Review exercise 1

1 a $z_1 - z_2$

$= 4 - 5\text{i} - p\text{i}$

$= 4 - (5 + p)\text{i}$

b $z_1 z_2$

$= (4 - 5\text{i})p\text{i}$

$= 4p\text{i} - 5p\text{i}^2$

$= 4p\text{i} + 5p$

$= 5p + 4p\text{i}$

c $\dfrac{z_1}{z_2}$

$= \dfrac{4 - 5\text{i}}{p\text{i}}$

$= \dfrac{\text{i}(4 - 5\text{i})}{-p}$

$= \dfrac{4\text{i} + 5}{-p}$

$= -\dfrac{5}{p} - \dfrac{4}{p}\text{i}$

2 a $z^3 - kz^2 + 3z$

$= z(z^2 - kz + 3)$

So if there are 2 imaginary roots, the

discriminant of $z^2 - kz + 3 < 0$

$\Rightarrow (-k)^2 - 12 < 0$

$k^2 < 12$

$-2\sqrt{3} < k < 2\sqrt{3}$

b $z^3 - 2z^2 + 3z = 0$

$\Rightarrow z(z^2 - 2z + 3) = 0$

$\Rightarrow z = 0, z = \dfrac{2 \pm \sqrt{-8}}{2}$

$\Rightarrow z = 0, z = 1 \pm \text{i}\sqrt{2}$

3

$$z = \frac{5 \pm \sqrt{25-52}}{2}$$
$$= \frac{5 \pm \sqrt{-27}}{2}$$
$$= \frac{5}{2} \pm \frac{3\sqrt{3}}{2}\text{i}$$

So $z_1, z_2 = \frac{5}{2} + \frac{3\sqrt{3}}{2}\text{i}, \frac{5}{2} - \frac{3\sqrt{3}}{2}\text{i}$

4 $(2-\text{i})x - (1+3\text{i})y - 7 = 0$

$\Rightarrow (2x - y - 7) + (-x - 3y)\text{i} = 0$

$\Rightarrow 2x - y = 7, x + 3y = 0$

$\Rightarrow x = 3, y = -1$

5 a
$$\frac{2+3\text{i}}{5+\text{i}} \times \frac{5-\text{i}}{5-\text{i}} = \frac{10 - 2\text{i} + 15\text{i} + 3}{26}$$
$$= \frac{13+13\text{i}}{26} = \frac{1}{2} + \frac{1}{2}\text{i}$$
$$= \frac{1}{2}(1+\text{i})$$
$$\lambda = \frac{1}{2}$$

$(5+\text{i})(5-\text{i}) = 5^2 + 1^2 = 26$ You should practise doing such calculations mentally.

You use the result from part **a** to simplify the working in part **b**.

b
$$\left(\frac{2+3\text{i}}{5+\text{i}}\right)^4 = \left[\frac{1}{2}(1+\text{i})\right]^4$$
$$= \frac{1}{16}(1 + 4\text{i} + 6\text{i}^2 + 4\text{i}^3 + \text{i}^4)$$
$$= \frac{1}{16}(1 + 4\text{i} - 6 - 4\text{i} + 1)$$
$$= \frac{1}{16} \times -4 = -\frac{1}{4}, \text{ a real number}$$

$(1+\text{i})^4$ is expanded using the binomial expansion

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$\text{i}^3 = \text{i}^2 \times \text{i} = -1 \times \text{i} = -\text{i}$

$\text{i}^4 = \text{i}^2 \times \text{i}^2 = -1 \times -1 = 1$

6 $-1+\text{i}$ is a root $\Rightarrow -1-\text{i}$ is also a root

$\Rightarrow (z+1-\text{i})(z+1+\text{i})$ is a factor

$\Rightarrow z^2 + 2z + 2$ is a factor

$\Rightarrow z^3 + 5z^2 + 8z + 6 = (z^2 + 2z + 2)(z+3)$

$\Rightarrow z = -3, -1 \pm \text{i}$

7 a $f(2-3i)=0$

$\Rightarrow (2-3i)^3-6(2-3i)^2+k(2-3i)-26=0$

$\Rightarrow 8-36i-54+27i-24+72i+54+2k-3ki-26=0$

Equating real coefficients $-42+2k=0 \Rightarrow k=21$

b $2+3i$ must also be a factor

$\Rightarrow (z-2+3i)(z-2-3i)=z^2-4z+13$ is a factor

$\Rightarrow z^3-6z^2+21z-26=(z^2-4z+13)(z-2)$

$\Rightarrow z=2,\ 2+3i$ are the other two factors

8 a $b-3=-1 \Rightarrow b=2$

$-4c=-16 \Rightarrow c=4$

$\Rightarrow z^4-z^3-6z^2-20z-16=(z^2-3z-4)(z^2+2z+4)$

b $z^4-z^3-6z^2-20z-16=(z-4)(z+1)(z^2+2z+4)$

$\Rightarrow z=4,-1,\dfrac{-2\pm\sqrt{12}}{2}$

$\Rightarrow z=4,-1,-1\pm\sqrt{3}i$

9 $(z-1-2i)(z-1+2i)$ must be a factor

$\Rightarrow z^2-2z+5$ is a factor

$\Rightarrow z^4-8z^3+27z^2-50z+50$

$=(z^2-2z+5)(z^2+kz+10)$

Equating coefficients of z^3

$-2+k=-8 \Rightarrow k=-6$

$\Rightarrow (z^2-2z+5)(z^2-6z+10)=0$

$\Rightarrow z=1\pm 2i,\dfrac{6\pm\sqrt{-4}}{2}$

$\Rightarrow z=1\pm 2i, 3\pm i$

10 a Comparing constant coefficients

$\alpha\times\dfrac{4}{\alpha}\times(\alpha+\dfrac{4}{\alpha}+1)=12$

$\Rightarrow 4(\alpha+\dfrac{4}{\alpha}+1)=12$

$\Rightarrow \alpha^2+4+\alpha=3\alpha$

$\Rightarrow \alpha^2-2\alpha+4=0$

$\Rightarrow \alpha=1\pm\sqrt{3}i$

So the roots are $1\pm\sqrt{3}i, 3$

10 b $\mathrm{f}(z) = (z-3)(z-1-\sqrt{3}\mathrm{i})(z-1-\sqrt{3}\mathrm{i})$

$= (z-3)(z^2-2z+4)$

$= z^3-5z^2+10z-12$

$\Rightarrow p=-5, q=10$

11 a $\dfrac{3z-1}{2-\mathrm{i}} = \dfrac{4}{1+2\mathrm{i}}$

$3z-1 = \dfrac{8-4\mathrm{i}}{1+2\mathrm{i}} \times \dfrac{1-2\mathrm{i}}{1-2\mathrm{i}}$

$= \dfrac{8-16\mathrm{i}-4\mathrm{i}-8}{5} = \dfrac{-20\mathrm{i}}{5} = -4\mathrm{i}$

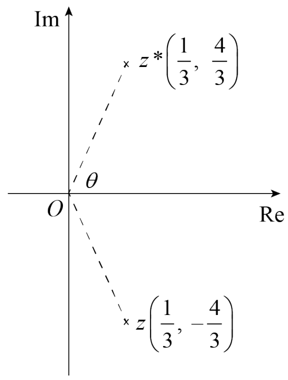
$3z = 1-4\mathrm{i}$

$z = \dfrac{1}{3} - \dfrac{4}{3}\mathrm{i}$

You multiply both sides of the equation by $2-\mathrm{i}$.

Then multiply the numerator and denominator by the conjugate complex of the denominator.

b



You place the points in the Argand diagram which represent conjugate complex numbers symmetrically about the real x-axis.

Label the points so it is clear which is the original number (z) and which is the conjugate (z^*).

c $|z|^2 = \left(\dfrac{1}{3}\right)^2 + \left(-\dfrac{4}{3}\right)^3 = \dfrac{1}{9} + \dfrac{16}{9} = \dfrac{17}{9}$

$|z| = \dfrac{\sqrt{17}}{3}$

$\tan\theta = \dfrac{\frac{4}{3}}{\frac{1}{3}} = 4 \Rightarrow \theta \approx 76^\circ$

z is in the fourth quadrant.

$\arg z = -76^\circ$, to the nearest degree.

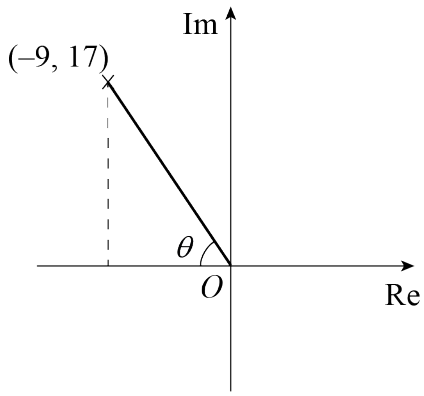
$z = \dfrac{\sqrt{17}}{3}\cos(-76^\circ) + \mathrm{i}\dfrac{\sqrt{17}}{3}\sin(-76^\circ)$

$z^* = \dfrac{\sqrt{17}}{3}\cos 76^\circ + \mathrm{i}\dfrac{\sqrt{17}}{3}\sin 76^\circ$

The diagram you have drawn in part **b** shows that z is in the fourth quadrant. There is no need to draw it again.

It is always true $|z^*| = |z|$ and $\arg z^* = -\arg z$, so you just write down the final answer without further working.

12 a



b $\tan\theta = \frac{17}{9} \Rightarrow \theta = 1.084...$

z is in the second quadrant.

$\arg z = \pi - 1.084\ldots = 2.057\ldots$

$= 2.06$, in radians to 2 d.p.

You have to give your answer to 2 decimal places. To do this accurately you must work to at least 3 decimal places. This avoids rounding errors and errors due to premature approximation.

c

$$w = \frac{25+35\text{i}}{z} = \frac{25+35\text{i}}{-9+17\text{i}} = \frac{25+35\text{i}}{-9+17\text{i}} \times \frac{-9-17\text{i}}{-9-17\text{i}}$$
$$= \frac{-225-425\text{i}-315\text{i}+595}{(-9)^2+17^2}$$
$$= \frac{370-740\text{i}}{370} = 1-2\text{i}$$

In this question, the arithmetic gets complicated. Use a calculator to help you with this. However, when you use a calculator, remember to show sufficient working to make your method clear.

13 a

$$z^2 = (2+\text{i})^2 = 4-4\text{i}+\text{i}^2$$
$$= 4-4\text{i}-1$$
$$= 3-4\text{i}\text{, as required.}$$

b From part **a**, the square roots of $3-4\text{i}$ are $2-\text{i}$ and $-2+\text{i}$.

Taking square roots of both sides of the equation $(z+\text{i})^2 = 3-4\text{i}$

$z+\text{i} = 2-\text{i} \Rightarrow z = 2-2\text{i}$

$z+\text{i} = -2+\text{i} \Rightarrow z = -2$

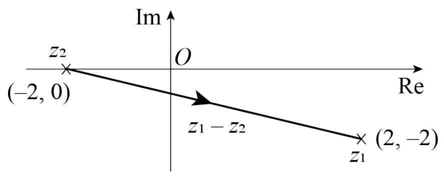
$z_1 = 2-2\text{i}$, say, and $z_2 = -2$

The square root of any number k, real or complex, is a root of $z^2 = k$. Hence, part **a** shows that one square root of $3-4\text{i}$ is $2-\text{i}$.

If one square root of $3-4\text{i}$ is $2-\text{i}$, then the other is $-(2-\text{i})$.

z_1 and z_2 could be the other way round but that would make no difference to $|z_1 - z_2|$ or $z_1 - z_2$, the expressions you are asked about in parts **d** and **e**.

13 c



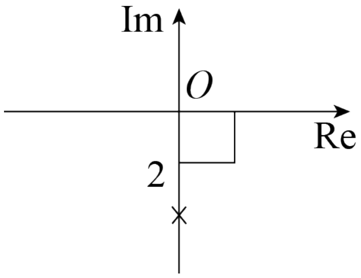
$z_1 - z_2$ can be represented on the diagram you drew in part **c** by the vector joining the point representing z_1 to the point representing z_2. The modulus of $z_1 - z_2$ is then just the length of the line joining these two points and this length can be found using coordinate geometry.

d Using the formula

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$= (2-(-2))^2 + (-2-0)^2$$
$$= 4^2 + 2^2 = 20$$

Hence $|z_1 - z_2| = \sqrt{20} = 2\sqrt{5}$

e $z_1 + z_2 = 2 - 2\text{i} - 2 = -2\text{i}$



The argument of any number on the negative imaginary axis is $-\frac{\pi}{2}$ or $-90°$.

$\arg(z_1 + z_2) = -\frac{\pi}{2}$

14 a $\text{g}(x) = x^3 - x^2 - 1$

$\text{g}(1.4) = -0.216$

$\text{g}(1.5) = 0.125$

There is a change of sign so there must be a root between $x = 1.4$ and $x = 1.5$

b $\text{g}(1.4655) = -0.00025...$

$\text{g}(1.4665) = 0.00326...$

Therefore $1.4655 < \alpha < 1.4665$ and so $\alpha = 1.466$ to 2 d.p.

15 a i $3x^2+4x-1=0$ has roots α and β

The sum of the roots is $\alpha+\beta=-\frac{4}{3}$

The product of the roots is $\alpha\beta=-\frac{1}{3}$

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$$
$$=\left(-\frac{4}{3}\right)^2-2\left(-\frac{1}{3}\right)$$
$$=\frac{22}{9}$$

ii $$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha^2\beta-3\alpha\beta^2$$
$$=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$$
$$=\left(-\frac{4}{3}\right)^3-3\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)$$
$$=-\frac{100}{27}$$

b Roots are $\frac{\alpha}{\beta^2}$ and $\frac{\beta}{\alpha^2}$

Sum of the roots is:

$$\frac{\alpha}{\beta^2}+\frac{\beta}{\alpha^2}=\frac{\alpha^3+\beta^3}{\alpha^2\beta^2}$$
$$=\frac{\alpha^3+\beta^3}{(\alpha\beta)^2}$$
$$=\frac{\left(-\frac{100}{27}\right)}{\left(-\frac{1}{3}\right)^2}$$
$$=-\frac{100}{3}$$

Product of the roots is:

$$\frac{\alpha}{\beta^2}\left(\frac{\beta}{\alpha^2}\right)=\frac{1}{\alpha\beta}$$
$$=\frac{1}{\left(-\frac{1}{3}\right)}$$
$$=-3$$

So

$$x^2+\frac{100}{3}x-3=0$$
$$3x^2+100x-9=0$$

16 a $2x^2 + 5x - 4 = 0$ has roots α and β

The sum of the roots is $\alpha + \beta = -\frac{5}{2}$

The product of the roots is $\alpha\beta = -2$

b When roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$

The sum of the roots is:

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{\left(-\frac{5}{2}\right)}{(-2)} = \frac{5}{4}$$

The product of the roots is:

$$\frac{1}{\alpha}\left(\frac{1}{\beta}\right) = \frac{1}{\alpha\beta} = \frac{1}{(-2)} = -\frac{1}{2}$$

So

$$x^2 - \frac{5}{4}x - \frac{1}{2} = 0$$

$$4x^2 - 5x - 2 = 0$$

17 a i $x^2 - 3x + 1 = 0$ has roots α and β

The sum of the roots is $\alpha + \beta = 3$

The product of the roots is $\alpha\beta = 1$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (3)^2 - 2(1) = 7$$

ii $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha^2\beta - 3\alpha\beta^2$

$= (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= (3)^3 - 3(1)(3)$

$= 18$ as required

iii $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$

$= (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2$ as required

17 b When roots are $\alpha^3 - \beta$ and $\beta^3 - \alpha$

The sum of the roots is:

$$\alpha^3 - \beta + \beta^3 - \alpha = \alpha^3 + \beta^3 - (\alpha + \beta)$$
$$= 18 - 3$$
$$= 15$$

The product of the roots is:

$$(\alpha^3 - \beta)(\beta^3 - \alpha) = \alpha^3\beta^3 - \alpha^4 - \beta^4 + \alpha\beta$$
$$= (\alpha\beta)^3 - (\alpha^4 + \beta^4) + \alpha\beta$$
$$= (\alpha\beta)^3 - \left((\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2\right) + \alpha\beta$$
$$= (1)^3 - \left((7)^2 - 2(1)^2\right) + (1)$$
$$= -45$$

So $x^2 - 15x - 45 = 0$

18 P has coordinates (x, y)

Let the distance from (x, y) to $(5, 0)$ be d

$$|d| = \sqrt{(x-5)^2 + (y-0)^2}$$

Let the distance from (x, y) to the line $x = -5$ also be d

$$|d| = \sqrt{(x+5)^2 + (y-y)^2}$$
$$= x + 5$$

Equating both values of d gives:

$$\sqrt{(x-5)^2 + (y-0)^2} = x + 5$$
$$(x-5)^2 + y^2 = (x+5)^2$$
$$y^2 = (x+5)^2 - (x-5)^2$$
$$= x^2 + 10x + 25 - x^2 + 10x - 25$$
$$= 20x$$

So the locus of P is of the form $y^2 = 4ax$ with $a = 5$

19 a $y^2 = 16x$

A parabola of the form $y^2 = 4ax$ has the focus at $(a, 0)$

So $y^2 = 16x$ has the focus S at $(4, 0)$

19 b P has coordinates (16, 16)
The gradient of PS is:

$$m_{PS} = \frac{y_P - y_S}{x_P - x_S}$$
$$= \frac{16-0}{16-4}$$
$$= \frac{4}{3}$$

To find the equation of PS use

$y - y_1 = m(x - x_1)$ with $m = \frac{4}{3}$ at (4, 0)

$$y - 0 = \frac{4}{3}(x-4)$$
$$3y = 4x - 16$$
$$4x - 3y - 16 = 0$$

c $4x - 3y - 16 = 0$ meets $y^2 = 16x$ at Q

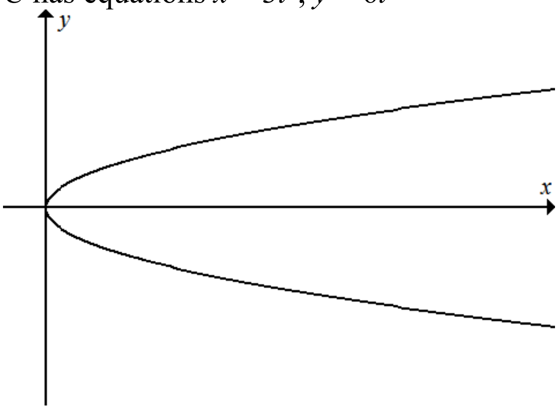
$$y^2 = 16x \Rightarrow x = \frac{y^2}{16}$$

Substituting $x = \frac{y^2}{16}$ into $4x - 3y - 16 = 0$ gives:

$$4\left(\frac{y^2}{16}\right) - 3y - 16 = 0$$
$$y^2 - 12y - 64 = 0$$
$$(y-16)(y+4) = 0$$
$$y = 16 \text{ or } y = -4$$

When $y = -4$, $x = 1$ so Q has coordinates $(1, -4)$

20 a C has equations $x = 3t^2$, $y = 6t$



20 b $y = x - 72$ meets C at A and B

Substituting $x = 3t^2$ and $y = 6t$ into $y = x - 72$ gives:

$6t = 3t^2 - 72$

$t^2 - 2t - 24 = 0$

$(t+4)(t-6) = 0$

$t = -4$ or $t = 6$

When $t = -4$

$x = 3t^2$

$= 3(-4)^2$

$= 48$

$y = 6t$

$= 6(-4)$

$= -24$

So A is the point (48, −24)

When $t = 6$

$x = 3t^2$

$= 3(6)^2$

$= 108$

$y = 6t$

$= 6(6)$

$= 36$

So B is the point (108, 36)

The length of AB is given by:

$|AB| = \sqrt{(108-48)^2 + (36-(-24))^2}$

$= \sqrt{60^2 + 60^2}$

$= 60\sqrt{2}$

21 A parabola of the form $y^2 = 4ax$ has the directrix at $x + a = 0$

So the parabola $y^2 = 12x$ has the directrix at $x + 3 = 0$

If P and Q lie on the parabola at a distance of 8 from the directrix then they both have x-coordinates of 5.

When $x = 5$

$y^2 = 12x$

$= 12(5)$

$= 60$

$y = \pm 2\sqrt{15}$

So P is the point $(5, 2\sqrt{15})$ and Q is the point $(5, -2\sqrt{15})$

The distance PQ is $4\sqrt{15}$

22 a $P(2, 8)$ lies on $y^2 = 4ax$

Substituting (2, 8) into $y^2 = 4ax$ gives:

$(8)^2 = 4(2)a$

$a = 8$

So $y^2 = 32x$

b $y^2 = 32x \Rightarrow y = 4\sqrt{2}x^{\frac{1}{2}}$

The tangent of a point to the curve is:

$\frac{dy}{dx} = 2\sqrt{2}x^{-\frac{1}{2}}$

$= \frac{2\sqrt{2}}{\sqrt{x}}$

At $x = 2$

$\frac{dy}{dx} = \frac{2\sqrt{2}}{(\sqrt{2})}$

$= 2$

To find the equation of the tangent use

$y - y_1 = m(x - x_1)$ with $m = 2$ at (2, 8)

$y - 8 = 2(x - 2)$

$y = 2x + 4$

c The tangent cuts the x-axis at $y = 0$

Substituting $y = 0$ into $y = 2x + 4$ gives:

$(0) = 2x + 4$

$x = -2$

So X has coordinates (−2, 0)

The tangent cuts the y-axis at $x = 0$

Substituting $x = 0$ into $y = 2x + 4$ gives:

$y = 2(0) + 4$

$y = 4$

So Y has coordinates (0, 4)

$\text{Area}_{OXY} = \frac{1}{2}(2)(4)$

$= 4$

23 a P has coordinates (3, 4) and lies on $xy = 12$
Q has coordinates (−2, 0)
The gradient of l is

$$m_l = \frac{y_P - y_Q}{x_P - x_Q}$$
$$= \frac{4-0}{3-(-2)}$$
$$= \frac{4}{5}$$

To find the equation of l use

$y - y_1 = m(x - x_1)$ with $m = \frac{4}{5}$ at (−2, 0)

$$y - 0 = \frac{4}{5}(x+2)$$
$$y = \frac{4}{5}x + \frac{8}{5}$$

b $y = \frac{4}{5}x + \frac{8}{5}$ cuts $xy = 12$ at the point R

Substituting $y = \frac{12}{x}$ into $y = \frac{4}{5}x + \frac{8}{5}$ gives:

$$\frac{12}{x} = \frac{4}{5}x + \frac{8}{5}$$
$$60 = 4x^2 + 8x$$
$$x^2 + 2x - 15 = 0$$
$$(x-3)(x+5) = 0$$
$$x = 3 \text{ or } x = -5$$

When $x = -5$, $y = -\frac{12}{5}$

So R has coordinates $\left(-5, -\frac{12}{5}\right)$

24 a P has coordinates (12, 3) and lies on $xy = 36$

$xy = 36 \Rightarrow y = 36x^{-1}$

The gradient of the tangent to a point on the curve is:

$$\frac{dy}{dx} = -36x^{-2}$$
$$= -\frac{36}{x^2}$$

At $x = 12$

$$\frac{dy}{dx} = -\frac{36}{(12)^2}$$
$$= -\frac{1}{4}$$

To find the equation of the tangent at (12, 3) use

$y - y_1 = m(x - x_1)$ with $m = -\frac{1}{4}$ at (12, 3)

$$y - 3 = -\frac{1}{4}(x - 12)$$
$$4y - 12 = -x + 12$$
$$x + 4y - 24 = 0$$

b The tangent cuts the x-axis at M and the y-axis at N

At M, $y = 0$

Substituting $y = 0$ into $x + 4y - 24 = 0$ gives:

$$x + 4(0) - 24 = 0$$
$$x = 24$$

Therefore M has coordinates (24, 0)

At N, $x = 0$

Substituting $x = 0$ into $x + 4y - 24 = 0$ gives:

$$(0) + 4y - 24 = 0$$
$$y = 6$$

Therefore N has coordinates (0, 6)

The length of MN is given by:

$$|MN| = \sqrt{(24 - 0)^2 + (0 - 6)^2}$$
$$= 6\sqrt{17}$$

25 C has equations $x = 8t,\ y = \frac{16}{t}$

The line $y = \frac{1}{4}x + 4$ intersects C at A and B

Substituting $x = 8t$ and $y = \frac{16}{t}$ into $y = \frac{1}{4}x + 4$ gives:

$$\left(\frac{16}{t}\right) = \frac{1}{4}(8t) + 4$$

$$16 = 2t^2 + 4t$$

$$t^2 + 2t - 8 = 0$$

$$(t-2)(t+4) = 0$$

$t = 2$ or $t = -4$

When $t = 2$

$$x = 8(2)$$
$$= 16$$

and

$$y = \frac{16}{(2)}$$
$$= 8$$

So A has coordinates (16, 8)

When $t = -4$

$$x = 8(-4)$$
$$= -32$$

and

$$y = \frac{16}{(-4)}$$
$$= -4$$

So B has coordinates (−32, −4)

The midpoint of AB is found using:

$$\left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}\right) = \left(\frac{16 + (-32)}{2}, \frac{8 + (-4)}{2}\right)$$
$$= (-8, 2)$$

So M has coordinates (−8, 2)

26 a $P(24t^2, 48t)$ lies on $y^2 = 96x$ and on $xy = 144$

Substituting $(24t^2, 48t)$ into $xy = 144$ gives:

$$(24t^2)(48t) = 144$$

$$t^3 = \frac{144}{1152}$$

$$t = \frac{1}{2}$$

Therefore P has coordinates (6, 24)

26 b $y^2 = 96x \Rightarrow y = 4\sqrt{6}x^{\frac{1}{2}}$

The gradient of a tangent to a point on the parabola is:

$$\frac{dy}{dx} = 2\sqrt{6}x^{-\frac{1}{2}}$$

$$= \frac{2\sqrt{6}}{x^{\frac{1}{2}}}$$

At $x = 6$

$$\frac{dy}{dx} = \frac{2\sqrt{6}}{(6)^{\frac{1}{2}}}$$

$$= 2$$

To find the equation of the tangent at (6, 24) use

$y - y_1 = m(x - x_1)$ with $m = 2$ at (6, 24)

$$y - 24 = 2(x - 6)$$

$$y = 2x + 12$$

27 a $P(9, 8)$ and $Q(6, 12)$ lie on $xy = 72$

The gradient of PQ is found using:

$$m_{PQ} = \frac{y_P - y_Q}{x_P - x_Q}$$

$$= \frac{8 - 12}{9 - 6}$$

$$= -\frac{4}{3}$$

To find the equation of PQ:

$y - y_1 = m(x - x_1)$ with $m = -\frac{4}{3}$ at (6, 12)

$$y - 12 = -\frac{4}{3}(x - 6)$$

$$3y - 36 = -4x + 24$$

$4x + 3y = 60$ as required

27 b The tangent at R is parallel to PQ therefore the gradient at R is $-\frac{4}{3}$

$xy = 72 \Rightarrow y = 72x^{-1}$

The gradient of a tangent to a point on the curve is:

$$\frac{dy}{dx} = -72x^{-2}$$
$$= -\frac{72}{x^2}$$

Equating both values of the tangent gives:

$$-\frac{72}{x^2} = -\frac{4}{3}$$
$$4x^2 = 216$$
$$x = \pm 3\sqrt{6}$$

When $x = 3\sqrt{6},\ y = 4\sqrt{6}$

When $x = -3\sqrt{6},\ y = -4\sqrt{6}$

So the possible coordinates of R are $\left(3\sqrt{6}, 4\sqrt{6}\right)$ and $\left(-3\sqrt{6}, -4\sqrt{6}\right)$

28 a The point $\left(3t, \frac{3}{t}\right)$ lies on $xy = 9$

$xy = 9 \Rightarrow y = 9x^{-1}$

The gradient of a tangent to a point on the curve is:

$$\frac{dy}{dx} = -9x^{-2}$$
$$= -\frac{9}{x^2}$$

At $x = 3t$

$$\frac{dy}{dx} = -\frac{9}{(3t)^2}$$
$$= -\frac{1}{t^2}$$

To find the equation of the tangent at $\left(3t, \frac{3}{t}\right)$ use:

$y - y_1 = m(x - x_1)$ with $m = -\frac{1}{t^2}$ at $\left(3t, \frac{3}{t}\right)$

$$y - \frac{3}{t} = -\frac{1}{t^2}(x - 3t)$$
$$t^2 y - 3t = -x + 3t$$

$x + t^2 y = 6t$ as required

28 b The tangent at $\left(3t, \frac{3}{t}\right)$ cuts the x-axis at A and the y-axis at B

At A, $y = 0$
Substituting $y = 0$ into $x + t^2 y = 6t$ gives:

$x + t^2(0) = 6t$

$x = 6t$

So A has coordinates $(6t, 0)$
At B, $x = 0$
Substituting $x = 0$ into $x + t^2 y = 6t$ gives:

$(0) + t^2 y = 6t$

$y = \frac{6}{t}$

So B has coordinates $\left(0, \frac{6}{t}\right)$

$$\text{Area}_{OAB} = \frac{1}{2}(6t)\left(\frac{6}{t}\right)$$
$$= 18$$

Therefore the area of triangle OAB is constant

29 a The point $\left(ct, \frac{c}{t}\right)$ lies on $xy = c^2$

$xy = c^2 \Rightarrow y = c^2 x^{-1}$

The gradient of a tangent to a point on the curve is:

$$\frac{dy}{dx} = -c^2 x^{-2}$$
$$= -\frac{c^2}{x^2}$$

At $x = ct$

$$\frac{dy}{dx} = -\frac{c^2}{(ct)^2}$$
$$= -\frac{1}{t^2}$$

At $\left(ct, \frac{c}{t}\right)$ the tangent has gradient $-\frac{1}{t^2}$ so the normal has gradient t^2

To find the equation of the normal at $\left(ct, \frac{c}{t}\right)$ use:

$y - y_1 = m(x - x_1)$ with $m = t^2$ at $\left(ct, \frac{c}{t}\right)$

$y - \frac{c}{t} = t^2(x - ct)$

$ty - c = t^3 x - ct^4$

$t^3 x - ty - c(t^4 - 1) = 0$ as required

29 b The normal at P meets the line $y = x$ at G

Substituting $y = x$ into $t^3x - ty - c(t^4 - 1) = 0$ gives:

$$t^3x - t(x) - c(t^4 - 1) = 0$$

$$tx(t^2 - 1) = c(t^4 - 1)$$

$$x = \frac{c(t^4 - 1)}{t(t^2 - 1)}$$

$$= \frac{c(t^2 + 1)}{t}$$

and since $x = y$, G has coordinates $\left(\frac{c(t^2+1)}{t}, \frac{c(t^2+1)}{t}\right)$

The length of PG is found using:

$$|PG| = \sqrt{\left(ct - \frac{c(t^2+1)}{t}\right)^2 + \left(\frac{c}{t} - \frac{c(t^2+1)}{t}\right)^2}$$

$$= \sqrt{\left(\frac{ct^2 - c(t^2+1)}{t}\right)^2 + \left(\frac{c - c(t^2+1)}{t}\right)^2}$$

$$= \sqrt{\left(\frac{-c}{t}\right)^2 + \left(-\frac{ct^2}{t}\right)^2}$$

$$|PG|^2 = \left(\frac{-c}{t}\right)^2 + (-ct)^2$$

$$= \frac{c^2}{t^2} + c^2t^2$$

$$= c^2\left(\frac{1}{t^2} + t^2\right) \text{ as required}$$

30 a The point $\left(ct, \frac{c}{t}\right)$ lies on $xy = c^2$

$xy = c^2 \Rightarrow y = c^2x^{-1}$

The gradient of a tangent to a point on the curve is:

$$\frac{dy}{dx} = -c^2x^{-2}$$

$$= -\frac{c^2}{x^2}$$

At $x = ct$

$$\frac{dy}{dx} = -\frac{c^2}{(ct)^2}$$

$$= -\frac{1}{t^2}$$

To find the equation of the tangent at $\left(ct, \frac{c}{t}\right)$ use:

$y - y_1 = m(x - x_1)$ with $m = -\frac{1}{t^2}$ at $\left(ct, \frac{c}{t}\right)$

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

$$t^2y - ct = -x + ct$$

$x + t^2y = 2ct$ as required

b Tangents are drawn from (−3, 3) to $xy = 16$

Substituting (−3, 3) into $x + t^2y = 2ct$ gives:

$$(-3) + t^2(3) = 2ct$$

$$3t^2 - 2ct - 3 = 0$$

Comparing $xy = 16$ to $xy = c^2$ gives $c = 4$, since the formula for a rectangular hyperbola assumes c to be a positive constant

When $c = 4$ the equation of the tangent is:

$$3t^2 - 2(4)t - 3 = 0$$

$$3t^2 - 8t - 3 = 0$$

$$(3t + 1)(t - 3) = 0$$

$$t = -\frac{1}{3} \text{ or } t = 3$$

When $c = 4$ and $t = -\frac{1}{3}$

$$\left(ct, \frac{c}{t}\right) = \left(-\frac{4}{3}, -12\right)$$

When $c = 4$ and $t = 3$

$$\left(ct, \frac{c}{t}\right) = \left(12, \frac{4}{3}\right)$$

So the coordinates of the points where the tangent meets the curve are

$\left(-\frac{4}{3}, -12\right)$ and $\left(12, \frac{4}{3}\right)$

31 $P\left(at^2, 2at\right)$, $t > 0$ lies on $y^2 = 4ax$

$y^2 = 4ax \Rightarrow y = 2\sqrt{a}x^{\frac{1}{2}}$

The gradient of a tangent to a point on the curve is:

$\frac{dy}{dx} = \sqrt{a}x^{-\frac{1}{2}}$

$= \frac{\sqrt{a}}{x^{\frac{1}{2}}}$

At $x = at^2$

$\frac{dy}{dx} = \frac{\sqrt{a}}{\left(at^2\right)^{\frac{1}{2}}}$

$= \frac{1}{t}$

To find the equation of the tangent use:

$y - y_1 = m\left(x - x_1\right)$ with $m = \frac{1}{t}$ at $\left(at^2, 2at\right)$

$y - 2at = \frac{1}{t}\left(x - at^2\right)$

$ty - 2at^2 = x - at^2$

$x - ty + at^2 = 0$

The tangent cuts the x-axis at T where $y = 0$

Substituting $y = 0$ into $x - ty + at^2 = 0$ gives:

$x - t\left(0\right) + at^2 = 0$

$x = -at^2$

So T has coordinates $(-at^2, 0)$

The length of PT is found using:

$\left|PT\right| = \sqrt{\left(at^2 - \left(-at^2\right)\right)^2 + \left(2at - 0\right)^2}$

$= \sqrt{4a^2t^4 + 4a^2t^2}$

$= \sqrt{4a^2t^2\left(t^2 + 1\right)}$

$= 2at\sqrt{\left(t^2 + 1\right)}$

Since the gradient of the tangent at $\left(at^2, 2at\right)$ is $\frac{1}{t}$ the gradient of the normal is $-t$

To find the equation of the normal use:

$y - y_1 = m\left(x - x_1\right)$ with $m = -t$ at $\left(at^2, 2at\right)$

$y - 2at = -t\left(x - at^2\right)$

$y - 2at = -tx + at^3$

$tx + y - 2at - at^3 = 0$

The normal cuts the x-axis at N where $y = 0$

Substituting $y = 0$ into $tx + y - 2at - at^3 = 0$ gives:

$tx + 0 - 2at - at^3 = 0$

$x = a\left(2 + t^2\right)$

So N has coordinates $(a(2 + t^2),0)$

The length of PN is found using:

$$|PN| = \sqrt{\left(at^2 - a(2+t^2)\right)^2 + \left(2at - 0\right)^2}$$

$$= \sqrt{4a^2 + 4a^2t^2}$$

$$= 2a\sqrt{\left(t^2 + 1\right)}$$

$$\frac{|PT|}{|PN|} = \frac{2at\sqrt{\left(t^2+1\right)}}{2a\sqrt{\left(t^2+1\right)}}$$

$$= t$$

32 a $P\left(ap^2, 2ap\right)$, $p > 0$ lies on $y^2 = 4ax$

$y^2 = 4ax \Rightarrow y = 2\sqrt{a}x^{\frac{1}{2}}$

The gradient of a tangent to a point on the curve is:

$$\frac{dy}{dx} = \sqrt{a}x^{-\frac{1}{2}}$$

$$= \frac{\sqrt{a}}{x^{\frac{1}{2}}}$$

At $x = ap^2$

$$\frac{dy}{dx} = \frac{\sqrt{a}}{\left(ap^2\right)^{\frac{1}{2}}}$$

$$= \frac{1}{p}$$

To find the equation of the tangent use:

$y - y_1 = m\left(x - x_1\right)$ with $m = \frac{1}{p}$ at $\left(ap^2, 2ap\right)$

$y - 2ap = \frac{1}{p}\left(x - ap^2\right)$

$py - 2ap^2 = x - ap^2$

$py = x + ap^2$ as required

32 b The tangents at $P\left(ap^2, 2ap\right)$ and $Q\left(aq^2, 2aq\right)$ meet at N

The tangent at P has equation:

$$py = x + ap^2 \Rightarrow y = \frac{x}{p} + ap$$

The tangent at Q has equation:

$$qy = x + aq^2 \Rightarrow y = \frac{x}{q} + aq$$

To find the coordinates of N, equate the equations:

$$\frac{x}{p} + ap = \frac{x}{q} + aq$$

$$\frac{x}{p} - \frac{x}{q} = aq - ap$$

$$\frac{qx - px}{pq} = a\left(q - p\right)$$

$$x\left(q - p\right) = apq\left(q - p\right)$$

As $p \neq x,\ x = apq$

When $x = apq$

$$y = \frac{\left(apq\right)}{q} + aq$$

$$= a\left(p + q\right)$$

So N has coordinates $(apq, a(p + q))$

c Since N lies on $y = 4a$

$a(p + q) = 4a$

$p = 4 - q$

33 a $P\left(at^2, 2at\right)$, $t > 0$ lies on $y^2 = 4ax$

$y^2 = 4ax \Rightarrow y = 2\sqrt{a}x^{\frac{1}{2}}$

The gradient of a tangent to a point on the curve is:

$\frac{dy}{dx} = \sqrt{a}x^{-\frac{1}{2}}$

$= \frac{\sqrt{a}}{x^{\frac{1}{2}}}$

At $x = at^2$

$\frac{dy}{dx} = \frac{\sqrt{a}}{\left(at^2\right)^{\frac{1}{2}}}$

$= \frac{1}{t}$

At $\left(at^2, 2at\right)$ the gradient of the tangent is $\frac{1}{t}$ so the gradient of the normal is $-t$

To find the equation of the normal use:

$y - y_1 = m\left(x - x_1\right)$ with $m = -t$ at $\left(at^2, 2at\right)$

$y - 2at = -t\left(x - at^2\right)$

$y - 2at = -tx + at^3$

$y + tx = 2at + at^3$ as required

33 b The normal meets the curve again at Q

$y + tx = 2at + at^3 \Rightarrow x = -\frac{y}{t} + 2a + at^2$

Substituting $x = -\frac{y}{t} + 2a + at^2$ into $y^2 = 4ax$ gives:

$y^2 = 4a\left(-\frac{y}{t} + 2a + at^2\right)$

$y^2 + \frac{4ay}{t} - 8a^2 - 4a^2t^2 = 0$

We know that $y = 2at$ is one solution to this equation.

So, use the fact that $(y - 2at)$ is a factor to rewrite the left-hand side

$(y - 2at)\left(y + 2at + \frac{4a}{t}\right) = 0$

$y = 2at$ or $y = -2at - \frac{4a}{t} = \frac{-2a(t^2 + 2)}{t}$

When $y = \frac{-2a\left(t^2 + 2\right)}{t}$

Substituting $y = \frac{-2a\left(t^2 + 2\right)}{t}$ into $y^2 = 4ax$ gives:

$4ax = \left(\frac{-2a\left(t^2 + 2\right)}{t}\right)^2$

$= \frac{4a^2(t^2 + 2)^2}{t^2}$

$x = \frac{a(t^2 + 2)^2}{t^2}$

So Q has coordinates $\left(\frac{a\left(t^2 + 2\right)^2}{t^2}, \frac{-2a\left(t^2 + 2\right)}{t}\right)$

34 a The point $\left(ct, \frac{c}{t}\right)$ lies on $xy = c^2$

$xy = c^2 \Rightarrow y = c^2x^{-1}$

The gradient of a tangent to a point on the curve is:

$\frac{dy}{dx} = -c^2x^{-2} = -\frac{c^2}{x^2}$

At $x = ct$

$\frac{dy}{dx} = -\frac{c^2}{(ct)^2}$

$= -\frac{1}{t^2}$

At $x = ct$ the gradient of the tangent is $-\frac{1}{t^2}$ so the gradient of the normal is t^2

To find the equation of the normal at $\left(ct, \frac{c}{t}\right)$ use:

$y - y_1 = m(x - x_1)$ with $m = t^2$ at $\left(ct, \frac{c}{t}\right)$

$y - \frac{c}{t} = t^2(x - ct)$

$y - \frac{c}{t} = t^2x - ct^3$

$y = t^2x + \frac{c}{t} - ct^3$ as required

b The normal meets the equation again at Q

Substituting $y = t^2x + \frac{c}{t} - ct^3$ into $xy = c^2$ gives:

$x\left(t^2x + \frac{c}{t} - ct^3\right) = c^2$

$t^2x^2 + \frac{cx}{t} - ct^3x - c^2 = 0$

$x^2 + \left(\frac{c}{t^3} - ct\right)x - \frac{c^2}{t^2} = 0$

$(x - ct)\left(x + \frac{c}{t^3}\right) = 0$

$x = ct$ or $x = -\frac{c}{t^3}$

Substituting $x = -\frac{c}{t^3}$ into $xy = c^2$ gives:

$\left(-\frac{c}{t^3}\right)y = c^2 \Rightarrow y = -ct^3$

So Q is the point $\left(-\frac{c}{t^3}, -ct^3\right)$

34 c P has coordinates $\left(ct, \frac{c}{t}\right)$ and Q has coordinates $\left(-\frac{c}{t^3}, -ct^3\right)$

The midpoint of PQ is found using:

$$\left(\frac{x_P + x_Q}{2}, \frac{y_P + y_Q}{2}\right) = \left(\frac{ct + \left(-\frac{c}{t^3}\right)}{2}, \frac{\frac{c}{t} + \left(-ct^3\right)}{2}\right)$$

$$= \left(\frac{c\left(t^4 - 1\right)}{2t^3}, \frac{c\left(1 - t^4\right)}{2t}\right)$$

So $X = \frac{c\left(t^4 - 1\right)}{2t^3}$ and $Y = \frac{c\left(1 - t^4\right)}{2t}$

$$\frac{X}{Y} = \frac{\frac{c\left(t^4 - 1\right)}{2t^3}}{\frac{c\left(1 - t^4\right)}{2t}}$$

$$= \frac{2ct\left(t^4 - 1\right)}{2ct^3\left(1 - t^4\right)}$$

$$= -\frac{1}{t^2}$$ as required

35 a The point $P\left(cp, \frac{c}{p}\right)$ lies on $xy = c^2$

$xy = c^2 \Rightarrow y = c^2x^{-1}$

The gradient of a tangent to a point on the curve is:

$$\frac{dy}{dx} = -c^2x^{-2}$$

$$= -\frac{c^2}{x^2}$$

At $x = cp$

$$\frac{dy}{dx} = -\frac{c^2}{(cp)^2}$$

$$= -\frac{1}{p^2}$$

To find the equation of the tangent at $\left(cp, \frac{c}{p}\right)$ use:

$y - y_1 = m\left(x - x_1\right)$ with $m = -\frac{1}{p^2}$ at $\left(cp, \frac{c}{p}\right)$

$$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$$

$$p^2y - pc = -x + cp$$

$p^2y = -x + 2cp$ as required

35 b The tangents at $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ meet at N

The tangent at P has equation:

$p^2 y = -x + 2cp \Rightarrow x = 2cp - p^2 y$

The tangent at Q has equation:

$q^2 y = -x + 2cq \Rightarrow x = 2cq - q^2 y$

To find the coordinates of N, equate the equations:

$2cp - p^2 y = 2cq - q^2 y$

$y(q^2 - p^2) = 2c(q - p)$

$y = 2c\frac{(q-p)}{(q^2 - p^2)} = 2c\frac{(q-p)}{(q-p)(q+p)}$

$y = \frac{2c}{p+q}$ as required

c P has coordinates $\left(cp, \frac{c}{p}\right)$ and Q has coordinates $\left(cq, \frac{c}{q}\right)$

The gradient of PQ is found using:

$$m_{PQ} = \frac{y_P - y_Q}{x_P - x_Q}$$

$$= \frac{\frac{c}{p} - \frac{c}{q}}{cp - cq}$$

$$= \frac{q - p}{pq(p-q)}$$

$$= -\frac{1}{pq}$$

N has y-coordinate $\frac{2c}{p+q}$. Substituting this into the equation $x = 2cp - p^2 y$ gives:

$$x = 2cp - p^2\left(\frac{2c}{p+q}\right)$$

$$x = \frac{2cp(p+q) - 2cp^2}{p+q}$$

$$= \frac{2cpq}{p+q}$$

O has coordinates (0, 0) and N has coordinates $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$

The gradient of ON is found using:

$$m_{ON}=\frac{y_N-y_O}{x_N-x_O}$$

$$=\frac{\frac{2c}{p+q}}{\frac{2cpq}{p+q}}$$

$$=\frac{2c(p+q)}{2cpq(p+q)}$$

$$=\frac{1}{pq}$$

Since m_{PQ} and m_{ON} are perpendicular

$$m_{PQ}\times m_{ON}=-1$$

$$-\frac{1}{pq}\times\frac{1}{pq}=-1$$

$$p^2q^2=1$$

36 a $P(ap^2,2ap)$, $p\neq 0$ lies on $y^2=4ax$

$$y^2=4ax\Rightarrow y=2\sqrt{a}x^{\frac{1}{2}}$$

The gradient of a tangent to a point on the curve is:

$$\frac{dy}{dx}=\sqrt{a}x^{-\frac{1}{2}}$$

$$=\frac{\sqrt{a}}{x^{\frac{1}{2}}}$$

At $x=ap^2$

$$\frac{dy}{dx}=\frac{\sqrt{a}}{(ap^2)^{\frac{1}{2}}}$$

$$=\frac{1}{p}$$

At $x=ap^2$ the tangent has gradient $\frac{1}{p}$ so the normal has gradient $-p$

To find the equation of the normal use:

$y-y_1=m(x-x_1)$ with $m=-p$ at $(ap^2,2ap)$

$$y-2ap=-p(x-ap^2)$$

$$y-2ap=-px+ap^3$$

$y+px=2ap+ap^3$ as required

36 b The normal meets the curve again at $Q\left(aq^2, 2aq\right)$

Substituting $\left(aq^2, 2aq\right)$ into $y + px = 2ap + ap^3$ gives:

$$(2aq) + p\left(aq^2\right) = 2ap + ap^3$$

$$apq^2 + 2aq - ap\left(2 + p^2\right) = 0$$

$$q^2 + \frac{2q}{p} - \left(2 + p^2\right) = 0$$

We know that $q = p$ would give one solution to this equation.

So, use the fact that $(q - p)$ is a factor to rewrite the left-hand side:

$$(q - p)\left(q + \left(p + \frac{2}{p}\right)\right) = 0$$

$q = p$ or $q = -p - \dfrac{2}{p}$

Since $q \neq p$

$$q = -p - \frac{2}{p}$$

c The midpoint of PQ is $\left(\dfrac{125}{18}a, -3a\right)$

P has coordinates $P\left(ap^2, 2ap\right)$ and Q has coordinates $P\left(aq^2, 2aq\right)$

Find the midpoint of PQ using:

$$\left(\frac{x_P + x_Q}{2}, \frac{y_P + y_Q}{2}\right) = \left(\frac{ap^2 + aq^2}{2}, \frac{2ap + 2aq}{2}\right)$$

$$= \left(\frac{a\left(p^2 + q^2\right)}{2}, a(p + q)\right)$$

Comparing $\left(\dfrac{125a}{18}, -3a\right)$ to $\left(\dfrac{a\left(2p^2 + 4 + \dfrac{4}{p^2}\right)}{2}, -\dfrac{2a}{p}\right)$

$$-3a = a(p + q)$$

$$p + q = -3$$

Since $q = -p - \dfrac{2}{p}$, $p + \left(-p - \dfrac{2}{p}\right) = -3$

$$p = \frac{2}{3}$$

37 a A parabola of the form $y^2 = 4ax$ has the focus at $(a, 0)$
So $y^2 = 32x$ has the focus S at (8, 0)

b A parabola of the form $y^2 = 4ax$ has the directrix at $x + a = 0$
So $y^2 = 32x$ has the directrix at $x + 8 = 0$

c P has coordinates (2, 8) and Q has coordinates (32, −32)
The gradient of PQ is found using:

$$m_{PQ} = \frac{y_P - y_Q}{x_P - x_Q}$$

$$= \frac{8-(-32)}{2-32}$$

$$= -\frac{4}{3}$$

To find the equation of PQ use:

$y - y_1 = m(x - x_1)$ with $m = -\frac{4}{3}$ at (2, 8)

$$y - 8 = -\frac{4}{3}(x-2)$$

$$3y - 24 = -4x + 8$$

$$4x + 3y - 32 = 0$$

If S lies on $4x+3y-32=0$ then (8, 0) will satisfy $4x+3y-32=0$
Substituting (8, 0) into $4x+3y-32=0$ gives:

$$4(8)+3(0)-32=0$$

$$0=0$$

So S lies on PQ

37 d $y^2 = 32x \Rightarrow y = \pm 4\sqrt{2}x^{\frac{1}{2}}$

The gradient of a tangent to point on the curve is:

$$\frac{dy}{dx} = \pm 2\sqrt{2}x^{-\frac{1}{2}}$$

$$= \pm\frac{2\sqrt{2}}{x^{\frac{1}{2}}}$$

At P, $x = 2$

$$\frac{dy}{dx} = \frac{2\sqrt{2}}{(2)^{\frac{1}{2}}}$$

$$= 2$$

To find the equation of the tangent at P use:

$y - y_1 = m(x - x_1)$ with $m = 2$ at (2, 8)

$y - 8 = 2(x - 2)$

$y = 2x + 4$

At Q, $x = 32$

Since Q is the point (32, −32), use $\frac{dy}{dx} = -\frac{2\sqrt{2}}{x^{\frac{1}{2}}}$

$$\frac{dy}{dx} = -\frac{2\sqrt{2}}{(32)^{\frac{1}{2}}}$$

$$= -\frac{1}{2}$$

To find the equation of the tangent at Q use:

$y - y_1 = m(x - x_1)$ with $m = \frac{1}{2}$ at (32, −32)

$y + 32 = -\frac{1}{2}(x - 32)$

$y = -\frac{1}{2}x - 16$

Equate the tangents to find the point where they meet:

$2x + 4 = -\frac{1}{2}x - 16$

$4x + 8 = -x - 32$

$5x = -40$

$x = -8$

Therefore D lies on the directrix

Challenge

1 $x^2 + 2\mathrm{i}x + 5 = 0$

$(x+\mathrm{i})^2 - \mathrm{i}^2 + 5 = 0$

$(x+\mathrm{i})^2 = -6$

$x + \mathrm{i} = \pm\mathrm{i}\sqrt{6}$

$x = \mathrm{i} \pm \mathrm{i}\sqrt{6}$

$x = \mathrm{i}\left(1 \pm \sqrt{6}\right)$

2 $ax^2 + bx + c = 0$ has roots α and β

$\alpha^4 + \beta^4 = \left(\alpha^2 + \beta^2\right)^2 - 2(\alpha\beta)^2$

$\alpha^4 + \beta^4 = -\frac{79}{16}$ and $\alpha^2 + \beta^2 = -\frac{7}{4}$

Substituting gives:

$-\frac{79}{16} = \left(-\frac{7}{4}\right)^2 - 2(\alpha\beta)^2$

$2(\alpha\beta)^2 = \frac{49}{16} + \frac{79}{16}$

$\alpha\beta = \pm 2$

Since $\alpha\beta > 0$, $\alpha\beta = 2$

$(\alpha+\beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$

$\alpha^2 + \beta^2 = -\frac{7}{4}$ and $\alpha\beta = 2$

Substituting gives:

$(\alpha+\beta)^2 = -\frac{7}{4} + 2(2)$

$= \frac{9}{4}$

$\alpha + \beta = \pm\frac{3}{2}$

The equation is:

$2x^2 - 3x + 4 = 0$ or $2x^2 + 3x + 4 = 0$