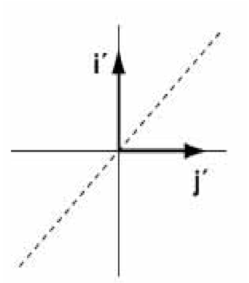
Exercise 6D

1 a $\mathbf{AB} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



Reflection in $y = x$

b $\mathbf{BA} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Reflection in $y = x$

c $\mathbf{AC} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ Enlargement scale factor -2 centre (0, 0)

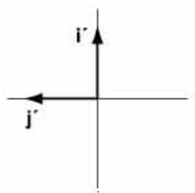
d $\mathbf{A}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Identity (No transformation)

[This can be thought of as a rotation of 180° +180°=360°]

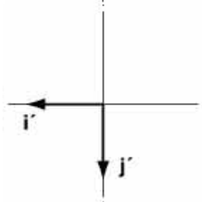
e $\mathbf{C}^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

Enlargement scale factor 4 centre (0, 0)

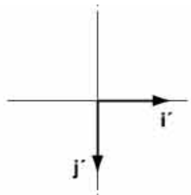
2 a Rotation of 90° anticlockwise $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



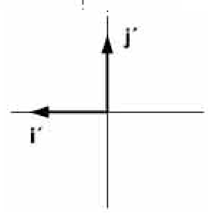
Rotation of 180° about (0, 0) $B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$



Reflection in x-axis $C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



Reflection in y-axis $D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



2 b i $BC = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $(= D)$

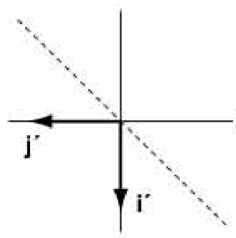
Reflection in y-axis

ii $CB = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $(= D)$

Reflection in y-axis

iii $CD = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $(= B)$

Rotation of 180° about (0, 0)



iv $AD = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Reflection in line $y = -x$.

v $BB = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Rotation of 360° about (0, 0) or Identity

vi $DAC = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $(= A)$

Rotation of 90° anticlockwise about (0, 0)

vii $\text{CBD} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Identify – no transformation

3 a $\mathbf{RS}=\begin{pmatrix}3 & 0\\0 & -2\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}=\begin{pmatrix}0 & 3\\2 & 0\end{pmatrix}$

Reflection in $y = x$ with a stretch by scale factor 3 parallel to the x-axis and by scale factor 2 parallel to the y-axis.

b $\mathbf{RT}=\begin{pmatrix}3 & 0\\0 & -2\end{pmatrix}\begin{pmatrix}5 & 0\\0 & 5\end{pmatrix}=\begin{pmatrix}15 & 0\\0 & -10\end{pmatrix}$

Stretch by scale factor 15 parallel to the x-axis and by scale factor -10 parallel to the y-axis

c $\mathbf{TS}=\begin{pmatrix}5 & 0\\0 & 5\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}=\begin{pmatrix}0 & 5\\-5 & 0\end{pmatrix}$

Enlargement by scale factor 5 about (0,0) and rotation through 270° anti-clockwise.

d $\mathbf{TR}=\begin{pmatrix}5 & 0\\0 & 5\end{pmatrix}\begin{pmatrix}3 & 0\\0 & -2\end{pmatrix}=\begin{pmatrix}15 & 0\\0 & -10\end{pmatrix}$

Stretch by scale factor 15 parallel to the x-axis and by scale factor –10 parallel to the y-axis.

e $\mathbf{ST}=\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}5 & 0\\0 & 5\end{pmatrix}=\begin{pmatrix}0 & 5\\-5 & 0\end{pmatrix}$

Enlargement by scale factor 5 about (0,0) and rotation through 270° anti-clockwise.

f $\mathbf{RST}=\begin{pmatrix}3 & 0\\0 & -2\end{pmatrix}\begin{pmatrix}0 & 5\\-5 & 0\end{pmatrix}=\begin{pmatrix}0 & 15\\10 & 0\end{pmatrix}$

Reflection in $y = x$ with a stretch by scale factor 15 parallel to the x-axis and by scale factor 10 parallel to the y-axis.

4 a $\mathbf{A}=\begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}-2 & 0\\0 & -2\end{pmatrix}$, $\mathbf{C}=\begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}$

b i $\mathbf{AB}=\begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}\begin{pmatrix}-2 & 0\\0 & -2\end{pmatrix}=\begin{pmatrix}-4 & 0\\0 & -6\end{pmatrix}$

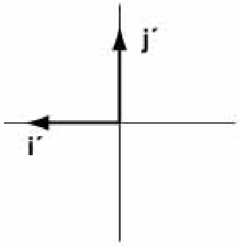
ii $\mathbf{AC}=\begin{pmatrix}2 & 0\\0 & 3\end{pmatrix}\begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}=\begin{pmatrix}8 & 0\\0 & 12\end{pmatrix}$

iii $\mathbf{CB}=\begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}\begin{pmatrix}-2 & 0\\0 & -2\end{pmatrix}=\begin{pmatrix}-8 & 0\\0 & -8\end{pmatrix}$

iv $\mathbf{C}^2=\begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}\begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}=\begin{pmatrix}16 & 0\\0 & 16\end{pmatrix}$

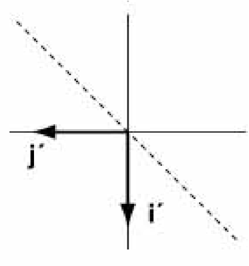
v $\mathbf{ABC}=\begin{pmatrix}-4 & 0\\0 & -6\end{pmatrix}\begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}=\begin{pmatrix}-16 & 0\\0 & -24\end{pmatrix}$

5 Reflection in y-axis



$\therefore$ Matrix is $\mathbf{Y}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$

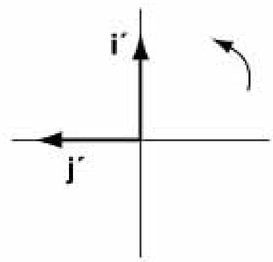
Reflection in $y=-x$



$\therefore$ Matrix $\mathbf{R}=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$

$$\mathbf{RY}=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

$$\begin{pmatrix}1\\0\end{pmatrix}\to\begin{pmatrix}0\\1\end{pmatrix}$$
$$\begin{pmatrix}0\\1\end{pmatrix}\to\begin{pmatrix}-1\\0\end{pmatrix}$$



i.e Rotation of 90° anticlockwise about (0, 0).

6 $\mathbf{T}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $\mathbf{U}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$

$$\mathbf{UT}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\mathbf{TU}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\neq\mathbf{UT}$$

7 a $\mathbf{PQ}=\begin{pmatrix}-4 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}k & 0\\ 0 & k\end{pmatrix}=\begin{pmatrix}-4k & 0\\ 0 & 2k\end{pmatrix}$

b Stretch by scale factor $-4k$ parallel to the x-axis and by scale factor $2k$ parallel to the y-axis.

c $\mathbf{QP}=\begin{pmatrix}k & 0\\ 0 & k\end{pmatrix}\begin{pmatrix}-4 & 0\\ 0 & 2\end{pmatrix}=\begin{pmatrix}-4k & 0\\ 0 & 2k\end{pmatrix}=\mathbf{PQ}$ (from part **a**)

8 a $\mathbf{A}^2=\begin{pmatrix}3 & 0\\ 0 & 4\end{pmatrix}\begin{pmatrix}3 & 0\\ 0 & 4\end{pmatrix}=\begin{pmatrix}9 & 0\\ 0 & 16\end{pmatrix}$

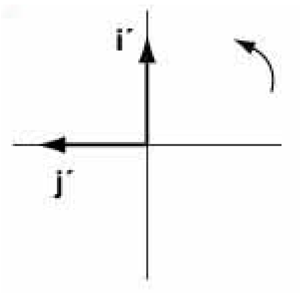
b Stretch by scale factor 9 parallel to the x-axis and by scale factor 16 parallel to the y-axis.

8 c $\mathbf{B}^2 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$

Stretch by scale factor a^2 parallel to the x-axis and by scale factor b^2 parallel to the y-axis.

9 a $\mathbf{R}^2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b



i.e. $\mathbf{R}^2$ represents rotation of 90° anticlockwise about (0, 0)

c **R** represents a rotation of 45° anticlockwise about (0, 0)

d $\mathbf{R}^8$ will represent rotation of $8 \times 45° = 360°$

This is equivalent to no transformation

$\therefore \quad \mathbf{R}^8 = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

10 a $\det \mathbf{M} = \frac{9}{2} + \frac{9}{2} = 9$

$\Rightarrow k = -3$ since $k < 0$

b $\mathbf{M} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$

$\Rightarrow \theta = 45^\circ$

11 $\mathbf{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ -5 & 0 \end{pmatrix}$

$\det \mathbf{M} = 25$

$\Rightarrow$ Area of $T = 75 \div 25 = 3$

12 a $\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

12 b $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

c $\mathbf{TU} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

13 a $\mathbf{A}^2 = \begin{pmatrix} k & \sqrt{3} \\ \sqrt{3} & -k \end{pmatrix}\begin{pmatrix} k & \sqrt{3} \\ \sqrt{3} & -k \end{pmatrix} = \begin{pmatrix} k^2+3 & 0 \\ 0 & k^2+3 \end{pmatrix}$

b Enlargement centre (0,0) with scale factor k^2+3

14 $\mathbf{P}^2 = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}\begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ab-ba \\ ab-ba & b^2+a^2 \end{pmatrix}$

$= \begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix}$

Enlargement centre (0,0) with scale factor $\mathbf{a}^2+\mathbf{b}^2$

Challenge

a $\mathbf{P}^2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$= \begin{pmatrix} \cos^2\theta-\sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta \end{pmatrix} = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$

b Two successive anticlockwise rotations about the origin by an angle θ are equivalent to a single anticlockwise rotation by an angle 2θ.