Chapter review 6

1 a $\mathbf{Y} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b $\mathbf{AB} = \mathbf{Y} \Rightarrow \mathbf{A} = \mathbf{YB}^{-1}$

$\mathbf{B} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$

det $\mathbf{B}$ = 3 – 4

$\mathbf{B}^{-1} = -\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$

$\mathbf{A} = \mathbf{YB}^{-1}$

$= -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$

$= -\begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$

$= \begin{pmatrix} -2 & 3 \\ -1 & 2 \end{pmatrix}$

c **ABABABAB** = **YYYY**
Since Y represents an anticlockwise rotation of 90° about O
ABABABAB represents four successive anticlockwise rotations of 90° about O, which will map all points onto themselves.

Therefore, **ABABABAB** $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2 a $\mathbf{R} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{E} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$\mathbf{C} = \mathbf{ER}$

$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

Reflection in x-axis together with an enlargement with scale factor 2, centre (0, 0).
Transformation can also be described as a stretch of scale factor 2 parallel to the x-axis and –2 parallel to the y-axis.

2 b det $\mathbf{C} = (2)(-2) - (0)(0) = -4$

$\mathbf{C}^{-1} = -\frac{1}{4}\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

$\mathbf{C}^{-1}$ represents a reflection in x-axis together with an enlargement with scale factor $\frac{1}{2}$, centre (0, 0).
Transformation can also be described as a stretch of scale factor $\frac{1}{2}$ parallel to the x-axis and $-\frac{1}{2}$ parallel to the y-axis.

3 a $\mathbf{P} = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$

Let $\mathbf{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(reflection in the line $y = x$)
has det $\mathbf{R} = -1$

$\mathbf{R}^{-1} = -\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\mathbf{P} = \mathbf{TR}$

$\mathbf{T} = \mathbf{PR}^{-1}$

$= \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

b $\mathbf{T}\begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$

$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$

$\begin{pmatrix} -3k \\ -2k \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$

$k = -3$

3 c A general point on the line $y = mx$ is given by (x, mx)

$$\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} -3x \\ -3mx \end{pmatrix}$$

If $y = mx$ is invariant under **T**, then the point $(-3x, -3mx)$ must lie on $y = mx$.

$-3mx = m(-3x)$ for all $m \in \mathbb{R}$.

Therefore, all lines of the form $y = mx$ are invariant under **T.**

4 a $\mathbf{M}\begin{pmatrix} 2\sqrt{3} & -2 \\ 2 & 2\sqrt{3} \end{pmatrix}$

det **M** = 16

Therefore, area scale factor = 16

Since area is not affected by rotation, scale factor of enlargement = $\sqrt{\det \mathbf{A}} = 4$ (or –4)

b $$\begin{pmatrix} 2\sqrt{3} & -2 \\ 2 & 2\sqrt{3} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 2\sqrt{3} & -2 \\ 2 & 2\sqrt{3} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4\cos\theta & -4\sin\theta \\ 4\sin\theta & 4\cos\theta \end{pmatrix}$$

$4\cos\theta = 2\sqrt{3}$ and $4\sin\theta = 2$

$\cos\theta = \dfrac{\sqrt{3}}{2}$ and $\sin\theta = \dfrac{1}{2}$

$\theta = 30°$

so the rotation is 30° anticlockwise about (0, 0)

(or 210° if a value of –4 was used for part **a**.)

4 c From **a,** det **M** = 16

So, $\mathbf{M}^{-1} = \dfrac{1}{16}\begin{pmatrix} 2\sqrt{3} & 2 \\ -2 & 2\sqrt{3} \end{pmatrix}$

$$\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\mathbf{M}^{-1}\mathbf{M}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{M}^{-1}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \frac{1}{16}\begin{pmatrix} 2\sqrt{3} & 2 \\ -2 & 2\sqrt{3} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \frac{1}{16}\left(2\sqrt{3}a + 2b,\ -2a + 2\sqrt{3}b\right)$$

So P has coordinates $\left(\dfrac{\sqrt{3}a + b}{8}, \dfrac{\sqrt{3}b - a}{8}\right)$

5 a $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

A represents reflection in the line $y = x$

B represents a rotation through 270° anticlockwise (or 90° clockwise) about (0, 0)

b $$\mathbf{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -p \\ q \end{pmatrix}$$

Therefore the image is $(-p, q)$

6 a $\mathbf{M} = \begin{pmatrix} -4 & 3 \\ 1 & -2 \end{pmatrix}$

det $\mathbf{M} = 5$

Therefore area scale factor is 5 and the area of the original triangle is $\frac{110}{5} = 22$

The triangle has original coordinates $(k, 2)$, $(6, 2)$ and $(6, 7)$

$\text{Area} = \frac{1}{2}bh$

$22 = \frac{1}{2}(b)(5)$

$b = \frac{44}{5}$

So $k = 6 \pm \frac{44}{5}$

$k = -2.8$ or $k = 14.8$

b A general point on the line $x + 3y = 0$ is $\left(x, -\frac{1}{3}x\right)$

$$\begin{pmatrix} -4 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ -\frac{1}{3}x \end{pmatrix} = \begin{pmatrix} -5x \\ \frac{5}{3}x \end{pmatrix}$$

$\left(-5x, \frac{5}{3}x\right)$ lies on the line $x + 3y = 0$ therefore it is invariant under **M**

7 a $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$

$\mathbf{P} = \mathbf{AB}$

$$\mathbf{P} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -4 & 0 \\ 0 & 3 \end{pmatrix}$$

b det $\mathbf{P} = -12$

Therefore area scale factor is 12 and the area of the original triangle is $\frac{60}{12} = 5$

8 a $\mathbf{P} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

det $\mathbf{P} = a^2$

$$\mathbf{P}^{-1} = \frac{1}{a^2}\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{pmatrix}$$

b $\mathbf{P}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{pmatrix}\begin{pmatrix} 4 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{4}{a} \\ \frac{7}{a} \end{pmatrix}$$

So A has coordinates $\left(\frac{4}{a}, \frac{7}{a}\right)$

9 $\mathbf{P} = \begin{pmatrix} -1 & 2 \\ -5 & 8 \end{pmatrix}$ represents **U**

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ represents an anticlockwise rotation through 90° about the origin

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -5 & 8 \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ -1 & 2 \end{pmatrix}$$

Let $\begin{pmatrix} 5 & -8 \\ -1 & 2 \end{pmatrix}$ be the matrix **T**

det $\mathbf{T} = 2$

$$\mathbf{T}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 8 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ \frac{1}{2} & \frac{5}{2} \end{pmatrix}$$

10 a $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & -1 \\ 3 & -2 \end{pmatrix}$

$\mathbf{P} = \mathbf{BA}$

$$\mathbf{P} = \begin{pmatrix} 4 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 1 \\ -3 & 2 \end{pmatrix}$$

b det $\mathbf{P} = -5$

Therefore the area scale factor is 5 and the area of the original triangle is $\frac{35}{5} = 7$

c $\mathbf{Q}$ is the inverse of $\mathbf{P}\begin{pmatrix} -4 & 1 \\ -3 & 2 \end{pmatrix}$

det $\mathbf{P} = -5$

$$\mathbf{P}^{-1} = -\frac{1}{5}\begin{pmatrix} 2 & -1 \\ 3 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{2}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

Challenge

a Let the point P have coordinates (a, b)

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ b \end{pmatrix}$$

Since the x-coordinate is the same as the y-coordinate, it must lie on the line $y = x$.

b Since $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ maps any point onto $y = x$

$\begin{pmatrix} 0 & 1 \\ 0 & m \end{pmatrix}$ will map any point onto $y = mx$

c For $c \neq 0$, then the line $ax + by = c$ does not go through the origin. Hence the origin cannot be mapped to itself, and the transformation is not linear.