Exercise 8D

1 $n=1;\ \text{LHS}=\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}^1=\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$

$\text{RHS}=\begin{pmatrix}1 & 2(1)\\0 & 1\end{pmatrix}=\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$

As LHS = RHS, the matrix equation is true for $n = 1$.
Assume that the matrix equation is true for $n = k$.

i.e. $\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}^k=\begin{pmatrix}1 & 2k\\0 & 1\end{pmatrix}$

With $n = k + 1$ the matrix equation becomes

$$\begin{aligned}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}^{k+1}&=\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}^k\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\\&=\begin{pmatrix}1 & 2k\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\\&=\begin{pmatrix}1+0 & 2+2k\\0+0 & 0+1\end{pmatrix}.\\&=\begin{pmatrix}1 & 2(k+1)\\0 & 1\end{pmatrix}\end{aligned}$$

Therefore the matrix equation is true when $n = k + 1$.
If the matrix equation is true for $n = k$ then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$ it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$, by mathematical induction.

2 $n=1$; $\text{LHS}=\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}^1=\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}$

$\text{RHS}=\begin{pmatrix}2(1)+1 & -4(1)\\ 1 & -2(1)+1\end{pmatrix}=\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}$

As LHS = RHS, the matrix equation is true for $n = 1$.
Assume that the matrix equation is true for $n = k$.

i.e. $\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}^k=\begin{pmatrix}2k+1 & -4k\\ k & -2k+1\end{pmatrix}$.

With $n = k + 1$ the matrix equation becomes

$$\begin{aligned}\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}^{k+1}&=\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}^k\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}\\ &=\begin{pmatrix}2k+1 & -4k\\ k & -2k+1\end{pmatrix}\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}\\ &=\begin{pmatrix}6k+3-4k & -8k-4+4k\\ 3k-2k+1 & -4k+2k-1\end{pmatrix}\\ &=\begin{pmatrix}2k+3 & -4k-4\\ k+1 & -2k-1\end{pmatrix}\\ &=\begin{pmatrix}2(k+1)+1 & -4(k+1)\\ (k+1) & -2(k+1)+1\end{pmatrix}\end{aligned}$$

Therefore the matrix equation is true when $n = k + 1$.
If the matrix equation is true for $n = k$ then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

3 $n=1$; LHS $=\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}^1=\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}$

RHS $=\begin{pmatrix}2^1 & 0\\2^1-1 & 1\end{pmatrix}=\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}$

As LHS=RHS, the matrix equation is true for $n = 1$.
Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}^k=\begin{pmatrix}2^k & 0\\2^k-1 & 1\end{pmatrix}$

With $n = k + 1$ the matrix equation becomes

$$\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}^{k+1}=\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}^k\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}$$

$$=\begin{pmatrix}2^k & 0\\2^k-1 & 1\end{pmatrix}\begin{pmatrix}2 & 0\\1 & 1\end{pmatrix}$$

$$=\begin{pmatrix}2(2^k)+0 & 0+0\\2(2^k)-2+1 & 0+1\end{pmatrix}$$

$$=\begin{pmatrix}2^{1(2^k)} & 0\\2^{1(2^k)}-1 & 1\end{pmatrix}$$

$$=\begin{pmatrix}2^{k+1} & 0\\2^{k+1}-1 & 1\end{pmatrix}$$

Therefore the matrix equation is true when $n = k + 1$.
If the matrix equation is true for $n = k$ then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

4 a $n = 1$; LHS $= \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^1 = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$

$$\text{RHS} = \begin{pmatrix} 4(1)+1 & -8(1) \\ 2(1) & 1-4(1) \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$$

As LHS = RHS, the matrix equation is true for $n = 1$.
Assume that the matrix equation is true for $n = k$.

ie. $\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^k = \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix}$.

With $n = k + 1$ the matrix equation becomes

$$\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^{k+1} = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}^k \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 4k+1 & -8k \\ 2k & 1-4k \end{pmatrix}\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} 20k+5-16k & -32-8+24k \\ 10k+2-8k & -16k-3+12k \end{pmatrix}$$

$$= \begin{pmatrix} 4k+5 & -8k-8 \\ 2k+2 & -4k-3 \end{pmatrix}$$

$$= \begin{pmatrix} 4(k+1)+1 & -8(k+1) \\ 2(k+1) & 1-4(k+1) \end{pmatrix}$$

Therefore the matrix equation is true when $n = k + 1$.
If the matrix equation is true for $n = k$, then it is shown to be true for $n = k + 1$. As the matrix equation is true for $n = 1$, it is now also true for all $n \geqslant 1$ and $n \in \mathbb{Z}^+$ by mathematical induction.

b $n = 6$

5 a Basis: $n = 1$: LHS = RHS $= \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}$

Assumption: $M^k = \begin{pmatrix} 2k & 5(2k-1) \\ 0 & 1 \end{pmatrix}$

Induction: With $n = k + 1$ the matrix equation becomes

$$M^{k+1} = M^k \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2k & 5(2k-1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2k+1 & 5\times 2k+5(2k-1) \\ 0 & 1 \end{pmatrix}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.
Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

b $\begin{pmatrix} 2^{-n} & 5(2^{-n}-1) \\ 0 & 1 \end{pmatrix}$