Chapter review 8

1 Let $f(n) = 9^n - 1$, where $n \in \mathbb{Z}^+$.

$\therefore$ $f(1) = 9^1 - 1 = 8$, which is divisible by 8.

$\therefore$ $f(n)$ is divisible by 8 when $n = 1$.

Assume that for $n = k$,

$f(k) = 9^k - 1$ is divisible by 8 for $k \in \mathbb{Z}^+$.

$$\begin{aligned}\therefore f(k+1) &= 9^{k+1} - 1\\ &= 9^k.9^1 - 1\\ &= 9(9^k) - 1\end{aligned}$$

$$\begin{aligned}\therefore f(k+1) - f(k) &= [9(9^k) - 1] - [9^k - 1]\\ &= 9(9^k) - 1 - 9^k + 1\\ &= 8(9^k)\end{aligned}$$

$$\therefore f(k+1) = f(k) + 8(9^k)$$

As both f (k) and $8(9^k)$ are divisible by 8 then the sum of these two terms must also be divisible by 8. Therefore f (n) is divisible by 8 when $n = k + 1$.

If f (n) is divisible by 8 when $n = k$, then it has been shown that f (n) is also divisible by 8 when $n = k + 1$. As f (n) is divisible by 8 when $n = 1$, f (n) is also divisible by 8 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

2 **a** $\mathbf{B}^2 = \mathbf{BB} = \begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 3\end{pmatrix} = \begin{pmatrix}1+0 & 0+0\\0+0 & 0+9\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 9\end{pmatrix}$

$\mathbf{B}^3 = \mathbf{B}^2\mathbf{B} = \begin{pmatrix}1 & 0\\0 & 9\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 3\end{pmatrix} = \begin{pmatrix}1+0 & 0+0\\0+0 & 0+27\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 27\end{pmatrix}$

b As $\mathbf{B}^2 = \begin{pmatrix}1 & 0\\0 & 3^2\end{pmatrix}$ and $\mathbf{B}^3 = \begin{pmatrix}1 & 0\\0 & 3^3\end{pmatrix}$, we suggest that $\mathbf{B}^n = \begin{pmatrix}1 & 0\\0 & 3^n\end{pmatrix}$.

c $n = 1$; $\text{LHS} = \begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}^1 = \begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}$

$\text{RHS} = \begin{pmatrix}1 & 0\\0 & 3^1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

i.e. $\begin{pmatrix}1 & 0\\0 & 3\end{pmatrix}^k = \begin{pmatrix}1 & 0\\0 & 3^k\end{pmatrix}$

With $n = k + 1$ the matrix equation becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{k} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 3^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 1+0 & 0+0 \\ 0+0 & 0+3(3^k) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 3^{k+1} \end{pmatrix}$$

Therefore the matrix is true when $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

3 Basis: $n = 1$: LHS $= 3 \times 1 + 4 = 7$; RHS $= \frac{1}{2} \times 1(3 \times 1 + 11) = 7$

Assumption:

$$\sum_{r=1}^{k}(3r+4) = \frac{1}{2}k(3k+11)$$

Induction:

$$\sum_{r=1}^{k+1}(3r+4) = \sum_{r=1}^{k}(3r+4) + 3(k+1) + 4$$

$$= \frac{1}{2}k(3k+11) + 3(k+1) + 4 = \frac{1}{2}(3k^2 + 17k + 14)$$
$$= \frac{1}{2}(k+1)(3(k+1)+11)$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

4 a $n=1$: LHS $=\begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^1 = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}$

$$= \begin{pmatrix} 8(1)+1 & 16(1) \\ -4(1) & 1-8(1) \end{pmatrix} = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.

Assume that the matrix equation is true for $n = k$.

i.e. $\begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^k = \begin{pmatrix} 8k+1 & 16k \\ -4k & 1-8k \end{pmatrix}$.

With $n = k + 1$ the matrix equation becomes

$$\begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^{k+1} = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}^k \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} 8k+1 & 16k \\ -4k & 1-8k \end{pmatrix}\begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}$$

$$= \begin{pmatrix} 72k+9-64k & 128k+16-112k \\ -36k-4+32k & -64k-7+56k \end{pmatrix}$$

$$= \begin{pmatrix} 8k+9 & 16k+16 \\ -4k-4 & -8k-7 \end{pmatrix}$$

$$= \begin{pmatrix} 8(k+1)+1 & 16(k+1) \\ -4(k+1) & 1-8(k+1) \end{pmatrix}$$

Therefore the matrix equation is true when $n = k + 1$.

If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

b $\det(\mathbf{A}^n) = (8n+1)(1-8n) - -64n^2$

$$= 8n - 64n^2 + 1 - 8n + 64n^2$$

$$= 1$$

$$\mathbf{B} = (\mathbf{A}^n)^{-1} = \frac{1}{1}\begin{pmatrix} 1-8n & -16n \\ 4n & 8n+1 \end{pmatrix}$$

So $\mathbf{B} = \begin{pmatrix} 1-8n & -16n \\ 4n & 8n+1 \end{pmatrix}$

5 a $\mathrm{f}(n+1) = 5^{2(n+1)-1} + 1$

$= 5^{2n+2-1} + 1$

$= 5^{2n-1}.5^2 + 1$

$= 24(5^{2n-1}) + 1$

$\therefore\ \mathrm{f}(n+1) - \mathrm{f}(n) = [25(5^{2n-1}) + 1] - [5^{2n-1} + 1]$

$= 25(5^{2n-1}) + 1 - (5^{2n-1}) - 1$

$= 24(5^{2n-1})$

Therefore, $\mu = 24$.

b $\mathrm{f}(n) = 5^{2n-1} + 1$, where $n \in \mathbb{Z}^+$.

$\therefore \mathrm{f}(1) = 5^{2(1)-1} + 1 = 5^1 + 1 = 6$, which is divisible by 6.

$\therefore$ f(n) is divisible by 6 when $n = 1$.

Assume that for $n = k$,

$\mathrm{f}(k) = 5^{2k-1} + 1$ is divisible by 6 for $k \in \mathbb{Z}^+$.

Using **a**, $\mathrm{f}(k+1) - \mathrm{f}(k) = 24(5^{2k-1})$

$\therefore\ \mathrm{f}(k+1) = \mathrm{f}(k) + 24(5^{2k-1})$

As both f(k) and $24(5^{2k-1})$ are divisible by 6 then the sum of these two terms must also be divisible by 6. Therefore f(n) is divisible by 6 when $n = k + 1$.

If f(n) is divisible by 6 when $n = k$, then it has been shown that f(n) is also divisible by 6 when $n = k + 1$. As f(n) is divisible by 6 when $n = 1$, f(n) is also divisible by 6 for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

6 Let $f(n)=7^n+4^n+1$, where $n\in\mathbb{Z}^+$.

$\therefore\ f(1)=7^1+4^1+1=7+4+1=12$, which is divisible by 6.

$\therefore$ f(n) is divisible by 6 when n = 1.

Assume that for $n = k$,

$f(k)=7^k+4^k+1$ is divisible by 6 for $k\in\mathbb{Z}^+$.

$$\begin{aligned}\therefore\ f(k+1)&=7^{k+1}+4^{k+1}+1\\&=7^k\times7^1+4^k\times4^1+1\\&=7(7^k)+4(4^k)+1\end{aligned}$$

$$\begin{aligned}\therefore\ f(k+1)-f(k)&=[7(7^k)+4(4^k)+1]-[7^k+4^k+1]\\&=7(7^k)+4(4^k)+1-7^k-4^k-1\\&=6(7^k)+3(4^k)\\&=6(7^k)+3(4^{k-1}).4^1\\&=6(7^k)+12(4^{k-1})\\&=6[7^k+2(4)^{k-1}]\end{aligned}$$

$$\therefore\ f(k+1)=f(k)+6[7^k+2(4)^{k-1}]$$

As both f(k) and $6[7^k+2(4)^{k-1}]$ are divisible by 6 then the sum of these two terms must also be divisible by 6.

Therefore f(n) is divisible by 6 when $n = k + 1$.

If f(n) is divisible by 6 when $n = k$, then it has been shown that f(n) is also divisible by 6 when $n = k + 1$. As f(n) is divisible by 6 when n = 1, f(n) is also divisible by 6 for all $n\geqslant1$ and $n\in\mathbb{Z}^+$ by mathematical induction.

7 <u>Basis:</u> n = 1: LHS = 1 × 5 = 5; RHS = $\frac{1}{6}\times1\times2\times(2+13)=5$

<u>Assumption:</u> $\sum_{r=1}^{k}r(r+4)=\frac{1}{6}k(k+1)(2k+13)$

<u>Induction:</u>

$$\begin{aligned}\sum_{r=1}^{k+1}r(r+4)&=\sum_{r=1}^{k}r(r+4)+(k+1)(k+5)\\&=\frac{1}{6}k(k+1)(2k+13)+(k+1)(k+5)\\&=\frac{1}{6}(2k^3+21k^2+49k+30)=\frac{1}{6}(k+1)(k+2)(2(k+1)+13)\end{aligned}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

<u>Conclusion:</u> The statement holds for all $n\in\mathbb{Z}^+$.

8 **a** Basis: $n = 1$: LHS = 1 + 4 = 5; RHS = $\frac{1}{3} \times 1 \times 3 \times 5 = 5$

Assumption:

$$\sum_{r=1}^{2k} r^2 = \frac{1}{3}k(2k+1)(4k+1)$$

Induction:

$$\sum_{r=1}^{2(k+1)} r^2 = \sum_{r=1}^{2k} r^2 + (2k+1)^2 + (2k+2)^2$$

$$= \frac{1}{3}k(2k+1)(4k+1) + (2k+1)^2 + (2k+2)^2$$

$$= \frac{1}{3}(8k^3 + 30k^2 + 37k + 15)$$

$$= \frac{1}{3}(k+1)(2(k+1)+1)(4(k+1)+1)$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

b Using **a** and the formula for $\sum_{r=1}^{n} r^2$

$$\frac{1}{6} \times 2n(2n+1)(4n+1) = \frac{1}{6}kn(n+1)(2n+1)$$

$$2n(2n + 1) + (4n + 1) = kn(n + 1)(2n + 1)$$

$$16n^3 + 12n^3 + 12n^2 + 2n = k(2n^3 + 3n^2)$$

$$\Rightarrow k = \frac{2n(8n^2+6n+1)}{n(2n^2+3n+1)} = \frac{2(2n+1)(4n+1)}{(2n+1)(n+1)} = \frac{8n+2}{n+1}$$

$$\Rightarrow kn + k = 8n + 2 \Rightarrow n(k-8) = 2 - k \Rightarrow n = \frac{2-k}{k-8}$$

9 Basis: $u_n = \frac{3^n - 1}{2} \Rightarrow n = 1:\ u_1 = \frac{3^1 - 1}{2} = 1$ as given

For $n = 2$: $u_2 = \frac{3^2 - 1}{2} = 4$ from the general statement

$u_2 = 3u_1 + 1 = 3\times1+1 = 4$ from the recurrence relation.

Therefore, u_n is true for $n = 1$ and $n = 2$.

Assumption: Assume u_n is true for $n = k$, where $k \in \mathbb{Z}^+$

$$u_k = \frac{3^k - 1}{2}$$

Induction: Using the recurrence relation

$$u_{k+1} = 3u_k + 1$$
$$= 3\left(\frac{3^k - 1}{2}\right) + 1$$
$$= \frac{3^{k+1}}{2} - \frac{3}{2} + 1$$
$$= \frac{3^{k+1} - 1}{2}$$

This is the same expression that the general statement gives for u_{k+1}.

Therefore the general statement, u_n, is true for $n = k + 1$.

Conclusion: If u_n is true when $n = k$, then it has been shown that u_n is also true when $n = k + 1$. As u_n is true for $n = 1$ and $n = 2$ then u_n is true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

10 a $u_{n+1} = \frac{3u_n - 1}{4}$

$$u_1 = 2$$
$$u_2 = \frac{3(2) - 1}{4} = \frac{5}{4}$$
$$u_3 = \frac{3\left(\frac{5}{4}\right) - 1}{4} = \frac{11}{16}$$
$$u_4 = \frac{3\left(\frac{11}{16}\right) - 1}{4} = \frac{17}{64}$$
$$u_5 = \frac{3\left(\frac{17}{64}\right) - 1}{4} = -\frac{13}{256}$$

10 b Basis: $u_n = 4\left(\frac{3}{4}\right)^n - 1 \Rightarrow n = 1: u_1 = 4\left(\frac{3}{4}\right)^1 - 1 = 2$ as given

$n = 2$: $u_2 = 4\left(\frac{3}{4}\right)^2 - 1 = \frac{5}{4}$ from the general statement

$u_2 = \frac{3u_1 - 1}{4} = \frac{3(2)-1}{4} = \frac{5}{4}$ from the recurrence relation.

Therefore, u_n is true for $n = 1$ and $n = 2$.

Assumption: Assume u_n is true for $n = k$, where $k \in \mathbb{Z}^+$

$$u_k = 4\left(\frac{3}{4}\right)^k - 1$$

Induction: Using the recurrence relation

$$u_{k+1} = \frac{3u_k - 1}{4}$$

$$= \frac{3\left[4\left(\frac{3}{4}\right)^k - 1\right] - 1}{4}$$

$$= \frac{12\left(\frac{3}{4}\right)^k - 3 - 1}{4}$$

$$= \frac{12\left(\frac{3}{4}\right)^k}{4} + \frac{-3-1}{4}$$

$$= 3\left(\frac{3}{4}\right)^k - 1$$

$$= \left(4 \times \frac{3}{4}\right)\left(\frac{3}{4}\right)^k - 1$$

$$= 4\left(\frac{3}{4}\right)^{k+1} - 1$$

This is the same expression that the general statement gives for u_{k+1}.

Therefore the general statement, u_n, is true for $n = k + 1$.

Conclusion: If u_n is true when $n = k$, then it has been shown that u_n is also true when $n = k + 1$. As u_n is true for $n = 1$ and $n = 2$ then u_n is true for all $n \geq 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

11 a <u>Basis:</u> $n = 1$: LHS = RHS = $\begin{pmatrix} 2c & 1 \\ 0 & c \end{pmatrix}$

<u>Assumption:</u>

$$\mathbf{M}^k = \mathbf{c}^k \begin{pmatrix} 2^k & \dfrac{2^k - 1}{c} \\ 0 & 1 \end{pmatrix}$$

<u>Induction:</u>

$$\mathbf{M}^{k+1} = \mathbf{M}^k \begin{pmatrix} 2c & 1 \\ 0 & c \end{pmatrix}$$

$$= c^k \begin{pmatrix} 2^k & \dfrac{2^k - 1}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2c & 1 \\ 0 & c \end{pmatrix} = c^{k+1} \begin{pmatrix} 2^k & \dfrac{2^k - 1}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & \dfrac{1}{c} \\ 0 & 1 \end{pmatrix}$$

$$= c^{k+1} \begin{pmatrix} 2^{k+1} & \dfrac{2^k + 2^k - 1}{c} \\ 0 & 1 \end{pmatrix} = c^{k+1} \begin{pmatrix} 2^{k+1} & \dfrac{2^{k+1} - 1}{c} \\ 0 & 1 \end{pmatrix}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

<u>Conclusion:</u> The statement holds for all $n \in \mathbb{Z}^+$.

b Consider $n = 1$: det $\mathbf{M} = 50 \Rightarrow 2c^2 = 50$
So $c = 5$, since c is positive.

Challenge

a Basis: $n = 1$: LHS = RHS = $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Assumption:

$$\mathbf{M}^k = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}$$

Induction:

$$\mathbf{M}^{k+1} = \mathbf{M}^k \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos k\theta\ \cos\theta\ -\sin k\theta\sin\theta & -\cos\theta\sin\theta\ \ -\sin\theta\cos\theta \\ \sin k\theta\ \cos\theta\ +\cos k\theta\sin\theta & -\sin k\theta\sin\theta\ +\cos k\theta\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\left((k+1)\theta\right) & -\sin\left((k+1)\theta\right) \\ \sin\left((k+1)\theta\right) & \cos\left((k+1)\theta\right) \end{pmatrix}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

b The matrix M represents a rotation through angle θ, and so $\mathbf{M}^n$ represents a rotation through angle $n\theta$.