Exercise 8A

1 Basis: When $n = 1$: LHS = 1; RHS = $\frac{1}{2}(1)(1 + 1) = 1$

Assumption:

$$\sum_{r=1}^{k} r = \tfrac{1}{2}k(k+1)$$

Induction:

$$\sum_{r=1}^{k+1} r = \sum_{r=1}^{k} r + (k+1) = \tfrac{1}{2}k(k+1) + (k+1)$$

$$= \tfrac{1}{2}k(k+1) + (k+1) = \tfrac{1}{2}(k+1)(k+2)$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

2 Basis: When $n = 1$: LHS = 1; RHS = $\frac{1}{4}(1)^2(1 + 1)^2 = 1$

Assumption:

$$\sum_{r=1}^{k} r^3 = \tfrac{1}{4}k^2(k+1)^2$$

Induction:

$$\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k+1)^3 = \tfrac{1}{4}k^2(k+1)^2 + (k+1)^3$$

$$= \tfrac{1}{4}(k+1)^2\left(k^2 + 4(k+1)\right) = \tfrac{1}{4}(k+1)^2(k+2)^2$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

3 **a** Basis: $n = 1$: LHS = 0; RHS = $\frac{1}{3}(1)(1+1)(1-1) = 0$

Assumption:

$$\sum_{r=1}^{k} r(r-1) = \tfrac{1}{3}k(k+1)(k-1)$$

Induction:

$$\sum_{r=1}^{k+1} r(r-1) = \sum_{r=1}^{k} r(r-1) + (k+1)k$$

$$= \tfrac{1}{3}k(k+1)(k-1) + k(k+1)$$

$$= \tfrac{1}{3}k(k+1)(k-1+3) = \tfrac{1}{3}k(k+1)(k+2)$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

b Hence the required expression is

$$\sum_{r=1}^{2n+1} r(r-1) = \tfrac{1}{3}(2n+1)((2n+1)+1)((2n+1)-1)$$

$$= \tfrac{4}{3}n(2n+1)(n+1)$$

4 **a** Basis: $n = 1$: LHS = 2; RHS = $(1)^2(1+1) = 2$

Assumption:

$$\sum_{r=1}^{k} r(3r-1) = k^2(k+1)$$

Induction:

$$\sum_{r=1}^{k+1} r(3r-1) = \sum_{r=1}^{k} r(3r-1) + (k+1)(3k+2)$$

$$= k^2(k+1) + (k+1)(3k+2)$$

$$= (k+1)(k^2+3k+2) = (k+1)^2(k+2)$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

b We need to solve the equation

$$4n^2(n+1) = \frac{n^2(n+1)^2}{4}$$

Rearranging and cancelling the common factor $n^2(n+1)$ gives

$$(n+1) = 16$$

$$n = 15$$

5 a Basis: $n = 1$: LHS $= \frac{1}{2}$; RHS $= 1 - \frac{1}{2} = \frac{1}{2}$

Assumption:

$$\sum_{r=1}^{k}\left(\tfrac{1}{2}\right)^r = 1 - \frac{1}{2^k}$$

Induction:

$$\sum_{r=1}^{k+1}\left(\tfrac{1}{2}\right)^r = \sum_{r=1}^{k}\left(\tfrac{1}{2}\right)^r + \frac{1}{2^{k+1}} = \frac{1}{2^k} + \frac{1}{2^{k+1}}$$

$$= 1 - \frac{2}{2^{k+1}} + \frac{1}{2^{k+1}} = 1 - \frac{1}{2^{k+1}}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

b Basis: $n = 1$: LHS $=1 \times 1!$; RHS $= (1 + 1)! - 1 = 1$

Assumption:

$$\sum_{r=1}^{k+1} r(r!) = (n+1) - 1$$

Induction:

$$\sum_{r=1}^{k+1} r(r!) = \sum_{r=1}^{k} r(r!) + (k+1)(k+1)!$$

$$= (k + 1)! - 1 + (k + 1)(k + 1)!$$

$$= (k + 1)!\,(k + 2) - 1 = ((k + 1) + 1)! - 1)$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

5 **c** Basis: $n = 1$: LHS $= \frac{4}{1\times 3} = \frac{4}{3}$; RHS $= \frac{1\times 8}{2\times 3} = \frac{4}{3}$

Assumption:

$$\sum_{r=1}^{k} \frac{4}{r(r+2)} = \frac{k(3k+5)}{(k+1)(k+2)}$$

Induction:

$$\sum_{r=1}^{k+1} \frac{4}{r(r+2)} = \sum_{r=1}^{k} \frac{4}{r(r+2)} + \frac{4}{(k+1)(k+3)}$$

$$= \frac{k(3k+5)}{(k+1)(k+2)} + \frac{4}{(k+1)(k+3)}$$

$$= \frac{k(3k+5)(k+3)}{(k+1)(k+2)(k+3)} + \frac{4(k+2)}{(k+1)(k+2)(k+3)}$$

$$= \frac{k(3k+5)(k+3)+4(k+2)}{(k+1)(k+2)(k+3)} + \frac{(k+1)(3k+8)}{(k+2)(k+3)}$$

$$= \frac{(k+1)(3(k+1)+5)}{((k+1+1)((k+1)+2))}$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.

6 **a** The student has just stated and not shown that the statement is true for $n = k + 1$.

b e.g. $n = 2$: LHS $= (1 + 2)^2 = 9$; RHS $= 1^2 + 2^2 \neq 9$, so that LHS $\neq$ RHS.

7 **a** The student has not completed the basis step.

b e.g. $n = 1$: LHS $= 1$; RHS $= \frac{1}{2}(1^2 + 1 + 1) = \frac{3}{2} \neq 1$

Challenge

Basis: $n = 1$: LHS $= (-1)^1 \times 1^2$; RHS $= \frac{1}{2}(-1)^1(1)(1+1)$

Assumption:

$$\sum_{r=1}^{k} (-1)^r r^2 = \tfrac{1}{2}(-1)^k k(k+1)$$

Induction:

$$\sum_{r=1}^{k+1} (-1)^r r^2 = \sum_{r=1}^{k} (-1)^r r^2 + (-1)^{k+1}(k+1)^2$$

$$= \tfrac{1}{2}(-1)k(k+1) + (-1)^{k+1}(k+1)^2$$

$$= \tfrac{1}{2}(-1)^{k+1}(k+1)(-k+2(k+1))$$

$$= \tfrac{1}{2}(-1)^{k+1}(k+1)(k+2)$$

So if the statement holds for $n = k$, it holds for $n = k + 1$.

Conclusion: The statement holds for all $n \in \mathbb{Z}^+$.