1}(2r-1)^2$ consists of just one term.
That is $(2r-1)^2$ with 1 substituted for r.

The right-hand side becomes

$$\frac{1}{3}\times 1(2-1)(2+1) = \frac{1}{3}\times 1\times 1\times 3 = 1$$

The left-hand side and the right-hand side are equal and so the summation is true for $n=1$.

Assume the summation is true for $n=k$.

That is $\sum_{r=1}^{k}(2r-1)^2 = \frac{1}{3}k(2k-1)(2k+1)\ldots\ldots*$

The sum from 1 to $k+1$ is the sum from 1 to k plus one extra term. In this case, the extra term is found by replacing the r in $(2r-1)^2$ by $k+1$.

$$\sum_{r=1}^{k+1}(2r-1)^2 = \sum_{r=1}^{k}(2r-1)^2 + (2(k+1)-1)^2$$
$$= \sum_{r=1}^{k}(2r-1)^2 + (2k+1)^2$$
$$= \frac{1}{3}k(2k-1)(2k+1) + \frac{3}{3}(2k+1)^2, \text{using } *$$
$$= \frac{1}{3}(2k+1)[k(2k-1)+3(2k+1)]$$
$$= \frac{1}{3}(2k+1)[2k^2+5k+3]$$
$$= \frac{1}{3}(2k+1)(k+1)(2k+3)$$
$$= \frac{1}{3}(k+1)(2(k+1)-1)(2(k+1)+1)$$

Multiplying out the brackets would give you an awkward cubic expression which would be difficult to factorise. Look for any common factors and take them outside a bracket. Here $(2k+1)$ is a common factor.

This expression is $\frac{1}{3}n(2n-1)(2n+1)$ with each n replaced by $k+1$.

This is the result obtained by substituting $n=k+1$ into the right-hand side of the summation and so the summation is true for $n=k+1$.

The summation is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.

By mathematical induction the summation is true for all positive integers n.

20 $\sum_{r=1}^{n} a_r = \sum_{r=1}^{n} r(r+1)(2r+1) = \frac{1}{2}n(n+1)^2(n+2)$

Let $n=1$.

The left-hand side becomes

$$\sum_{r=1}^{1} r(r+1)(2r+1) = 1\times 2\times 3 = 6$$

The right-hand side becomes

$$\frac{1}{2}\times 1\times 2^2\times 3 = 6$$

The left-hand side and the right-hand side are equal and so the summation is true for $n=1$.

All inductions need to be shown to be true for a small number, usually 1.

Assume the summation is true for $n=k$.

That is $\sum_{r=1}^{k} r(r+1)(2r+1) = \frac{1}{2}k(k+1)^2(k+2)$ *

$$\sum_{r=1}^{k+1} r(r+1)(2r+1) = \sum_{r=1}^{k} r(r+1)(2r+1) + (k+1)(k+2)(2k+3)$$

$$= \frac{1}{2}k(k+1)^2(k+2) + \frac{2}{2}(k+1)(k+2)(2k+3), \text{using } *$$

$$= \frac{1}{2}(k+1)(k+2)[k(k+1)+2(2k+3)]$$

$$= \frac{1}{2}(k+1)(k+2)[k^2+5k+6]$$

$$= \frac{1}{2}(k+1)(k+2)(k+2)(k+3)$$

$$= \frac{1}{2}(k+1)(k+2)^2(k+3)$$

$$= \frac{1}{2}(k+1)((k+1)+1)^2((k+1)+2)$$

Fractions need to be expressed to the same denominator before factorising. The form of the answer shows that you need to have $\frac{1}{2}$ as a common factor and it helps you to write $\frac{2}{2}$ before the second term on the right-hand side of the summation.

This expression is $\frac{1}{2}n(n+1)^2(2n+1)$ with each n replaced by $k+1$.

This is the result obtained by substituting $n=k+1$ into the right-hand side of the summation and so the summation is true for $n=k+1$.

The summation is true for $n=1$, and if it is true for $n=k$, then it is true for $n=k+1$.

By mathematical induction the summation is true for all positive integers n.

21 $\sum_{r=1}^{n} r^2(r-1) = \frac{1}{12}n(n-1)(n+1)(3n+2)$

Let $n=1$.

The left-hand side becomes

$$\sum_{r=1}^{1} r^2(r-1) = 1^2 \times (1-1) = 0$$

$\sum_{r=1}^{1} r^2(r-1)$ consists of just one term. That is $r^2(r-1)$ with 1 substituted for r. In this case, because of the bracket, this clearly gives 0.

The right-hand side becomes

$$\frac{1}{12}\times 1\times(1-1)\times(1+1)\times(3+2)$$

$$=\frac{1}{12}\times 1\times 0\times 2\times 5 = 0$$

The left-hand side and the right-hand side are equal and so the summation is true for $n=1$.

Assume the summation is true for $n=k$.

That is $\sum_{r=1}^{k} r^2(r-1) = \frac{1}{12}k(k-1)(k+1)(3k+2)\ldots\ldots *$

$$\sum_{r=1}^{k+1} r^2(r-1) = \sum_{r=1}^{k} r^2(r-1) + (k+1)^2(k+1-1)$$

$$=\frac{1}{12}k(k-1)(k+1)(3k+2)+\frac{12}{12}k(k+1)^2, \text{using} *$$

The common factors in these two terms are $\frac{1}{12}$, k and $(k+1)$.

$$=\frac{1}{12}k(k+1)[(k-1)(3k+2)+12(k+1)]$$

$$=\frac{1}{12}k(k+1)[3k^2-k-2+12k+12]$$

$$=\frac{1}{12}k(k+1)[3k^2+11k+10]$$

$$=\frac{1}{12}(k+1)k(k+2)(3k+5)$$

$$=\frac{1}{12}(k+1)((k+1)-1)((k+1)+1)(3(k+1)+2)$$

Rearrange this expression so that it is the right-hand side of the summation with n replaced by $k+1$.

This is the result obtained by substituting $n=k+1$ into the right-hand side of the summation and so the summation is true for $n=k+1$.

The summation is true for $n=1$, and, if it is true for $n=k$, then it is true for $n=k+1$.

By mathematical induction the summation is true for all positive integers n.

22 a $\mathrm{f}(n) = 3^{4n} + 2^{4n+2}$

$\mathrm{f}(k+1) - \mathrm{f}(k) = 3^{4(k+1)} + 2^{4(k+1)+2} - (3^{4k} + 2^{4k+2})$

$= 3^{4k+4} - 3^{4k} + 2^{4k+6} - 2^{4k+2}$

$= 3^{4k}(3^4 - 1) + 2^{4k}(2^6 - 2^2)$

$= 3^{4k} \times 80 + 2^{4k} \times 60$

$= 3^{4k-1} \times 3 \times 80 + 2^{4k} \times 60$

$= 240 \times 3^{4k-1} + 60 \times 2^{4k}$

$= 15(16 \times 3^{4k-1} + 4 \times 2^{4k})$ *

For all $k \in \mathbb{Z}^+$, $(16 \times 3^{4k-1} + 4 \times 2^{4k})$ is an integer, and, hence, $\mathrm{f}(k+1) - \mathrm{f}(k)$ is divisible by 15.

> At this stage $\mathrm{f}(k+1) - \mathrm{f}(k)$ is clearly divisible by 10 (and 20), but since 80 is not a multiple of 15, to obtain that the expression is divisible by 15 you have to remove a further factor of 3 by writing 3^{4k} as $3^{4k-1} \times 3^1$.

> This shows that 15 is a factor of $\mathrm{f}(k+1) - \mathrm{f}(k)$ and this is the equivalent to showing that $\mathrm{f}(k+1) - \mathrm{f}(k)$ is exactly divisible by 15. Note that the result would not be true for negative integers k as, for example, 4×2^{4k} would be a fraction less than one.

b Let $n = 1$

$\mathrm{f}(1) = 3^4 + 2^6 = 81 + 64 = 145 = 5 \times 29$

So $\mathrm{f}(n)$ is divisible by 5 for $n = 1$.

Assume that $\mathrm{f}(k)$ is divisible by 5.

It would follow that $\mathrm{f}(k) = 5m$, where m is an integer.

From *

$\mathrm{f}(k+1) = \mathrm{f}(k) + 15(16 \times 3^{4k-1} + 4 \times 2^{4k})$

$= 5m + 15(16 \times 3^{4k-1} + 4 \times 2^{4k})$

$= 5(m + 3(16 \times 3^{4k-1} + 4 \times 2^{4k}))$

So $\mathrm{f}(k+1)$ is divisible by 5.

$\mathrm{f}(n)$ is divisible by 5 for $n = 1$, and, if it is divisible by 5 for $n = k$, then it divisible by 5 for $n = k+1$.

By mathematical induction, $\mathrm{f}(n)$ is divisible by 5 for all $n \in \mathbb{Z}^+$.

> An expression which is divisible by 15 is certainly divisible by 5, which is all that is required for part **b**.

> If both $\mathrm{f}(k)$ and $15(16 \times 3^{4k-1} + 4 \times 2^{4k})$ are divisible by 5, then their sum, $\mathrm{f}(k+1)$ is divisible by 5.

> Although $\mathrm{f}(k+1) - \mathrm{f}(k)$ is divisible by 15, $\mathrm{f}(n)$ is never divisible by 15 for any n. After completing part **a**, you might misread the question for part **b** and attempt to prove that 15 was a factor of $\mathrm{f}(n)$. It is always necessary to read questions carefully.

23 a $f(n) = 24 \times 2^{4n} + 3^{4n}$

$f(n+1) - f(n)$
$= 24 \times 2^{4(n+1)} + 3^{4(n+1)} - 24 \times 2^{4n} - 3^{4n}$

This is an acceptable answer for part **a**. However, reading ahead, the question concerns divisibility by 5. So it is sensible to further work on this expression and show that it is divisible by 5.

b $f(n+1) - f(n)$
$= 24 \times 2^{4n+4} - 24 \times 2^{4n} + 3^{4n+4} - 3^{4n}$
$= 24 \times 2^{4n}(2^4 - 1) + 3^{4n}(3^4 - 1)$
$= 24 \times 2^{4n} \times 15 + 3^{4n} \times 80$
$= 5(72 \times 2^{4n} + 16 \times 3^{4n}) \ldots *$

In the middle of a question it is easy to forget that, in all inductions, you need to show that the result is true for a small number. This is usually 1 but this question asks you to show a result is true for all non-negative integers and 0 is a non-negative integer, so you should begin with 0.

Let $n = 0$

$f(0) = 24 \times 2^0 + 3^0 = 24 + 1 = 25$

So $f(n)$ is divisible by 5 for $n = 0$.

Assume that f (k) is divisible by 5. It would follow that $f(k) = 5m$, where m is an integer.

From*, substituting $n = k$ and rearranging,

$f(k+1) = f(k) + 5(72 \times 2^{4n} + 16 \times 3^{4n})$
$= 5m + 5(72 \times 2^{4n} + 16 \times 3^{4n})$
$= 5(m + 72 \times 2^{4n} + 16 \times 3^{4n})$

So $f(k+1)$ is divisible by 5.

$f(n)$ is divisible by 5 for $n = 0$, and, if it is divisible by 5 for $n = k$, then it divisible by 5 for $n = k + 1$.

By mathematical induction, $f(n)$ is divisible by 5 for all non-negative integers n.

24 Let $f(n) = 7^n + 4^n + 1$

> The question gives no label to the function $7^n + 4^n + 1$. Since you are going to have to refer to this function a number of times in your solution, it helps if you call it f(n).

Let $n = 1$

$f(1) = 7^1 + 4^1 + 1 = 12$

12 is divisible by 6, so $f(n)$ is divisible by 6 for $n = 1$.

Consider $f(k+1) - f(k)$

> This question gives you no hint to help you. With divisibility questions, it often helps to consider $f(k+1) - f(k)$ and try and show that this divides by the appropriate number, here 6. It does not always work and there are other methods which often work just as well or better.

$$
\begin{aligned}
f(k+1) - f(k) &= 7^{k+1} + 4^{k+1} + 1 - (7^k + 4^k + 1) \\
&= 7^{k+1} - 7^k + 4^{k+1} - 4^k \\
&= 7^k(7-1) + 4^k(4-1) \\
&= 6 \times 7^k + 3 \times 4^k \\
&= 6 \times 7^k + 3 \times 4 \times 4^{k-1} \\
&= 6(7^k + 2 \times 4^{k-1}) \ldots \ *
\end{aligned}
$$

So 6 is a factor of $f(k+1) - f(k)$.

Assume that f(k) is divisible by 6.

It would follow that $f(k) = 6m$, where m is an integer.

From *

$$
\begin{aligned}
f(k+1) &= f(k) + 6(7^k + 2 \times 4^{k-1}) \\
&= 6m + 6(7^k + 2 \times 4^{k-1}) \\
&= 6(m + 7^k + 2 \times 4^{k-1})
\end{aligned}
$$

> If both f(k) and $6(7^k + 2 \times 4^{k-1})$ are divisible by 6, then their sum, $f(k+1)$ is divisible by 6. You could write this down instead of the working shown here.

So $f(k+1)$ is divisible by 6.

f(n) is divisible by 6 for $n = 1$, and, if it is divisible by 6 for $n = k$, then it divisible by 6 for $n = k + 1$.

By mathematical induction, f(n) is divisible by 6 for all positive integers n.

25 Let $f(n) = 4^n + 6n - 1$

The question gives no label to the function $4^n + 6n - 1$. Since you are going to have to refer to this function a number of times in your solution, it helps if you call it f(n).

Let $n = 1$

$f(1) = 4^1 + 6 - 1 = 9$

So f(n) is divisible by 9 for $n = 1$.

Assume that f(k) is divisible by 9,

Then, for some integer m,

$f(k) = 4^k + 6k - 1 = 9m$

With divisibility questions, it often helps to consider $f(k+1) - f(k)$ and try and show that this divides by the appropriate number, here 9. This will work in this question if you choose to do it. However the method shown here is, for this question, a neat one and you need to be aware of various alternative methods. No particular method works every time.

Rearranging

$4^k = 9m - 6k + 1 \ldots *$

$$\begin{aligned} f(k+1) &= 4^{k+1} + 6(k+1) - 1 \\ &= 4 \times 4^k + 6k + 5 \\ &= 4 \times (9m - 6k + 1) + 6k + 5 \\ &= 36m - 24k + 4 + 6k + 5 \\ &= 36m - 18k + 9 \\ &= 9(4m - 2k + 1) \end{aligned}$$

Here you substitute the expression for 4^k in * for the 4^k in your expression for $f(k+1)$.

This is divisible by 9.

f(n) is divisible by 9 for $n = 1$, and, if it is divisible by 9 for $n = k$, then it divisible by 9 for $n = k + 1$.

By mathematical induction, f(n) is divisible by 9 for all $n \in \mathbb{Z}^+$.

26 Let $f(n) = 3^{4n-1} + 2^{4n-1} + 5$

Let $n = 1$

$f(1) = 3^3 + 2^3 + 5 = 27 + 8 + 5 = 40 = 10 \times 4$

So f (n) is divisible by 10 for $n = 1$.

Consider $f(k+1) - f(k)$

$$\begin{aligned} f(k+1) - f(k) &= 3^{4k+3} + 2^{4k+3} - 5 - (3^{4k-1} + 2^{4k-1} - 5) \\ &= 3^{4k+3} - 3^{4k-1} + 2^{4k+3} - 2^{4k-1} \\ &= 3^{4k-1}(3^4 - 1) + 2^{4k-3}(2^6 - 2^2) \\ &= 3^{4k-1} \times 80 + 2^{4k-3} \times 30 \\ &= 10(8 \times 3^{4k-1} + 3 \times 2^{4k-3}) \ldots * \end{aligned}$$

When you replace n by $k+1$ in, for example, 3^{4n-1} you get $3^{4(k+1)-1} = 3^{4k+4-1} = 3^{4k+3}$.

If we were to simplify $2^{4k+3} - 2^{4k-1}$ as far as possible, we could write $2^{4k-1}\left(2^4 - 1\right)$ in the next line.

However, $2^4 - 1 = 15$ which is not divisible by 10, so we only simplify the expression to $2^{4k-3}(2^6 - 2^2)$ which is divisible by 10.

Assume that f (k) is divisible by 10.

It would follow that $f(k) = 10m$, where m is an integer.

From *

$$\begin{aligned} f(k+1) &= f(k) + 10(8 \times 3^{4k-1} + 3 \times 2^{4k-3}) \\ &= 10m + 10(8 \times 3^{4k-1} + 3 \times 2^{4k-3}) \\ &= 10(m + (8 \times 3^{4k-1} + 3 \times 2^{4k-3})) \end{aligned}$$

If both f (k) and $10(8 \times 3^{4k-1} + 3 \times 2^{4k-3})$ are divisible by 10, then their sum, f (k+1) is divisible by 10. If you preferred, you could write this down instead of the working shown here.

So $f(k+1)$ is divisible by 10.

f(n) is divisible by 10 for $n = 1$, and, if it is divisible by 10 for $n = k$, then it divisible by 10 for $n = k + 1$.

By mathematical induction, f(n) is divisible by 10 for all positive integers n.

27 $\mathbf{A}^n = \begin{pmatrix} 1 & (2^n - 1)c \\ 0 & 2^n \end{pmatrix}$

Let $n = 1$

$\mathbf{A}^1 = \begin{pmatrix} 1 & (2^1 - 1)c \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & c \\ 0 & 2 \end{pmatrix}$

You need to begin by showing the result is true for $n = 1$. You substitute $n = 1$ into the printed expression for $\mathbf{A}^n$ and check that you get the matrix **A** as given in the question.

This is A, as defined in the question, so the result is true for $n = 1$.

Assume the result is true for $n = k$.

That is $\mathbf{A}^k = \begin{pmatrix} 1 & (2^k - 1)c \\ 0 & 2^k \end{pmatrix}$…… *

$\mathbf{A}^{k+1} = \mathbf{A}^k.\mathbf{A}$

$= \begin{pmatrix} 1 & (2^k - 1)c \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 1 & c \\ 0 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & c + 2(2^k - 1)c \\ 0 & 2 \times 2^k \end{pmatrix}$

$= \begin{pmatrix} 1 & c + 2^{k+1}c - 2c \\ 0 & 2^{k+1} \end{pmatrix}$

$= \begin{pmatrix} 1 & (2^{k+1} - 1)c \\ 0 & 2^{k+1} \end{pmatrix}$

Keep in mind as you multiply out the matrices that you are aiming at the expression $\mathbf{A}^n = \begin{pmatrix} 1 & (2^n - 1)c \\ 0 & 2^n \end{pmatrix}$ with each n replaced by $k + 1$.

$2 \times 2^k = 2^1 \times 2^k = 2^{k+1}$ by one of the laws of indices.
You use this twice.

This is the result obtained by substituting $n = k + 1$ into the result $\mathbf{A}^n = \begin{pmatrix} 1 & (2^n - 1)c \\ 0 & 2^n \end{pmatrix}$ and so the result is true for $n = k + 1$.

The result is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k + 1$.

By mathematical induction the result is true for all positive integers n.

28 $\mathbf{A}^n = \begin{pmatrix} 2n+1 & n \\ -4n & -2n+1 \end{pmatrix}$

Let $n=1$

$\mathbf{A}^1 = \begin{pmatrix} 2+1 & 1 \\ -4 & -2+1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$

You need to begin by showing the result is true for $n = 1$. You substitute $n = 1$ into the given expression for $\mathbf{A}^n$ and check that you get the matrix **A**, as given in the question.

This is A, as defined in the question, so the result is true for $n = 1$.

Assume the result is true for $n = k$.

That is $\mathbf{A}^k = \begin{pmatrix} 2k+1 & k \\ -4k & -2k+1 \end{pmatrix}$

The **induction hypothesis** is the result you are asked to prove with n replaced by k.

$\mathbf{A}^{k+1} = \mathbf{A}^k.\mathbf{A}$

$= \begin{pmatrix} 2k+1 & k \\ -4k & -2k+1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$

$= \begin{pmatrix} 3(2k+1)-4k & 2k+1-k \\ -12k-4(-2k+1) & -4k-(-2k+1) \end{pmatrix}$

$= \begin{pmatrix} 2k+3 & k+1 \\ -4k-4 & -2k-1 \end{pmatrix}$

$= \begin{pmatrix} 2(k+1)+1 & k+1 \\ -4(k+1) & -2(k+1)+1 \end{pmatrix}$

$\mathbf{A}^{k+1}$ is the matrix **A**, multiplied by itself k times, multiplied by **A** one more time. $\mathbf{A}^{k+1} = \mathbf{A}^k.\mathbf{A}^1 = \mathbf{A}^k.\mathbf{A}$. This is one of the index laws applied to matrices.

This is the result obtained by substituting $n = k+1$ into the result $\mathbf{A}^n = \begin{pmatrix} 2n+1 & n \\ -4n & -2n+1 \end{pmatrix}$ and so the result is true for $n = k+1$.

The result is true for $n = 1$, and, if it is true for $n = k$, then it is true for $n = k + 1$.

By mathematical induction the result is true for all positive integers n.

29 a He has not shown the general statement to be true for $k = 1$.

29 b

Let $f(n) = 2^{2n} - 1,$ where $n \in \mathbb{Z}^+$

$f(1) = 2^{2(1)} - 1 = 3,$ which is divisible by 3.

$f(n)$ is divisible by 3 when $n = 1$.

Assume true for $n = k$, so that $f(k) = 2^{2k} - 1$ is divisible by 3.

$$\begin{aligned} f(k+1) &= 2^{2(k+1)} - 1 \\ &= 2^{2k} \times 2^2 - 1 \\ &= 4(2^{2k}) - 1 \\ &= 4(f(k) + 1) - 1 \\ &= 4f(k) + 3 \end{aligned}$$

Therefore $f(n)$ is divisible by 3 when $n = k + 1$

If $f(n)$ is divisible by 3 when $n = k$, then it has been shown that $f(n)$ is also divisible by 3 when $n = k + 1$.

As $f(n)$ is divisible by 3 when $n = 1$, $f(n)$ is also divisible by 3 for all $n \in \mathbb{Z}^+$ by mathematical induction.

30 a $u_{n+1} = 2u_n + 1,\ n \in \mathbb{R},\ u_1 = 1$

$u_1 = 1$

$u_2 = 2(1) + 1 = 3$

$u_3 = 2(3) + 1 = 7$

b Basis: $u_n = 2^n - 1 \Rightarrow n = 1,\ u_1 = 2^1 - 1 = 1,\ n = 2,\ u_2 = 2^2 - 1 = 3$

and from the recurrence relation $u_2 = 2(1) + 1 = 3$

Thus the general statement for u_n is true for $n = 1$ and $n = 2$.

Assumption: Assume the general statement is true for $n = k \Rightarrow u_k = 2^k - 1$

Induction: Using the recurrence relation

$$\begin{aligned} u_{k+1} &= 2(2^k - 1) + 1 \\ &= 2^{k+1} - 2 + 1 \\ &= 2^{k+1} - 1 \end{aligned}$$

This is the same expression that the general statement gives for u_{k+1}.

Conclusion: If u_n is true when $n = k$, then it has been shown that u_n is also true when $n = k + 1$. As u_n is true for $n = 1$ and $n = 2$ then u_n is true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

31 Basis: $u_n = 5\left(6^{n-1}\right)+1 \Rightarrow n=1,\ u_1 = 5\left(6^{1-1}\right)+1=6,\ n=2,\ u_2 = 5\left(6^{2-1}\right)+1=31$

and from the recurrence relation $u_2 = 6\times 6-5=31$

Thus the general statement for u_n is true for $n = 1$ and $n = 2$.

Assumption: Assume the general statement is true for $n=k \Rightarrow u_k = 5\left(6^{k-1}\right)+1$

Induction: Using the recurrence relation

$$\begin{aligned} u_{k+1} &= 6u_k - 5 \\ &= 6\left[5\left(6^{k-1}\right)+1\right]-5 \\ &= 5(6)\left(6^{k-1}\right)+6-5 \\ &= 5\left(6^{k}\right)+1 \end{aligned}$$

This is the same expression that the general statement gives for u_{k+1}.

Conclusion: If u_n is true when $n = k$, then it has been shown that u_n is also true when $n = k + 1$. As u_n is true for $n = 1$ and $n = 2$ then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

Challenge

1 Basis: When $n = 1$, $r = 2$

$$2(1) \leqslant 2 \leqslant \frac{1}{2}\left(1^2 + 1 + 2\right) \text{ so the inequality holds true for } n = 1.$$

Assumption: Assume the inequality holds true for $n = k,\ k \in \mathbb{Z}^+$. Thus, for any diagram of k non-parallel lines: $2k \leqslant r_k \leqslant \frac{1}{2}\left(k^2 + k + 2\right)$

Induction:
Consider a new line being added to the diagram. Lines are non-parallel, therefore the additional line must cross all k previous lines.
The minimum number of 'crossings' will be 1, where all previous lines intersect at the same point and the new line passes through this point (or when there is only 1 previous line).
The maximum number of 'crossings' will occur when the new line crosses each previous line at a separate point. In this case, there will be k 'crossings'.
The new line will pass through one region before its first crossing, one region between each pair of crossings and one region after its last crossing. Thus, the number of regions it passes through will be equal to the number of crossings + 1. The new line will split each region it passes through into 2 new regions. Thus it will increase the number of regions in the diagram by the number of crossings + 1.
Therefore, the $(k + 1)$th line will add between 2 and $(k + 1)$ regions to the diagram, giving the inequality:

$$2 \leqslant r_{k+1} - r_k \leqslant k + 1$$

Adding this to the inequality $2k \leqslant r_k \leqslant \frac{1}{2}\left(k^2 + k + 2\right)$ gives:

$$2k + 2 \leqslant r_k + r_{k+1} - r_k \leqslant \frac{1}{2}\left(k^2 + k + 2\right) + k + 1$$

$$2(k+1) \leqslant r_{k+1} \leqslant \frac{1}{2}\left(k^2 + k + 2 + 2k + 2\right)$$

$$2(k+1) \leqslant r_{k+1} \leqslant \frac{1}{2}\left(k^2 + 2k + 1 + k + 1 + 2\right)$$

$$2(k+1) \leqslant r_{k+1} \leqslant \frac{1}{2}\left((k+1)^2 + (k+1) + 2\right)$$

So if the statement holds for $n = k$, it holds for $n = k + 1$

Conclusion: If the inequality, $2k \leqslant r_k \leqslant \frac{1}{2}\left(k^2 + k + 2\right)$, is true when $n = k$ then it has been shown that it is also true when $n = k + 1$. As the inequality is true for $n = 1$ then it is true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

2 $\begin{pmatrix}4 & 3\\3 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}8\\-7\end{pmatrix}$

Let $\mathbf{A}=\begin{pmatrix}4 & 3\\3 & -1\end{pmatrix}$

det $\mathbf{A}$ $=-13$

$$\mathbf{A}^{-1}=-\frac{1}{13}\begin{pmatrix}-1 & -3\\-3 & 4\end{pmatrix}=\frac{1}{13}\begin{pmatrix}1 & 3\\3 & -4\end{pmatrix}$$

$$\mathbf{A}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}8\\-7\end{pmatrix}$$

$$\Rightarrow \mathbf{A}^{-1}\mathbf{A}\begin{pmatrix}x\\y\end{pmatrix}=\mathbf{A}^{-1}\begin{pmatrix}8\\-7\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x\\y\end{pmatrix}=\mathbf{A}^{-1}\begin{pmatrix}8\\-7\end{pmatrix}$$

$$\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{13}\begin{pmatrix}1 & 3\\3 & -4\end{pmatrix}\begin{pmatrix}8\\-7\end{pmatrix}$$

$$=\begin{pmatrix}-1\\4\end{pmatrix}$$

So $x=-1$ and $y=4$