Exercise 5D

1 a

$$\det\begin{vmatrix}3 & -1\\-4 & 2\end{vmatrix} = 6-(-4)\times(-1)$$
$$= 6-4$$
$$= 2 \quad \neq 0$$

$\therefore$ the matrix is non-singular

So inverse is $\frac{1}{2}\begin{pmatrix}2 & 1\\4 & 3\end{pmatrix}$

or $\begin{pmatrix}1 & 0.5\\2 & 1.5\end{pmatrix}$

b $\det\begin{vmatrix}3 & 3\\-1 & -1\end{vmatrix} = -3-(-1)\times 3$
$$= -3+3$$
$$= 0$$

$\therefore$ matrix is singular.

c $\det\begin{vmatrix}2 & 5\\0 & 0\end{vmatrix} = 0-0$
$$= 0$$

$\therefore$ matrix is singular

d $\det\begin{vmatrix}1 & 2\\3 & 5\end{vmatrix} = 5-6$
$$= -1 \quad \neq 0$$

$\therefore$ matrix is non-singular

Inverse is $\frac{1}{-1}\begin{pmatrix}5 & -2\\-3 & 1\end{pmatrix} = \begin{pmatrix}-5 & 2\\3 & -1\end{pmatrix}$

1 e $\det\begin{vmatrix}6 & 3\\4 & 2\end{vmatrix} = 12-12$
$$= 0$$

$\therefore$ matrix is singular

f $\det\begin{vmatrix}4 & 3\\6 & 2\end{vmatrix} = 8-18$
$$= -10 \quad \neq 0$$

$\therefore$ matrix is non-singular

Inverse is $\frac{1}{-10}\begin{pmatrix}2 & -3\\-6 & 4\end{pmatrix}$
$$= \begin{pmatrix}-0.2 & 0.3\\0.6 & -0.4\end{pmatrix}$$

2 a Let $\mathbf{A} = \begin{pmatrix}a & 1+a\\1+a & 2+a\end{pmatrix}$

$$\det\ \mathbf{A} = 2a+a^2-(1+a)^2$$
$$= 2a+a^2-1-2a-a^2$$
$$= -1$$
$$\mathbf{A}^{-1} = \frac{1}{-1}\begin{pmatrix}2+a & -(1+a)\\-(1+a) & a\end{pmatrix}$$
$$= \begin{pmatrix}-(2+a) & (1+a)\\(1+a) & -a\end{pmatrix}$$

b Let $\mathbf{B} = \begin{pmatrix}2a & 3b\\-a & -b\end{pmatrix}$

$$\det\ \mathbf{B} = -2ab-(-a)\times 3b$$
$$= -2ab+3ab$$
$$= ab$$
$$\mathbf{B}^{-1} = \frac{1}{ab}\begin{pmatrix}-b & -3b\\a & 2a\end{pmatrix}$$
$$= \begin{pmatrix}-\frac{1}{a} & -\frac{3}{a}\\\frac{1}{b} & \frac{2}{b}\end{pmatrix} \quad \textbf{provided that } ab \neq 0$$

3 a
$$\mathbf{ABC} = 1$$
$$\Rightarrow \mathbf{A}^{-1}\mathbf{ABC} = \mathbf{A}^{-1}\mathbf{I}$$
$$\Rightarrow \mathbf{BC} = \mathbf{A}^{-1}$$
$$\Rightarrow \mathbf{BCC}^{-1} = \mathbf{A}^{-1}\mathbf{C}^{-1}$$
$$\Rightarrow \mathbf{B} = \mathbf{A}^{-1}\mathbf{C}^{-1} = (\mathbf{CA})^{-1}$$
$$\therefore \mathbf{B}^{-1} = \mathbf{CA}$$

b
$$\mathbf{CA} = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & -6 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$$
$$\therefore (\mathbf{CA})^{-1} = \frac{1}{-3+4}\begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$$
$$\mathbf{B} = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}$$

4 a
$$\mathbf{AB} = \mathbf{C}$$
$$\Rightarrow \mathbf{A}^{-1}\mathbf{AB} = \mathbf{A}^{-1}\mathbf{C}$$
$$\Rightarrow \mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$$

b
$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix} \Rightarrow \det \mathbf{A} = 6 - -4 = 10$$
$$\therefore \mathbf{A}^{-1} = \frac{1}{10}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}$$
$$\therefore \mathbf{B} = \mathbf{A}^{-1}\mathbf{C}$$
$$= \frac{1}{10}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 1 & 22 \end{pmatrix}$$
$$= \frac{1}{10}\begin{pmatrix} 10 & 40 \\ -10 & 20 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 4 \\ -1 & 2 \end{pmatrix}$$

5 a
$$\mathbf{BAC} = \mathbf{B}$$
$$\Rightarrow \mathbf{B}^{-1}\mathbf{BAC} = \mathbf{B}^{-1}\mathbf{B}$$
$$\Rightarrow \mathbf{AC} = \mathbf{I}$$
$$\Rightarrow \mathbf{A} = \mathbf{C}^{-1}.$$

5 b
$$\mathbf{C} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$
$$\det \mathbf{C} = 10 - 9 = 1$$
$$\therefore \mathbf{C}^{-1} = \frac{1}{1}\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$$
$$\therefore \mathbf{A} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}.$$

6 a
$$\mathbf{A} = \begin{pmatrix} 3a & b \\ 4a & 2b \end{pmatrix} \Rightarrow$$
$$\det \mathbf{A} = 6ab - 4ab = 2ab$$
$$\therefore \mathbf{A}^{-1} = \frac{1}{2ab}\begin{pmatrix} 2b & -b \\ -4a & 3a \end{pmatrix}.$$

b
$$\mathbf{B} = \mathbf{XA}$$
$$\Rightarrow \mathbf{BA}^{-1} = \mathbf{XAA}^{-1}$$
$$\therefore \mathbf{X} = \mathbf{BA}^{-1}$$

So
$$\mathbf{X} = \begin{pmatrix} -a & b \\ 3a & 2b \end{pmatrix}\begin{pmatrix} 2b & -b \\ -4a & 3a \end{pmatrix} \times \frac{1}{2ab}$$
$$= \frac{1}{2ab}\begin{pmatrix} -6ab & 4ab \\ -2ab & 3ab \end{pmatrix}$$
$$\therefore \mathbf{X} = \begin{pmatrix} -3 & 2 \\ -1 & \frac{3}{2} \end{pmatrix}$$