## Exercise 5C

**1 a** $\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} = 3\times 2 - 4\times(-1) = 10$

**b** $\begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix} = 4\times 2 - 2\times 1 = 6$

**c** $\begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} = (-2)\times 0 - 1\times 3 = -3$

**d** $\begin{vmatrix} -4 & -4 \\ 1 & 1 \end{vmatrix} = (-4)\times 1 - (-4)\times 1 = 0$

**e** $\begin{vmatrix} 7 & -4 \\ 0 & 3 \end{vmatrix} = 7\times 3 - (-4)\times 0 = 21$

**f** $\begin{vmatrix} -1 & -1 \\ -6 & -10 \end{vmatrix} = (-1)\times(-10) - (-1)\times(-6) = 4$

**2 a**

$\det\begin{vmatrix} a & 1+a \\ 3 & 2 \end{vmatrix} = 2a - 3(1+a)$

$= 2a - 3 - 3a$

$= -3 - a$

Matrix is singular for $a = -3$

**b** Let $\mathbf{A} = \begin{pmatrix} 1+a & 3-a \\ a+2 & 1-a \end{pmatrix}$

$\det \mathbf{A} = (1+a)(1-a) - (3-a)(a+2)$

$= 1 - a^2 - (-a^2 + a + 6)$

$= 1 - a^2 + a^2 - a - 6$

$= -a - 5$

$\det \mathbf{A} = 0 \Rightarrow a = -5$

**c** Let $\mathbf{B} = \begin{pmatrix} 2+a & 1-a \\ 1-a & a \end{pmatrix}$

$\det \mathbf{B} = 2a + a^2 - (1-a)^2$

$= 2a + a^2 - 1 + 2a - a^2$

$= 4a - 1$

$\det \mathbf{B} = 0 \Rightarrow a = \frac{1}{4}$

**3** $\det \mathbf{M} = \begin{vmatrix} -2 & 1-k \\ k-1 & k \end{vmatrix} = (-2)\times k - (1-k)\times(k-1)$

$= k^2 - 4k + 1$

For singular matrix $k^2 - 4k + 1 = 0$

$k = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4\times 1\times 1}}{2\times 1}$

$= \dfrac{4 \pm \sqrt{12}}{2}$

$k = 2 + \sqrt{3},\ k = 2 - \sqrt{3}$

**4** $\det \mathbf{P} = \begin{vmatrix} 3k & 4-k \\ k-2 & -k \end{vmatrix}$

$= 3k\times(-k) - (4-k)\times(k-2)$

$= -2k^2 - 6k + 8$

For singular matrix $2k^2 + 6k - 8 = 0$

$(k+4)(k-1) = 0$

$k = -4$ and $k = 1$

**5 a** $\mathbf{A} = \begin{pmatrix} a & 2a \\ b & 2b \end{pmatrix} \Rightarrow \det \mathbf{A} = 2ab - 2ab = 0$

$\mathbf{B} = \begin{pmatrix} 2b & -2a \\ -b & a \end{pmatrix} \Rightarrow \det \mathbf{B} = 2ab - 2ab = 0$

**b** $\mathbf{AB} = \begin{pmatrix} a & 2a \\ b & 2b \end{pmatrix}\begin{pmatrix} 2b & -2a \\ -b & a \end{pmatrix}$

$= \begin{pmatrix} 2ab - 2ab & -2a^2 + 2a^2 \\ 2b^2 - 2b^2 & -2ab + 2ab \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

**6 a**

$\det \mathbf{M} = \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = 1\times 1 - (-3)\times 2 = 7$

**6 b**

$$\det \mathbf{N} = \begin{vmatrix} -1 & k \\ 4 & 3 \end{vmatrix} = (-1)\times 1 - k\times 4 = -3-4k$$

$\det \mathbf{N} = 7$

$-3-4k = 7$

$4k = -10 \Rightarrow k = -2.5$

**c**

$$\mathbf{MN} = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & -2.5 \\ 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -13 & -11.5 \\ 2 & -2 \end{pmatrix}$$

**d**

$$\det \mathbf{MN} = \begin{vmatrix} -13 & -11.5 \\ 2 & -2 \end{vmatrix}$$

$$= (-13)\times(-2) - (-11.5)\times 2 = 49$$

$\det \mathbf{M} = 7$

$\det \mathbf{N} = -3 + (-4)\times(-2.5) = 7$

$\det \mathbf{MN} = 7\times 7 = 49$

**Challenge**

**a**

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 0 \text{ for singular matrix}$$

$ad - bc = 0 \Rightarrow ad = bc$

The possibilities are:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$$

**b**

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$