Exercise 7A

1 a $\sum_{r=0}^{3}(2r+1)=1+3+5+7$
$=16$

b $\sum_{r=1}^{40} r=\frac{1}{2}\times 40\times 41$
$=820$

c $\sum_{r=1}^{20} r=\frac{1}{2}\times 20\times 21$
$=210$

d $\sum_{r=1}^{99} r=\frac{1}{2}\times 99\times 100$
$=4950$

e $\sum_{r=10}^{40} r=\sum_{r=1}^{40} r-\sum_{r=1}^{9} r$
$=\frac{1}{2}\times 40\times 41-\frac{1}{2}\times 9\times 10$
$=820-45=775$

f $\sum_{r=100}^{200} r=\sum_{r=1}^{200} r-\sum_{r=1}^{99} r$
$=\frac{1}{2}\times 200\times 201-\frac{1}{2}\times 99\times 100$
$=20\,100-4950=15\,150$

g $\sum_{r=21}^{40} r=\sum_{r=1}^{40} r-\sum_{r=1}^{20} r$
$=\frac{1}{2}\times 40\times 41-\frac{1}{2}\times 20\times 21$
$=820-210=610$

h $\sum_{r=1}^{k} r+\sum_{r=k+1}^{80} r=\sum_{r=1}^{k} r+\sum_{r=1}^{80} r-\sum_{r=1}^{k} r$
$=\sum_{r=1}^{80} r$
$=\frac{1}{2}\times 80\times 81$
$=3240$

2 $\frac{1}{2}\times n\times(n+1)=528$
$n^2+n-1056=0$
$(n+33)(n-32)=0$
$n=32$

3 $\frac{1}{2}k(k+1)=\frac{1}{2}\times\frac{1}{2}\times 20\times 21$
$k^2+k-210=0$
$(k+15)(k-14)=0$
$k=14$

4 a $\sum_{r=1}^{2n-1} r=\frac{1}{2}\times(2n-1)\times(2n-1+1)$
$=\frac{1}{2}\times(2n-1)\times 2n$
$=n(2n-1)$

b $\sum_{r=n+1}^{2n-1} r=\sum_{r=1}^{2n-1} r-\sum_{r=1}^{n} r=n(2n-1)-\frac{1}{2}n(n+1)$
$=\frac{1}{2}n(4n-2-n-1)$
$=\frac{1}{2}n(3n-3)=\frac{3}{2}n(n-1)$

5 $\sum_{r=n-1}^{2n} r=\sum_{r=1}^{2n} r-\sum_{r=1}^{n-2} r$
$=\frac{1}{2}(2n)(2n+1)-\frac{1}{2}(n-2)(n-1)$
$=\frac{1}{2}(2n(2n+1)-(n-2)(n-1))$
$=\frac{1}{2}(3n^2+5n-2)$
$=\frac{1}{2}(n+2)(3n-1)$

6 a $\sum_{r=1}^{n^2} r-\sum_{r=1}^{n} r$
$=\frac{1}{2}n^2(n^2+1)-\frac{1}{2}n(n+1)$
$=\frac{1}{2}n(n(n^2+1)-(n+1))$
$=\frac{1}{2}n(n^3+n-n-1)$
$=\frac{1}{2}n(n^3-1)$

b $\sum_{r=10}^{81} r=\sum_{r=1}^{9^2} r-\sum_{r=1}^{9} r$
$=\frac{1}{2}n(n^3-1)$
$=\frac{1}{2}\times 9\times(9^3-1)$
$=3276$

7 a $\sum_{r=1}^{55}(3r-1)=3\sum_{r=1}^{55} r-\sum_{r=1}^{55} 1$
$=3\times\frac{1}{2}\times 55\times 56-55$
$=4565$

7 b $\sum_{r=1}^{90}(2-7r)=2\sum_{r=1}^{90}1-7\sum_{r=1}^{90}r$

$=2\times 90-7\times\frac{1}{2}\times 90\times 91$

$=-28\,485$

c $\sum_{r=1}^{46}(9+2r)=9\sum_{r=1}^{46}1+2\sum_{r=1}^{46}r$

$=9\times 46+2\times\frac{1}{2}\times 46\times 47$

$=2576$

8 a $\sum_{r=1}^{n}(3r+2)=3\sum_{r=1}^{n}r+2\sum_{r=1}^{n}1$

$=\frac{3}{2}n(n+1)+2n$

$=\frac{1}{2}n(3(n+1)+4)=\frac{1}{2}n(3n+7)$

b $\sum_{r=1}^{2n}(5r-4)=5\sum_{r=1}^{2n}r-4\sum_{r=1}^{2n}1$

$=\frac{5}{2}(2n)(2n+1)-4(2n)$

$=5n(2n+1)-8n$

$=10n^2-3n=n(10n-3)$

c $\sum_{r=1}^{n+2}(2r+3)=2\sum_{r=1}^{n+2}r+3\sum_{r=1}^{n+2}1$

$=(n+2)(n+3)+3(n+2)$

$=(n+2)((n+3)+3)$

$=(n+2)(n+6)$

d $\sum_{r=3}^{n}(4r+5)=4\sum_{r=3}^{n}4r+5\sum_{r=3}^{n}1$

$=4\left(\sum_{r=1}^{n}r-5\sum_{r=1}^{2}r\right)+5\left(\sum_{r=1}^{n}1-\sum_{r=1}^{2}1\right)$

$=4(\frac{1}{2}n(n+1)-3)+5(n-2)$

$=2n^2+7n-22=(2n+11)(n-2)$

9 a $\sum_{r=1}^{k}(4r-5)=4\sum_{r=1}^{k}r-5\sum_{r=1}^{k}1$

$=\frac{4}{2}k(k+1)-5k$

$=2k^2-2k-5k$

$=2k^2-3k$

b $2k^2-3k>4850\Rightarrow 2k^2-3k-4850>0$

$\Rightarrow(2k+97)(k-50)>0$

so $k>50$ since k is positive

$\Rightarrow k=51$

10 $\sum_{r=1}^{n}ar+b=a\sum_{r=1}^{n}r+\sum_{r=1}^{n}b$

$=a\times\frac{1}{2}n(n+1)+bn$

$=\frac{1}{2}an^2+\left(\frac{1}{2}a+b\right)n$

$=\frac{1}{2}n(an+(a+2b))$

Since $\sum_{r=1}^{n}\mathrm{f}(r)=\frac{1}{2}n(7n+1)$

$\Rightarrow a=7$

$\Rightarrow a+2b=1$

$\Rightarrow 7+2b=1\Rightarrow b=-3$

11 a $\sum_{r=1}^{4n-1}(3r+1)=3\sum_{r=1}^{4n-1}r+\sum_{r=1}^{4n-1}1$

$=\frac{3}{2}(4n-1)(4n)+(4n-1)$

$=6n(4n-1)+(4n-1)$

$=24n^2-6n+4n-1$

$=24n^2-2n-1$

b Substituting $n=25$ into the result from part **a**:
$24\times 25^2-2\times 25-1=14\,949$

12 a $\sum_{r=1}^{2k+1}(4-5r)=4\sum_{r=1}^{2k+1}1-5\sum_{r=1}^{2k+1}r$

$=4(2k+1)-\frac{5}{2}(2k+1)(2k+2)$

$=(2k+1)(4-5(k+1))$

$=(2k+1)(-1-5k)$

$=-(2k+1)(5k+1)$

b Substituting $k = 12$ into the result from part **a**:
$-25 \times 61 = -1525$

c $\sum_{r=1}^{15}(5r-4)=-\sum_{r=1}^{15}(4-5r)$

Substituting $k = 7$ into the result from part **a**:
$-(-15 \times 36) = 540$

13 Let $\mathrm{f}(r)=ar+b$

$\Rightarrow a\times\frac{1}{2}n(n+1)+bn$

$=\frac{1}{2}an^2+\left(\frac{1}{2}a+b\right)n$

Since $\sum_{r=1}^{n}\mathrm{f}(r)=n^2+4n$

$\Rightarrow \frac{1}{2}a=1\Rightarrow a=2$

and $\frac{1}{2}a+b=4\Rightarrow 1=b=4\Rightarrow b=3$

$\Rightarrow \mathrm{f}(r)=2r+3$

14 a $\sum_{r=1}^{n}ar+b=\frac{1}{2}an^2+\left(\frac{1}{2}a+b\right)n$

$\sum_{r=1}^{4}\mathrm{f}(r)=36\Rightarrow 8a+2a+4b=36$

$\Rightarrow 10a+4b=36\Rightarrow 5a+2b=18$

$\sum_{r=1}^{6}\mathrm{f}(r)=78\Rightarrow 18a+3a+6b=78$

$\Rightarrow 21a+6b=78\Rightarrow 7a+2b=26$

Solving simultaneously:

$2a=8\Rightarrow a=4$

$5\times4+2b=18\Rightarrow b=-1$

$\Rightarrow \mathrm{f}(r)=4r-1$

b $\sum_{r=1}^{10}\mathrm{f}(r)=\sum_{r=1}^{10}(2r-1)$

$=\frac{1}{2}\times4\times10^2+\left(\frac{1}{2}\times4-1\right)\times10$

$=200+10$

$=210$

Challenge

$\sum_{r=n}^{2n}(12-2r)=0$

$\Rightarrow \sum_{r=n}^{2n}12-\sum_{r=n}^{2n}2r=0$

$\sum_{r=n}^{2n}12=\sum_{r=n}^{2n}2r$

$\sum_{r=n}^{2n}12=\sum_{r=1}^{2n}12-\sum_{r=1}^{n-1}12=12(n+1)$

$\sum_{r=n}^{2n}2r=\sum_{r=1}^{2n}2r-\sum_{r=1}^{n-1}2r$

$=2\times\frac{1}{2}\times2n(2n+1)-2\times\frac{1}{2}(n-1)n$

$=3n^2+3n$

$\Rightarrow 3n^2+3n=12n+12$

$\Rightarrow n^2-3n-4=0$

$\Rightarrow (n-4)(n+1)=0$

$\Rightarrow n=4$