Exercise 7B

1 a $\sum_{r=1}^{4} r^2 = \frac{1}{6}\times 4\times 5\times 9 = 30$

b $\sum_{r=1}^{40} r^2 = \frac{1}{6}\times 40\times 41\times 81 = 22\,140$

c $\sum_{r=21}^{40} r^2 = \sum_{r=1}^{40} r^2 - \sum_{r=1}^{20} r^2$

$= \frac{1}{6}\times 40\times 41\times 81 - \frac{1}{6}\times 20\times 21\times 41$

$= 19270$

d $\sum_{r=1}^{99} r^3 = \frac{1}{4}\times 99^2\times 100^2 = 24\,502\,500$

e $\sum_{r=100}^{200} r^3 = \sum_{r=1}^{200} r^3 - \sum_{r=1}^{99} r^3$

$= \frac{1}{4}\times 200^2\times 201^2 - \frac{1}{4}\times 99^2\times 100^2$

$= 379\,507\,500$

f $\sum_{r=1}^{k} r^2 + \sum_{r=k+1}^{80} r^2 = \sum_{r=1}^{80} r^2$

$= \frac{1}{6}\times 80\times 81\times 161$

$= 173\,880$

2 a $\sum_{r=1}^{2n} r^2 = \frac{1}{6}(2n)(2n+1)(4n+1)$

$= \frac{1}{3}n(2n+1)(4n+1)$

b $\sum_{r=1}^{2n-1} r^2 = \frac{1}{6}(2n-1)(2n)(2(2n-1)+1)$

$= \frac{1}{6}\times 2n(2n-1)(4n-1)$

$= \frac{1}{3}n(2n-1)(4n-1)$

c $\sum_{r=n}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n-1} r^2$

$= \frac{1}{3}n(2n+1)(4n+1) - \frac{1}{6}(n-1)(n)(2(n-1)+1)$

$= \frac{1}{6}n(2(2n+1)(4n+1) - (n-1)(2n-1))$

$= \frac{1}{6}n(16n^2+12n+2-2n^2+3n-1)$

$= \frac{1}{6}n(14n^2+15n+1)$

$= \frac{1}{6}n(n+1)(14n+1)$

3 Let $n+k=m$

$\Rightarrow \sum_{r=1}^{m} r^3 = \frac{1}{4}m^2(m+1)^2$

and so $\sum_{r=1}^{n+k} r^3 = \frac{1}{4}(n+k)^2(n+k+1)^2$

4 a $\sum_{r=n+1}^{3n} r^3 = \sum_{r=1}^{3n} r^3 - \sum_{r=1}^{n} r^3$

$= \frac{1}{4}(3n)^2(3n+1)^2 - \frac{1}{4}n^2(n+1)^2$

$= \frac{1}{4}n^2(9(3n+1)^2 - (n+1)^2)$

$= \frac{1}{4}n^2(81n^2 - n^2 + 54n - 2n + 9 - 1)$

$= n^2(20n^2+13n+2)$

$= n^2(4n+1)(5n+2)$

b $\sum_{r=11}^{30} r^3 = 10^2\times 41\times 52 = 213\,200$

5 a $\sum_{r=n}^{2n} r^3 = \sum_{r=1}^{2n} r^3 - \sum_{r=1}^{n-1} r^3$

$= \frac{1}{4}(2n)^2(2n+1)^2 - \frac{1}{4}(n-1)^2n^2$

$= \frac{1}{4}n^2(4(2n+1)^2 - (n-1)^2)$

$= \frac{1}{4}n^2(16n^2+16n+4-n^2+2n-1)$

$= \frac{1}{4}n^2(15n^2+18n+3)$

$= \frac{3}{4}n^2(5n^2+6n+1)$

$= \frac{3}{4}n^2(n+1)(5n+1)$

b $\sum_{r=30}^{60} r^3 = \frac{3}{4}\times 30^2\times 31\times 151 = 3\,159\,675$

6 a $\sum_{m=1}^{30}(m^2-1) = \sum_{m=1}^{30} m^2 - \sum_{m=1}^{30} 1$

$= \sum_{m=1}^{30} m^2 - 30$

$= \frac{1}{6}\times 30\times 31\times 61 - 30 = 9425$

b $\sum_{r=1}^{40} r(r+4) = \sum_{r=1}^{40} r^2+4r = \sum_{r=1}^{40} r^2 + 4\sum_{r=1}^{40} r$

$= \frac{1}{6}\times 40\times 41\times 81 + 4\times\frac{1}{2}\times 40\times 41$

$= 22\,140 + 3280 = 25\,420$

c $\sum_{r=1}^{80} r(r^2+3) = \sum_{r=1}^{80}(r^3+3r) = \sum_{r=1}^{80} r^3 + 3\sum_{r=1}^{80} r$

$= \frac{1}{4}\times 80^2\times 81^2 + 3\times\frac{1}{2}\times 80\times 81$

$= 10\,497\,600 + 9720 = 10\,507\,320$

6 d $\sum_{r=11}^{35}(r^3-2)=\sum_{r=1}^{35}(r^3-2)-\sum_{r=1}^{10}(r^3-2)$

$=\sum_{r=1}^{35}r^3-\sum_{r=1}^{10}r^3-2\sum_{r=1}^{35}1-2\sum_{r=1}^{10}1$

$=\frac{1}{4}\times35^2\times36^2-\frac{1}{4}\times10^2\times11^2-2(35-10)$

$=396\,900-3025-50=393\,825$

7 a $\sum_{r=1}^{n}(r+2)(r+5)=\sum_{r=1}^{n}(r^2+7r+10)$

$=\sum_{r=1}^{n}r^2+7\sum_{r=1}^{n}r+10\sum_{r=1}^{n}1$

$=\frac{1}{6}n(n+1)(2n+1)+\frac{7}{2}n(n+1)+10n$

$=\frac{1}{6}n(2n^2+3n+1+21n+21+60)$

$=\frac{1}{3}n(n^2+12n+41)$

b $\sum_{r=10}^{50}(r+2)(r+5)$

$=\sum_{r=1}^{50}(r+2)(r+5)-\sum_{r=1}^{9}(r+2)(r+5)$

$=\frac{1}{3}\times50\times(50^2+600+41)$

$-\frac{1}{3}\times9\times(9^2+108+41)$

$=52\,350-690=51\,660$

8 a $\sum_{r=1}^{n}(r^2+3r+1)=\sum_{r=1}^{n}r^2+3\sum_{r=1}^{n}r+\sum_{r=1}^{n}1$

$=\frac{1}{6}n(n+1)(2n+1)+\frac{3}{2}n(n+1)+n$

$=\frac{1}{6}n(2n^2+3n+1+9n+9+6)$

$=\frac{1}{3}n(n^2+6n+8)=\frac{1}{3}n(n+2)(n+4)$

$a=2,\ b=4$

b $\sum_{r=19}^{40}(r^2+3r+1)$

$=\sum_{r=1}^{40}(r^2+3r+1)-\sum_{r=1}^{18}(r^2+3r+1)$

$=\frac{1}{3}\times40\times42\times44-\frac{1}{3}\times18\times20\times22$

$=22\,000$

9 a $\sum_{r=1}^{n}r^2(r-1)=\sum_{r=1}^{n}r^3-\sum_{r=1}^{n}r^2$

$=\frac{1}{4}n^2(n+1)^2-\frac{1}{6}n(n+1)(2n+1)$

$=\frac{1}{12}n(n+1)(3n(n+1)-2(2n+1))$

$=\frac{1}{12}n(n+1)(3n^2-n-2)$

9 b $\sum_{r=1}^{2n-1}r^2(r-1)$

$=\frac{1}{12}(2n-1)((2n-1)+1)(3(2n-1)^2$

$-(2n-1)-2)$

$=\frac{1}{12}\times2n(2n-1)(12n^2-14n+2)$

$=\frac{1}{3}n(2n-1)(6n^2-7n+1)$

10 a $\sum_{r=1}^{n}(r+1)(r+3)=\sum_{r=1}^{n}(r^2+4r+3)$

$=\sum_{r=1}^{n}r^2+4\sum_{r=1}^{n}r+3\sum_{r=1}^{n}1$

$=\frac{1}{6}n(n+1)(2n+1)+2n(n+1)+3n$

$=\frac{1}{6}n(2n^2+3n+1+12n+12+18)$

$=\frac{1}{6}n(2n^2+15n+31)$

b $\sum_{r=n+1}^{2n}(r+1)(r+3)$

$=\sum_{r=1}^{2n}(r+1)(r+3)-\sum_{r=1}^{n}(r+1)(r+3)$

$=\frac{1}{6}\times2n(8n^2+30n+31)$

$-\frac{1}{6}\times n\times(2n^2+15n+31)$

$=\frac{1}{6}n(16n^2+60n+62-2n^2-15n-31)$

$=\frac{1}{6}n(14n^2+45n+31)$

11 a $\sum_{r=1}^{n}(r+3)(r+4)=\sum_{r=1}^{n}r^2+7r+12$

$=\sum_{r=1}^{n}r^2+7\sum_{r=1}^{n}r+12\sum_{r=1}^{n}1$

$=\frac{1}{6}n(n+1)(2n+1)+\frac{7}{2}n(n+1)+12n$

$=\frac{1}{6}n(2n^2+3n+1+21n+21+72)$

$=\frac{1}{3}n(n^2+12n+47)$

b $\sum_{r=n+1}^{3n}(r+3)(r+4)$

$=\sum_{r=1}^{3n}(r+3)(r+4)-\sum_{r=1}^{n}(r+3)(r+4)$

$=\frac{1}{3}\times3n(9n^2+36n+47)$

$-\frac{1}{3}n(n^2+12n+47)$

$=\frac{1}{3}n(27n^2+108n+141-n^2-12n-47)$

$=\frac{1}{3}n(26n^2+96n+94)$

$=\frac{2}{3}n(13n^2+48n+47)$

12 a $\sum_{r=1}^{n} r(r+3)^2 = \sum_{r=1}^{n} r^3 + 6r^2 + 9r$

$= \sum_{r=1}^{n} r^3 + 6\sum_{r=1}^{n} r^2 + 9\sum_{r=1}^{n} r$

$= \frac{1}{4}n^2(n+1)^2 + n(n+1)(2n+1) + \frac{9}{2}n(n+1)$

$= \frac{1}{4}n(n+1)(n(n+1) + 4(2n+1) + 18)$

$= \frac{1}{4}n(n+1)(n^2 + 9n + 22)$

b $\sum_{r=10}^{20} r(r+3)^2$

$= \sum_{r=1}^{20} r(r+3)^2 - \sum_{r=1}^{9} r(r+3)^2$

$= \frac{1}{4}\times 20\times 21\times(20^2 + 180 + 22)$

$- \frac{1}{4}\times 9\times 10\times(9^2 + 81 + 22)$

$= 59\,070$

13 a $\sum_{r=1}^{kn} (2r-1) = 2\sum_{r=1}^{kn} r - \sum_{r=1}^{kn} 1$

$= kn(kn+1) - kn = k^2n^2$

b $\sum_{r=1}^{5n} (2r-1) = 25n^2 \Rightarrow \sum_{r=1}^{n} r^3 = 25n^2$

$\Rightarrow \frac{1}{4}n^2(n+1)^2 = 25n^2$

$\Rightarrow (n+1)^2 = 100$

$\Rightarrow n = 9$

14 a $\sum_{r=1}^{n} (r^3 - r^2) = \sum_{r=1}^{n} r^3 - \sum_{r-1}^{n} r^2$

$= \frac{1}{4}n^2(n+1)^2 - \frac{1}{6}n(n+1)(2n+1)$

$= \frac{1}{12}n(n+1)(3n(n+1) - 2(2n+1))$

$= \frac{1}{12}n(n+1)(3n^2 - n - 2)$

$= \frac{1}{12}n(n+1)(n-1)(3n+2)$

14 b $\sum_{r=1}^{n} 7r = \frac{7}{2}n(n+1)$

$\Rightarrow \frac{7}{2}n(n+1) = \frac{1}{12}n(n+1)(n-1)(3n+2)$

$\Rightarrow \frac{7}{2} = \frac{1}{12}(n-1)(3n+2)$

$\Rightarrow 42 = (n-1)(3n+2)$

$\Rightarrow 3n^2 - n - 44 = 0$

$\Rightarrow (3n+11)(n-4) = 0$

$\Rightarrow n = 4$

Challenge

a Let $\mathrm{f}_2(x) = ax + b$

$\sum_{r=1}^{n} \mathrm{f}_2(r) = \frac{1}{2}an(n+1) + bn$

$\Rightarrow \frac{1}{2}an(n+1) + bn = n^2$

$\Rightarrow \frac{1}{2}an^2 + \left(\frac{1}{2}a + b\right)n$

$a = 2,\ b = -1$

$\Rightarrow \mathrm{f}_2(x) = 2x - 1$

Let $\mathrm{f}_3(x) = ax^2 + bx + c$

$\sum_{r=1}^{n} \mathrm{f}_3(r) = \frac{1}{6}an(n+1)(2n+1) + \frac{1}{2}bn(n+1) + cn$

$\Rightarrow \frac{1}{6}an(n+1)(2n+1) + \frac{1}{2}bn(n+1) + cn = n^3$

$\Rightarrow \frac{1}{3}an^3 + \left(\frac{1}{2}a + \frac{1}{2}b\right)n^2 + \left(\frac{1}{6}a + \frac{1}{2}b + c\right)n = n^3$

$a = 3,\ b = -3,\ c = 1$

$\Rightarrow \mathrm{f}_3(x) = 3x^2 - 3x + 1$

Let $\mathrm{f}_4(x) = ax^3 + bx^2 + cx + d$

$\sum_{r=1}^{n} \mathrm{f}_4(r) = \frac{1}{4}an^2(n+1)^2 + \frac{1}{6}bn(n+1)(2n+1)$

$+ \frac{1}{2}cn(n+1) + dn$

$\Rightarrow \frac{1}{4}a = 1$

$\frac{1}{2}a + \frac{1}{3}b = 0$

$\frac{1}{4}a + \frac{1}{2}b + \frac{1}{2}c = 0$

$\frac{1}{6}b + \frac{1}{2}c + d = 0$

$a = 4,\ b = -6,\ c = 4,\ d = -1$

$\Rightarrow \mathrm{f}_4(x) = 4x^3 - 6x^2 + 4x - 1$

b Given $\mathrm{h}(x) = ax^3 + bx^2 + cx + d$,

$n\mathrm{h}(n) = an^4 + bn^3 + cn^2 + dn$

$= \sum_{r=1}^{n} (a\mathrm{f}_4(r) + b\mathrm{f}_3(r) + c\mathrm{f}_2(r) + d\mathrm{f}_1(r))$

$= \sum_{r=1}^{n} \mathrm{g}(r)$

for $\mathrm{g}(r) = a\mathrm{f}_4(r) + b\mathrm{f}_3(r) + c\mathrm{f}_2(r) + d\mathrm{f}_1(r)$