Exercise 6B

1 a $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

b $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ so $A' = (1, -3)$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ so $B' = (3, -3)$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ so $C' = (3, -2)$

2 a $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

b $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ so $P' = (-1, -1)$

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ so $Q' = (-3, -1)$

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$ so $R' = (-3, -2)$

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$ so $S' = (-1, -2)$

3 a $\begin{pmatrix} \cos 90^0 & -\sin 90^0 \\ \sin 90^0 & \cos 90^0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b $\begin{pmatrix} \cos 270^0 & -\sin 270^0 \\ \sin 270^0 & \cos 270^0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

c $\begin{pmatrix} \cos 45^0 & -\sin 45^0 \\ \sin 45^0 & \cos 45^0 \end{pmatrix}$

$= \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}$

3 d $\begin{pmatrix} \cos 210^0 & -\sin 210^0 \\ \sin 210^0 & \cos 210^0 \end{pmatrix}$

$= \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} -\sqrt{3} & 1 \\ -1 & -\sqrt{3} \end{pmatrix}$

e $\begin{pmatrix} \cos 225^0 & -\sin 225^0 \\ \sin 225^0 & \cos 225^0 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & -\sqrt{2} \end{pmatrix}$

4 a $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

So $A' = (-1,1)$, $B' = (-1,4)$, $C' = (-2,4)$

4 b

$$\begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}+1}{2} \\ \frac{1-\sqrt{3}}{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{4\sqrt{3}+1}{2} \\ \frac{4-\sqrt{3}}{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{4\sqrt{3}+2}{2} \\ \frac{4-2\sqrt{3}}{2} \end{pmatrix}$$

So $A' = \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}, \frac{1}{2} - \frac{\sqrt{3}}{2}\right)$,

$B' = \left(\text{-}2\sqrt{3} - \frac{1}{2}, 2\text{-}\frac{\sqrt{3}}{2}\right)$,

$C' = \left(-2\sqrt{3}\text{-}1, 2 - \sqrt{3}\right)$

5 a $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

So $P' = (2, -2)$,

$Q' = (3, -2)$, $R' = (3, -4)$,

$S' = (2, -4)$

5 b

$$\begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{5\sqrt{2}}{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{7\sqrt{2}}{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ -3\sqrt{2} \end{pmatrix}$$

So $P' = \left(0,\ -2\sqrt{2}\right)$,

$Q' = \left(\frac{\sqrt{2}}{2},\ -\frac{5\sqrt{2}}{2}\right)$, $R' = \left(-\frac{\sqrt{2}}{2}, -\frac{7\sqrt{2}}{2}\right)$,

$S' = \left(-\sqrt{2},\ -3\sqrt{2}\right)$

6 a **A** represents a reflection in the x-axis.
B represents a rotation through 270° anticlockwise about (0, 0).

b (3, −2)

c $b = a - 3b \Rightarrow a - 4b = 0$
$-a = 2a - 2b \Rightarrow 3a - 2b = 0$
$\Rightarrow a = 0, b = 0$

7 a Rotation through 225° anticlockwise about (0, 0)

b $-\frac{1}{\sqrt{2}}p + \frac{1}{\sqrt{2}}q = -\sqrt{2}$
$\Rightarrow -p + q = -2 \Rightarrow p - q = 2$
$-\frac{1}{\sqrt{2}}p = \frac{1}{\sqrt{2}}q = -2\sqrt{2}$
$\Rightarrow -p - q = -4 \Rightarrow p + q = 4$
$\Rightarrow p = 3, q = 1$

7 c $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

d $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -\sqrt{2} \\ -2\sqrt{2} \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

So $C' = (-3, -1)$

Rotation through 45° clockwise about (0,0)

8 a Reflection in the line $y = x$

b Since points on the line $y = x$ are invariant points, and the lines $y = x$ and $y = -x + k$ for any value of k are invariant lines, three different invariant lines are, for example, $y = x$, $y = -x$ and $y = -x + 1$

c $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

9 a $a = -0.5$

b $\cos\theta = -0.5 \Rightarrow \theta = 120^\circ$ or 240°

So possible matrices are

$\begin{pmatrix} \cos 120^\circ & -\sin 120^\circ \\ \sin 120^\circ & \cos 120^\circ \end{pmatrix} = \begin{pmatrix} -0.5 & -0.866 \\ 0.866 & -0.5 \end{pmatrix}$

$\begin{pmatrix} \cos 240^\circ & -\sin 240^\circ \\ \sin 240^\circ & \cos 240^\circ \end{pmatrix} = \begin{pmatrix} -0.5 & 0.866 \\ -0.866 & -0.5 \end{pmatrix}$

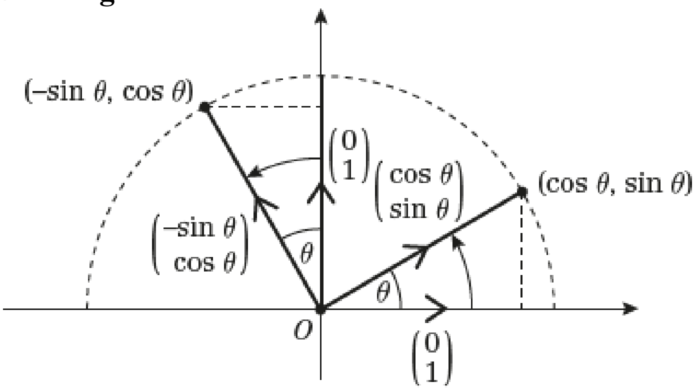
10 a $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b $a - 5b = -b \Rightarrow a = 4b$

$3b = a \Rightarrow a = 3b$

So $a = 0, b = 0$

Challenge



The diagram shows that rotating $\begin{pmatrix}1\\0\end{pmatrix}$ by θ takes it to $\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}$ and rotating $\begin{pmatrix}0\\1\end{pmatrix}$ by θ takes it to $\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}$. We know that the image of $\begin{pmatrix}1\\0\end{pmatrix}$ is the first column of the matrix representing the transformation and that the image of $\begin{pmatrix}0\\1\end{pmatrix}$ is the second column of the matrix representing the transformation. So the matrix representing the transformation is $\begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix}$